THE PROVING PROCESS IN MATHEMATICS CLASSROOM – METHOD AND RESULTS OF A VIDEO STUDY[1]–

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Mathematical proof is one of the most difficult topics for students to learn. Several empirical studies revealed different kinds of students' problems in this area. Our own research suggests that students' views on proofs and their abilities in proving are significantly influenced by their specific mathematics classrooms. However, the reasons for these differences in the students' performance remain unclear. Accordingly, we conducted a video study in order to analyse proof instruction in Germany. In this article we will present firstly a method for evaluating instruction and secondly some results that describe proving processes in mathematics classrooms at the lower secondary level from a mathematical perspective.

1. Introduction

Reasoning, proof and argumentation in the mathematics classroom is an important issue in mathematics education research. In the last years one could observe an increasing portion of empirical research on this subject. Moreover, reasoning and argumentation in mathematics was imbedded in international comparative studies like TIMSS or PISA (cf. Baumert et al., 1997; Deutsches PISA-Konsortium, 2001). Our own research adds to this and focuses on proving processes in the mathematics classroom. The aim of the video study, that we will present in this paper, is to describe the classroom conditions under which students learn mathematical proofs. In this article we present a method for evaluating proof instruction and first results of this video study. In particular, we analyse the proving process in the mathematics classroom from a mathematical perspective in a quantitative research design.

2. The role of proof in mathematics and in the mathematics classroom

Concerning the role of proof in mathematics and in the mathematics classroom we want to emphasize three important topics: proof as a social construct, the different functions of a proof, and the distinction between the process of proving and the proof as a product. Since a detailed discussion of these three topics will go beyond the scope of this article, we will restrict ourselves to the main ideas which should give an outline of the framework our research is embedded in.

There is an extensive discussion about the nature of proof in mathematics (e.g., Hanna & Jahnke, 1996). Though mathematics seems to be a strict and exact scientific discipline it cannot be denied that there are no clear definitions for basic notions like for example "proof". There exist probably some necessary conditions for a mathematical proof, but the acceptance of a proof takes place by unwritten rules of the mathematic community: „A proof becomes a proof after the social act of 'accepting it as a proof'" (Manin, 1977, p. 48). This is a fact that under different circumstances might also hold for the mathematics classroom (e.g. Herbst, 1998). The question

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whether a proof is accepted by the community depends on various factors. On the one hand, the proof has to validate whether a conjecture is true. On the other hand, the proof has to satisfy various other functions. As stressed by Hanna & Jahnke (1996), de Villiers (1990) and others, proving in mathematics is more than validation. Hanna & Jahnke (1996) describe eight different proof functions (like explanation, systematisation etc.). The function of validation, which for many people seems to be the most important one, is for mathematicians only one among others.

Mathematicians know through their own work, that the proving process and the proof as a product of this process must be distinguished. Sometimes the process of proving a theorem may take years and may include various approaches which may or may not lead to a success. In general, none of these efforts can be seen in the final product, that is in the formal written proof. Consequently, for the teaching and learning of proof it is not sufficient to show only the product. It is more important to stress the proof process. Boero (1999) described an expert model of the proof process. It is divided into different phases and gives insight into the combination of explorative empirical-inductive and hypothetical-deductive steps during the generation of a proof. We refer to Boero (1999) for the original description of this model; an adapted version for the analysis of proving processes in the mathematics classroom is given in Section 4.2.

3. Students' proof competencies – empirical results

In the last decade several empirical studies were conducted which gave an overview about students' mathematical competencies in different countries. In addition to studies like TIMSS and PISA which showed that in most countries comparatively few students perform well with proof items, there are results of several national studies which focus on the student competence in reasoning and proof (e.g., Healy & Hoyles, 1998; Lin, 2000; Reiss, Klieme & Heinze, 2001).

