THE NUMBER LINE AS A REPRESENTATION OF DECIMAL NUMBERS: A RESEARCH WITH SIXTH GRADE STUDENTS

Niki Michaelidou, Athanasios Gagatsis & Demetra Pitta-Pantazi
Department of Education, University of Cyprus

Based on Janvier's (1987) model on representations, in this paper we examine 12 year old students' understanding of the concept of decimal numbers. For this reason the study was conducted with the use of three kinds of tests related to decimal numbers. These three tests involved recognition, representation and translation tasks. In particular, the idea of the number line as a geometrical model is being discussed in respect to representations and translation between different representations. The application of the implicative statistical method demonstrated a compartmentalization of the different tasks and this signifies that there is a lack of coordination between recognition, representation and translation in decimal numbers.

INTRODUCTION

One of the aims of mathematics instruction is to achieve the understanding of mathematical concepts through the development of rich and well organized cognitive representations (Goldin, 1998; NCTM, 2000; DeWindt-King, & Goldin, 2003). In this study the term representation is interpreted as the tool used for representing mathematical ideas such as tables, equations and graphs (Confrey & Smith, 1991).

The first aim of this study is to investigate 12 year old students' understanding of addition of decimal numbers. Second, to explore the way these students understand and utilize the representations of decimal numbers as different concepts – modularization – or as different expressions of the same concept. The third aim of the study is to investigate which modes of representation of decimal numbers are the most difficult ones. Finally, the study is dealing with the differences in students' achievements concerning the issue addressed by the different exercises – concept and addition of decimals.

THEORETICAL BACKGROUND

Decimal numbers, representations and number line

According to Janvier (1987) the understanding of a mathematical concept passes through at least 3 stages:

1. The recognition of the mathematical concept among different representations
2. The flexible manipulation within a system of representation
3. The translation from one system of representation to another

In particular the ability to translate from one system of representation of a concept to another is of great importance, concerning mathematical problem solving and the learning of mathematics in general.

The concept of decimal numbers is included in mathematics curricula and it is considered to be of great significance especially due to its application and use in every day life. The specific concept can be linked to the concept of fractions, since decimals can be considered as the parts of the whole, a whole that has been divided into 10, 100, 1000, or some other number of parts that is in a power of 10. This is the reason why instruction should not approach decimals as an isolated mathematical concept. Decimal concepts need to be related to a variety of fraction ideas and to place value. Furthermore, teaching decimals in a comprehensive way should give students the opportunity to flexibly, utilize and link a variety of representations concerning the decimal numbers: linear models, such as the number line, manipulatives, surface, symbols and currency (Thomson & Walker, 1996).

Research findings often disagree in regard to the importance of number line as a didactical model in general and as a means of representing integers and rational numbers (Ernest, 1985; Behr, Lesh, Post, & Silver, 1987; Lesh, Post, & Behr, 1987; Raftopoulos, 2002; Gagatsis, Shiakalli, & Panaoura, 2003; Michaelidou & Gagatsis, 2003). For example, Thomson and Walker (2000), underline the fact that the knowledge of how to use the number line as a means to represent the decimals is quite necessary since it contributes to the development of concepts not only related to the identification and comparison of decimals but to the ability to perform operations as well.

Sometimes the disagreement of the researchers on the role of the number line is due to the fact that number line is not a common or standard representation but a geometrical model. In fact a directed straight line in Euclidean Geometry can serve as a geometrical model for addition, subtraction, multiplication and division of rational numbers as well as for constructing irrationals. Operations on real numbers are represented as operations on segments on the line. A straight line with a scale on it belongs to a mixed type of representations. On the one hand, it functions as a new comprehensive geometrical model with the rational numbers corresponding not only to directed segments and operators on them but also to a set of distinct points on the line. On the other hand, the scale can be used as a means of arithmetization. Points on the line can be numbered in such a way that differences of numbers measure distances of corresponding points. In this way, every geometric operation – like segment addition – can be translated into an arithmetic operation and carried out algorithmically (Gagatsis, et al., 2003).

Two ideas have been seminal to this study: The first is Janvier's classification on the understanding of a mathematical concept and the second is the idea of number line as a geometrical model. Thus we propose the following four research questions:

(a) Are 12-year old students able to recognize decimal numbers in different representation systems, verbal, symbolic and number line?

(b) Can 12-year old students flexibly manipulate the concept of decimals within a given representational system?

(c) Can 12 year old students accurately translate decimal concepts from one system of representation to another? And are some translations easier than others?

(d) What are the relationships among children's responses to tasks of recognition, representation and translation?

METHOD

A hundred and twenty 12-year-old students from three primary schools in Limassol, Cyprus participated in this study.

The study had three distinct phases. In phase 1, Test A was administrated to all 120 students. Test A, which had a multiple choice format aimed to examine students' ability to recognize and represent the concept of decimal numbers in a variety of different representations – line segment, number line and rectangular surface (Fig.1).

