UNDERSTANDING INVERSE FUNCTIONS: THE RELATIONSHIP BETWEEN TEACHING PRACTICE AND STUDENT LEARNING

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This study is a part of an ongoing research that attempts to explain the relationship between the teachers' instructional practices and students' learning in the context of functions. In this paper we report a case that shows significant differences between the achievements of two classes irrespective of the students' background training, the curricula taught, and the geographic or socioeconomic variables. Cross examination of the data suggest that these differences are attributable to the teachers' instructional practices.

Introduction

The influence of the teachers' instructional practices on students' learning has prompted considerable interest (see for, example, Brophy & Good, 1986; Leinhardt & Smith, 1985). Directing this interest is the belief that teachers play an active and direct role in the students' acquisition of knowledge. During the 1970s teacher's effectiveness was measured in a quantitative way through the analysis of data associated with the courses taken by the teachers during their undergraduate studies or with teachers' scores on standard tests (Fennema & Franke, 1992; Wilson, Shulman & Richert, 1987). Such an approach is often criticised and found deficient because it is not associated with the situation where the teaching and learning take place.

More recently there has been a tendency to use qualitative research to investigate teacher efficiency in producing desired learning outcomes (Leinhardt & Smith, 1985; Askew, Brown, Rhodes, William, & Johnson, 1996). Leinhardt & Smith reported that expert teachers who had deep understanding of the concept of fraction obtained better learning results with their classes than did novice teachers. Teaching approaches of the latter was characterised by the provision of procedural examples and explanations but an absence of explicit links between different aspects of the concept. Askew *et* al concluded that the students of teachers who provided conceptual explanations and identified links between the sub-concepts (connectionists) obtained relatively better learning results in comparison to those students whose teachers encouraged them discover mathematical ideas and principles by themselves or those who were the recipients of dispensed knowledge. This paper takes the interest further by examining the way in which two Turkish teachers introduce the concept of inverse function and relates this to the students' understanding of the notion.

Theoretical Framework

Our study is situated, in general, in the process-product paradigm. To examine the teachers' instructional practices we draw upon Shulman's (1986) notion of pedagogic content knowledge "*the ways of representing and formulating the subject that makes* *it comprehensible to others*" (p: 9). He suggests that such knowledge also includes the teachers' understanding of what makes the learning of certain topic easy or difficult for students, an understanding of the conceptions and preconceptions that students bring with them to the lessons and an awareness of students' misconceptions. We explain students' learning with reference to the APOS theory hypothesised by Dubinsky (1991) although we use only the first two aspects of this notion since we will show that the students did not appear to proceed to an object conception of inverse function. Dubinsky's notion of action refers to the repeatable mental or physical manipulations implemented upon an object to obtain a new one (Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas, & Vidakovic, 1996). In our context those students whose understanding is limited to the action conception would work out the rule of inverse function by inverting the process of a function step by step. A process conception of a mathematical idea is attained through interiorising actions, and this level of understanding enables students to have a conceptual control over a process without necessarily performing every step in that process (Breidenbach, Dubinsky, Hawks, & Nichols, 1992). In our case, those who attained a process conception are likely to deal with the concept of inverse function in the situations that do not involve an operational formula.

The Notion of Inverse Function—The Turkish Context

The Turkish mathematics curricula within which our study is situated presents the concept of inverse function through a definition: *"Consider that f and g are two functions. If* $(fog)(x)=I(x)\Rightarrow f$ *is the inverse function of g and g is the inverse function of f",* and symbolises this relation as $f^{-1}(x)=g(x)$ and $g^{-1}(x)=f(x)$ (Cetiner, Yildiz, & Kavcar, 2000). This definition involves the idea that 'an inverse function undoes what a function does'. In this sense, the notion of 'undoing' captures the underlying domain of inverse function (Even, 1991). The property of 'one-to-one and onto' is the basic criterion that a function must meet to be reversed. What makes this cognitively simple mathematical idea difficult for many of the students is the peculiarity of the representations. Whereas Venn diagrams, sets of ordered pairs, and Cartesian graphs are more able to elucidate the essence of this concept, the absence of an algebraic formula in such situations usually creates difficulties for the learners unless they have attained a process conception (Dubinsky & Harel, 1992). We believe that algebraic expressions are likely to shift the focus of attention from the notion of 'undoing' to the idea of an 'inverse operation' entailing the inversion of a sequence of algorithms in the process of a function by going from the end to the beginning.