In an ongoing study with 669 students in grade 7 and 8 of the German Gymnasium (high attaining students) we focussed on the question which cognitive and non-cognitive factors are influencing the students' ability to perform geometrical proofs (cf. Reiss, Hellmich & Reiss, 2002). From the outcomes of two tests we identified three levels of competency: (I) basic competency, (II) argumentative competency (one-step-argumentation) and (III) argumentative competency (combining several steps of argumentation). Low-achieving students were not able to solve any items on level III whereas students from the upper third performed well on level I and level II items and were showing a satisfying performance for level III tasks (cf. Reiss, Hellmich & Reiss, 2002, Table 1 for the grade 7 test). A deeper analysis of students' responses to the test items and additional interviews (Heinze, in preparation) indicate that high-achieving students have problems with combining arguments to a proof and with generating a proof idea whereas low-achieving students in addition have deficits in their declarative and methodological knowledge.

The previous results become even more interesting, if one considers the overall classroom level of performance. In both tests we saw enormous differences in the achievement level of the participating classes (range of the mean scores in grade 7: 22 – 68 % of the possible points, in grade 8: 12 – 59 %). A two-level analysis for the grade 8 test showed that 42.4% of the variance of the individual achievement can be explained by the classroom level. It may be assumed that an essential portion of this influence is based on instruction, since there are hardly any differences in other factors like the number of students per class, their social background etc.

There are not many studies analysing the specifics of the mathematics classroom. One of most important studies in this respect is the TIMS video study comparing classrooms in Germany, Japan, and the United States (Stiegler et al. 1999). The results characterised the typical German teaching style as guiding students through the development of a procedure by asking them to orally fill in relevant information. The teacher generally presents the problem at the black board, eliciting ideas and procedures from the class as work on the problem progresses (cf. Stiegler et al. 1999, p. 133). In the German sample of the TIMS video study proofs occur only in a few lessons. Until now there are no results of video studies focussing on the teaching of proofs in Germany[2]. However, proof in mathematics classroom in Germany is not a blind spot. For example, Knipping (2001) observed the teaching of the Pythagorean theorem and analysed the corresponding proving process on a micro level. The focus of her study was to understand the processes of argumentation and not the connection between the teaching of proof and the student achievement in proof.

4. Research questions and design

4.1. Research questions and sample

The empirical results in the previous section show that students have difficulties dealing with mathematical proofs. Moreover, the statistical analysis indicates that the students' proof competence is substantially influenced by classroom factors. Consequently, we have to focus on the situation in mathematics classrooms and, in particular, the lessons involving proof instruction. This is the starting point of the present video study which is concerned with the process of proving in actual mathematics classrooms. The study is guided by the following research questions:

- How are proofs taught in mathematics classrooms in Germany?
- Which aspects of proof are emphasized by the teachers in the proving process?
- Are there gaps in the proving process or aspects that are underemphasized?

The data of the presented study consists of 20 videotaped mathematics lessons with grade 8 students. The lessons are from eight different classes of four schools (German Gymnasium, i.e., high attaining students). In each class we videotaped between two and four consecutive lessons. The subject of all lessons was reasoning and proof in

[2] In addition to our study there is one other ongoing video project on proofs concerning the teaching of the Pythagorean theorem in German classrooms: http://www.dipf.de/projekte/qualitaetssicherung_pythagoras.htm.

geometry, in particular, in congruence geometry. The participating teachers were asked to provide their typical instruction. Asked by questionnaires after the lessons the vast majority of the students confirmed that the videotaped instruction was the instruction they were used to.

4.2 Design of the study

For the analysis and evaluation of the proving processes in the videotaped lessons we use the model of Boero (1999; cf. Section 2). Since Boero's model gives a detailed description of the mathematical proving process of experts, we found it necessary to modify the phases of Boero to the following coding categories.

1. Phase: The first phase consists of the exploration of the problem situation, the generation of a conjecture and the identification of different types of arguments for the plausibility of this conjecture. This phase is denoted as

treated well: all elements are presented by the students;
treated: either some of these elements are given by the teacher or the phase is very short;
treated badly: the teacher performs the first phase;
not treated: in other cases.