For example: Circle those diagrams where the shaded part presents 0.9

Line segment Number Line Rectangular Surface

Figure 1

Test B, included tasks dealing with addition of decimal numbers. Some of these items were presented exclusively in symbolic mode (i.e 0.04 +0.52 =). Some tasks were presented on a number line and aimed to examine students' ability to translate from the number line to the symbolic expression and vice versa (Fig.2).

Show on the number line the following mathematical expression: 0.3+ 0.5=

0 1

Figure 2

Test C included four word problems which involved decimal numbers. Sixty students were asked to solve the problems in any way they wished. The remaining sixty students were instructed to solve the problems using the number line.

> An athlete run a distance of 0.3 km and stopped to have some rest. Afterwards he ran a distance of 0.4 km. What is the total distance that the athlete had run?

The research design and the selection of the items are similar to the ones used in previous studies which examined the use of number line as a means for representing mathematical concepts (Michaelidou & Gagatsis, 2003; Shiakalli & Gagatsis, 2003).

Variables

<u>Test A: Recognition Item.</u> AL: Recognition on Number Line Item. ASUR: Recognition on Rectangular Surface Item. AS: Recognition on Line Segment Item.

Test B: Addition of Decimals Item, B-number (i.e B1): Addition of Decimals in Symbolic Mode Item. BL: Addition of Decimals Involving Translation from the Number Line to the Symbolic Expression. BS: Addition of Decimals Involving Translation from the Symbolic Expression to the Number Line.

RESULTS

We considered that – beyond frequencies of success (given by SPSS) – one appropriate statistical method to be used was the implicative statistical analysis of Regis Gras. This statistical method allows the examination not only of the difficulty level of the questions in the 3 questionnaires but also the relations between students' responses to the tasks in the 3 questionnaires. To this end we have used the statistical package CHIC (Gras, Peter, Briand, & Philippé, 1997). One of the diagrams produced by this method is the similarity diagram which represents groups of variables which are based on the similarity of students' responses to these variables.

The Understanding of the Concept of Decimal Numbers

The data suggest that students perform better in items including the rectangular surface concerning the recognition items (Table 1). Concerning the tasks involving the representation of decimals, students perform better in tasks where the given representation is the rectangular surface (Table 2). As far as the translation mode is concerned students perform better in translating from the number line to the symbolic expression in comparison to the translation from the symbolic expression to the number line (Table 3).

Success Percentage Concerning the Recognition of Decimals –Test A	
Representation	Percentage (%)
Line Segment	43
Number Line	46,7
Rectangular Surface	63,6

Table 1: Recognition of Decimals

Success Percentage Concerning the Representation of Decimals –Test A	
Representation	Percentage (%)
Number Line	83,5
Rectangular Surface	98,3

Table 2: Representation of Decimals

Success Percentage Concerning the Translation Ability –Test B	
Translation Mode	Percentage (%)
Number Line – Symbolic Expression	68.6
Symbolic Expression – Number Line	53.7

Table 3: Translation ability

Finally there was no significant different between the control and the experimental group ($a>0.05$) despite the fact that the experimental group was instructed to use the number line whilst the control group was instructed to solve the problems without any restriction regarding the solving procedure.

Relations between tasks-Modularization

The similarity diagram showed a formation of four groups of tasks (Figure 1). The first group (1L, A3SUR, AR6L, B4) involved tasks that examined students' ability to recognize and represent decimals. The second group (A1S, AR4SUR, B13S, B14L, B1, B6L, B9S, B12L, B10L, B11S) was comprised by tasks that explored students' translation ability concerning the addition of decimals. The third group (A2L, A2SUR, AR4L, AR5L) mainly consisted of tasks including number line as a means of representing decimals. Finally, the last group (AR6SUR, B5, B7S, B8L, B2, B3, B15S) consisted tasks involving addition of decimals. According to the above results the tasks of both Test A and B are modularized according to the kind the task - translation, recognition, representation, addition.

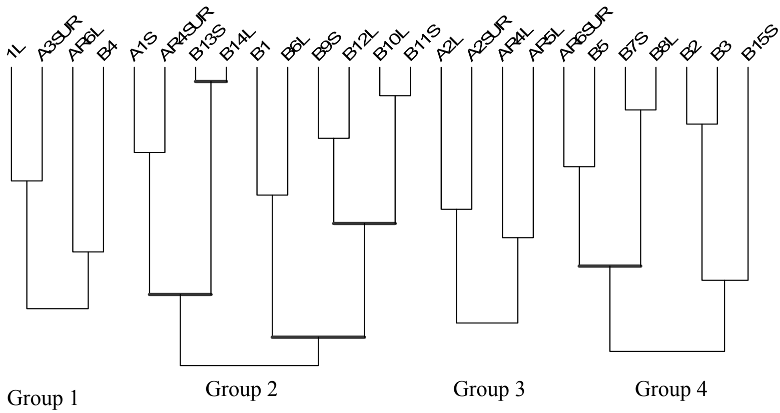

Figure 1: Similarity Diagram of Tasks included in Test A and Test B which examines Modularization