2. Phase: The second phase consists of the precise formulation of the conjecture according to the shared textual conventions. This phase is denoted as

treated well: the students formulate the conjecture (possibly corrected by the teacher);
treated: only the teacher gives a formulation of the conjecture;
treated badly: there are mistakes in the final version of the conjecture;
not treated: there is no formulation of the statement that is to be proved.

3. Phase: This is again an explorative phase that is based on the formulated conjecture. The aim is the identification of appropriate arguments for the conjecture and a rough planning of a proof strategy. We distinguish this phase in four subcategories: (1) reference to the assumptions, (2) investigation of the assumptions, (3) collection of further information and (4) generation of a proof idea. We denote this phase as

treated well: at least three of these subcategories are observed in this phase;
treated: two of these subcategories are observed;
treated badly: there is only one of these subcategories;
not treated: in all other cases.

4. Phase: Based on the proof idea and the selected arguments of Phase 3 it follows the combination of these arguments into a deductive chain that constitutes a sketch of the final proof. This phase can be performed only verbally or in connection with some written remarks; it is denoted as

treated well: students (supported by the teacher) give substantial contributions;
treated: it is presented mostly or exclusively by the teacher;
treated badly: there are big gaps or other deficits in the deductive chain;
not treated: in all other cases.

5. Phase: This is the last phase for the proving process in school mathematics. Here the chain of arguments of Phase 4 is written down according to the standards given in the respective mathematics classroom. In particular, it is important that this phase also gives a retrospective overview about the proof process. It is denoted as

treated well: all steps are written down and there is a retrospective summary of the proof process;
treated: the most important steps are written down and there is a retrospective summary;
treated badly: there are only some arguments written down, but no retrospective summary;
not treated: in all other cases.

For our study we applied this operationalised model of the proving process to all proofs we identified in the 20 videotaped lessons. For each proof we measured the time that was spent for the different phases and we determined the quality of each phase with respect to the categories described above. Hence, we achieved a characterisation of the proving process with respect to qualitative and quantitative criterions.

5. Results

According to the research questions (cf. Section 4.1) our aim is to describe the teaching of proof in mathematics classroom and, particularly, to describe steps in the proving processes that are emphasized or underemphasized. In the 20 videotaped mathematics lessons we identified 22 proofs. Each proof process was evaluated separately.

5.1. Time-based analysis

Diagram 1 gives an overview about the time portion (in percent) of each phase in the proof instruction (mean values). One can see that most of the time is spent on the fourth phase (about 36%), which is the organisation of arguments in a deductive chain. The first phase as an experimental phase which mainly consists of drawing and measurements takes about a quarter of the time. The important third phase, in which the exploration of the conjecture and the identification of arguments take place summarizes to 22 % of the proof time.

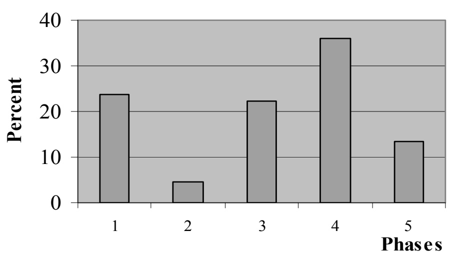


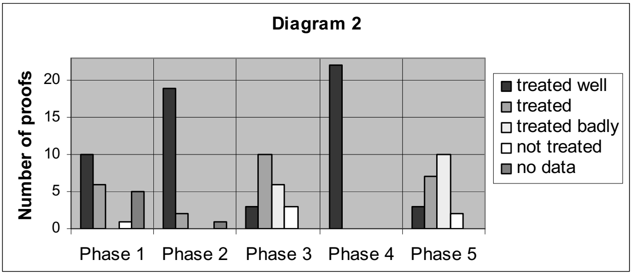
Diagram 1: Percent of time used per phase.

This diagram reflects the typical process of dealing with proof problems in the videotaped lessons. First of all, the students have to draw a geometrical figure and to make some observations on an experimental level. Afterwards a conjecture is discussed and formulated (Phase 2). In the third phase the students have the opportunity to propose some ideas for the proof. If the students are not able to generate a proof idea, then the teacher gives more and more hints. Then the proof is organized step by step on the chalk board. This takes place in a kind of classroom discourse, in which the teacher leads the students through the proof by specific questions. In other words, the students have to follow the proof the teacher has in his/her mind. The writing on the chalk board is frequently a collection of arguments as expected in Phase 4. Sometimes some parts are already very detailed as expected in Phase 5. Only in a few cases a genuine Phase 5 took place, in which additionally a retrospective summary is given.

5.2 Quality of proof instruction

In addition to the time-based results we also collected data about the quality of the proof phases (cf. Section 4.2). In some cases it was not possible to rate the complete proving process, since some phases were partly done in a lesson before that was not videotaped, or they were part of the homework for the observed lesson. This problem affected mainly Phase 1 and here, particularly, the drawings that were prepared at home.

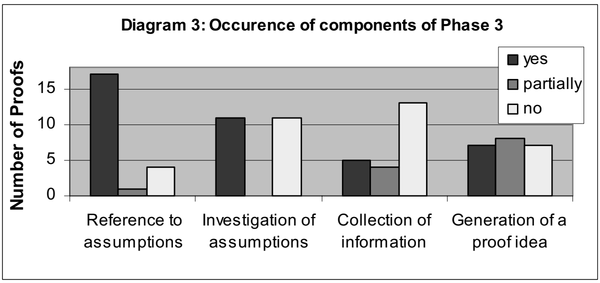
We can see in Diagram 2 that, in particular, the quality of Phase 2 and Phase 4 in the proof processes is quite good. Moreover, one can say, that Phase 1 is treated satisfactorily. Substantial deficits occur mainly in the Phases 3 and 5. The requirements of these phases are not satisfied in 9 and 12, respectively, of the 22 proofs. Their quality in the remaining proofs is in most cases rated as "treated" and only in a few cases as "treated well".



If we combine the time-based results of Section 5.1 with the quality-based results of this Section 5.2 we can identify clear deficits in the proof processes for the Phases 3 and 5. Both phases are important for the learning of mathematical proofs, however, Phase 3 as a phase of exploration of the conjecture, collecting additional information and generating a proof idea seems to be the most crucial phase for proof instruction on the lower secondary level. Therefore, we take a closer look on this phase.

5.3 A detailed analysis of Phase 3

As described in Section 4.2 we split Phase 3 into four different subcategories. In the videotaped lessons we analysed which of these subcategories occurred in the observed proofs. The results in Diagram 3 indicate that in general only one of the four components appeared: in 17 of 22 proofs the teacher and students



referred to the assumptions in Phase 3. Only half of the proof processes observed included an investigation of these assumptions. The collection of additional information did not appear in thirteen proof processes and was only partially included in four cases. Even the generation of a proof idea was given in only seven proofs and given partially in only eight proofs.

6. Discussion

Our analysis points up that essential phases in the proof process are neglected by the teachers. As already mentioned in Section 5.1 the typical proof process in the mathematics classroom is planned and controlled by the teacher. This means that the teacher leads the students through the "labyrinth" of the proof situation. The role of the students is to guess which direction the teacher has in mind. This kind of instruction is caused by the so called "fragend-entwickelnd" (questioning-developing) teaching style, which the German TIMSS video group identified as the most popular form of mathematics instruction in Germany (cf. Klieme, Schüner & Claussen, 2001). The problem with this kind of teaching is that there is no place for in-depth phases which are necessary in the proof process, e.g., for the exploration of the problem situation or the collection of additional information. In the videotaped lessons the first exploration phase (Phase 1) consists mainly of making drawings and measuring the lines or angles. The second in-depth phase (Phase 3) the students have no time for a deeper investigation or exploration of the situation (see Section 5). The consequence is that they get no real chance to solve the proof problems on their own. They have to follow the hints and questions of their teacher. As shown in Diagram 1 Phases 4 and 5 take most of the time in the proof processes (apart from the time for drawings in Phase 1). However, in general, these two phases do not coincide exactly with the description in the model of Boero. We observed more or less a mixture of the ideal Boero phases: the teacher elaborated on the proof step by step at the chalk board by asking questions and giving hints (i.e., applying the "fragend-entwickelnd" teaching style). From the students' perspective the proof splits into small steps which they have to deal with successively. Many students will finally loose the overview, since a retrospective summarizing of the proof was lacking in most lessons.