DISCUSSION

Understanding the Concept of Decimal Numbers

Low percentages of success in recognition items (Table 1) and translation items (Table 2) show that the representations of decimal numbers are not sufficiently developed and do not consist a unified whole. High percentages of success in representing the concept of decimals using a number line, a line segment and especially the rectangular surface can be attributed to instruction which usually gives great emphasis to isolated representations such as the rectangular surface. In this way the modularization of knowledge of decimal numbers is reinforced.

Moreover, the sub concept of part whole of rational numbers seems to dominate children's knowledge concerning rational numbers since students treat the number line which is infinite as a line segment from which they have to select a part. Consequently, students perform better in representation tasks since they were rectangular surfaces and number lines including the space from 0 to 1. On the contrary, concerning recognition items, students face difficulties especially when then number lines include spaces beyond 1.

Students' Performance in Different Modes of Translation

Students face difficulties in both kinds of translation tasks: from the number line to the symbolic expression and vice versa. Translation from number line to the symbolic expression seems to be easier than translation from the symbolic expression to the number line. This might be attributed to the fact that students are more familiar with the use of ruler, which is a linear representation, as a tool for measuring length. Consequently, they are more familiar with the translation from the number line to the symbolic expression.

Difference in performance concerning the two translation modes indicates that students deal with the two modes as if they are different concepts and not two different modes of the same concept. According to Janvier (1987) a translation involves two modes of representation: the source and the target (i.e number line Symbolic expression). To achieve directly and correctly a given translation, one has to transform the source "target-wise" or to look at it from a "target point of view". There are certain implications for instruction since the above comment suggests teaching strategies should emphasize to this two way procedure (Janvier, 1987).

Students' performance in Different Tasks

Students perform well in representation tasks but they face difficulties while dealing with the recognition tasks and the addition tasks. This difference in performance maybe due to the fact that among the recognition tasks were perceptual distractors, that is tasks with misleading information i.e number lines including spaces greater than the space from 0 to 1. This fact along with the fact that the sub concept of the rational number as a part whole is dominant in students' thought led students to the wrong selections. They treated number lines not as continuous models where a point obtains meaning if the position of two other points is determined, but as line segments from which they had to select a part. So students considered as correct representations, number lines which included spaces beyond the 0 to 1 space and they were representing numbers greater than the given decimal number.

Finally, students faced difficulties in tasks of addition of decimal numbers. This result indicates that addition tasks and especially the addition tasks that included translation procedures presupposed the ability of performing the operation of addition, the ability to represent the adders and the operation on a number line and the ability to manipulate and interpret the information represented by the number line and its features. Consequently, addition tasks might have caused a great cognitive load to the students dealing with them.

The performance of students varies according to the value of the digits that consist the adders. Tasks including adders which exclusively consist of tenths or hundredths seem to be easier than tasks that include adders which consist of both tenths and hundredths.

Number line and Problem Solving

According to the results there were no statistically significant differences between the means of success of students that solved the problems with no restriction to the solution procedure and the students that were asked to use the number line in order to solve the same problems.

Of course the fact that there are no significant differences among the two groups might equally indicate that the number line must have caused more difficulties to the some of the students of the experimental group. It is highly possible that the necessity to manipulate and interpret the number line and its features have been added to the cognitive load of the students and led them to erroneous approaches concerning the problems. In fact the number line is not a simple representation but a geometrical model (Gagatsis et al., 2003) which propose a continuous interchange between geometric and symbolic representation.

Compartmentalization or Modularization of Tasks

According to the results, the tasks of Test A and Test B form isolated groups based on the kind of issue addressed in each task. To be more specific, there are groups mainly consisting of recognition and representation tasks – tasks of Test A. In addition to this, there are groups in which the majority is translation tasks –tasks of Test B. Finally, there is modularization even among the addition tasks according to the number of the digits of the adders –tenths, hundredths. The modularization occurred underlines the fact that students have not developed a unified cognitive structure concerning the concept of decimals since their ideas about decimals seem to appear partial and isolated. More specifically the application of the implicative statistical method demonstrated a compartmentalization of the different tasks and this signifies that there is a lack of coordination between recognition, representation and translation in decimal numbers.

Concerning the last hypothesis, and the observed phenomenon of modularization of tasks, a suggestion should be that future research should look into organizing and conduction intervention programs. These intervention programs should deal with number line as a means of representing decimal numbers. In this way future studies will have the opportunity to investigate possible differences between control groups – students which are not involved in the program – and experimental groups –students involved in programs which focus on the manipulation and interpretation of number line as a means of representing decimal numbers.

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