Taking into account the results of our video analysis, we are not surprised that the proof competence of German students on the lower secondary level is poor (Section 3). We think that the students' problems are significantly influenced by the proof instruction as the results of the video study given in this article indicate. A systematic investigation of this "correlation" is in process. It requires a deeper analysis of the video data than the one described in this article.

We have to emphasize that we analysed the teaching of proof in the videotaped lessons only from a mathematical perspective. This means that in this study other questions and problems like the teaching style, the combination of classwork and seatwork, the participation of students etc. were not discussed. However, already the mathematical point of view discovered problems in proof instruction, since important topics concerning the proving process were not covered sufficiently.

Finally, we want to stress that we could identify examples of good practice in the lessons observed. In two classrooms all proof phases were treated well, and for each phase the teacher spent an appropriate portion of time.

7. References

Baumert, J. et al. (1997). *TIMSS – Mathematisch-naturwissenschaftlicher Unterricht im internationalen Vergleich.* Opladen: Leske + Budrich.

Boero, P. (1999). Argumentation and mathematical proof: A complex, productive, unavoidable relationship in mathematics and mathematics education. *International Newsletter on the Teaching and Learning of Mathematical Proof, 7/8.*

Deutsches PISA-Konsortium (2001). *PISA 2000: Basiskompetenzen von Schülerinnen und Schülern im internationalen Vergleich.* Opladen: Leske + Budrich.

Hanna, G. & Jahnke, H. N. (1996). Chapter 23: Proof and proving. In: Bishop, A. J., Clements, K., Keitel, C., Kilpatrick, J., Laborde, C. (Eds.) *International handbook of mathematics education. Vol. 4. Pt. 2* (877 - 908). Dordrecht: Kluwer.

Healy, L. & Hoyles, C. (1998). *Justifying and Proving in School Mathematics*. Technical report on the nationwide survey. Institute of Education, University of London.

Heinze, A. (in preparation). Schülerprobleme beim Lösen von geometrischen Beweisaufgaben – eine Interviewstudie.

Herbst, P. G. (1998). *What works as proofs in the mathematics class*. Doctoral dissertation, University of Georgia.

Klieme, E., Schüner, G. & Knoll, S. (2001). Mathematikunterricht in der Sekundarstufe I: „Aufgabenkultur" und Unterrichtsgestaltung. In: Bundesministerium für Bildung und Forschung (BMBF) (Ed.), *TIMSS – Impulse für Schule und Unterricht*. (43 - 57). Bonn: BMBF.

Knipping, C. (2001) Towards a comparative analysis of proof teaching. In: Heuvel-Panhuizen, v.d. (Ed.) Proceedings of the 25th Conference of the International Group for the Psychology of mathematics education, Bd. 3, Utrecht, 249-256.

Lin, F.L. (2000). An approach for developing well-tested, validated research of mathematics learning and teaching. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1 (pp. 84-88). Hiroshima: Hiroshima University.

Manin, Y. (1977). *A Course in Mathematical Logic*. New York: Springer.

Reiss, K., Hellmich, F. & Reiss, M. (2002) Reasoning and proof in geometry: Prerequisites of knowledge acquisition in secondary school students. In A.D. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education*. Volume IV (pp. 113-120). Norwich (Great Britain). University.

Reiss, K., Klieme, E. & Heinze, A. (2001). Prerequisites for the understanding of proofs in the geometry classroom. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, 97-104). Utrecht: Utrecht University.

Stigler, J. et al. (1999). *The TIMSS videotape classroom study*. U.S. Department of Education. National Center for Education Statistics, Washington,DC: U.S. Government Printing Office. http://nces.ed.gov/timss.

de Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras*, 24, p. 17-24.