Method

This study was conducted in Turkey. The research participants were two high school teachers, Ahmet with 25 years teaching experience and Mehmet with 24 years teaching experience (the names are altered), and their 9th grade students. Data about the teaching practices were obtained through classroom observations. Each teacher was observed teaching the concept of inverse functions. All the lessons were audio taped and field notes were taken to record the critical information as well as the visual aspects of the lesson that the audiotape could not detect. Data about the students learning comes from two sources: pre-test and post-test questionnaires.

Preceding the courses a pre-test questionnaire was administered to the students to assess their initial levels of understanding of function, in general, and inverse function, in particular. After completion of the course a post-test was conducted to observe the progress students had made as result of the instructional treatment. The questions presented in this paper were used in the questionnaires in an open-ended form to encourage the students to write down their actual reasoning about the problems at hand.

Results

The results are presented in two ways. First we consider the overall approaches of the two teachers in teaching the concept of inverse function, and secondly we consider the responses of students from each of their classes (Ahmet Class A and Mehmet Class B) to two questions that focus on the notion of the inverse function.

The two teachers display substantial difference in their approaches to the essence of the concept, and this manifests itself in every aspect of their instructional discourse. Ahmet's teaching is centred on the notion of 'undoing'. In this respect, his first and purposeful attempt is to strengthen the students' understanding of 'one-to-one and onto' condition before the formal instruction. Diversity as well as development in the use of representations that started with Venn diagrams and went through a sequence that included the use of sets of ordered pairs, graphs, and algebraic expressions, were indicators of his expertise and essential to his determination to align the logic of the concept to the students' comprehension. Connections between ideas as well as between representations were a distinctive feature of his instruction. Ahmet's teaching was exemplified by his tendency to encourage his students to examine the concept through conceptually focused and cognitively challenging tasks. He believes that algebraic expressions, especially linear ones, are not productive to explicate the essence of an inverse function.

In contrast, Mehmet's teaching could be described as action oriented practices. He focused on teaching algorithmic skills and the acquisition of procedural rules. As his teaching developed, it became clear that these rules and skills were regarded by him as essential in enabling his students to reverse an algebraic function. However, such skills didn't help them to meaningfully deal with the concept in various situations. He made use of the students' previous knowledge and offered several analogies from daily life situations to encourage the students' acquisition of these procedural skills. Cartesian graphs and sets of ordered pairs were absent in his teaching. The ultimate goal of his instruction appears to be the alignment of the logic of 'inverse operation' to the procedural knowledge of 'doing' ("Find the inverse of…"), but not the conceptual knowledge of 'undoing'. To reach this target he worked on ritual tasks and consistently provided procedural explanations through the implementation of a 'focused questioning teaching strategy'.

From full analysis of the data we summarise the critical aspects of the teachers' instructional practices in the table below.

Ahmet	Mehmet
Preliminary Consideration Prepared students for the concept of inverse inverse function before formal introduction	
Introduction Explained the necessity of 'one-to-one' and 'onto' condition with reference to the definition of the function and through several examples in the form of Venn diagrams…	Provided several analogies from the daily life situation to explain the way of inverting a sequence of operations in the process of the function…
Development Examined the concept of inverse function a through the Venn diagrams, sets of ordered pairs, graphs, and algebraic expressions…	Concept examined through Venn diagrams and algebraic expressions. Sets of ordered pairs and graphs ignored. With reference to the definition used a single example in the form of Venn diagram to explain the necessity of 'one-to-one and onto' condition.
Expansion Making use of the students' knowledge of 'inverse operation' when teaching linear functions in algebraic forms… Attempted to expand the students' understanding of inverse function as 'undoing' what a function does through conceptually focused and cognitively challenging tasks…	Did not engage students with conceptually focused and cognitively challenging tasks… Largely confined the notion of inverse function to the idea of 'inverse operation'…
Pedagogical Characteristics Displaying a mixed approach (connectionist & discovery) as a teaching strategy…	Implementing a focused questioning method as a teaching strategy…

Table 1: Salient aspects observed in teachers' instructional practices.

Prior to the course all of the students were asked to demonstrate their ability to reverse a process after being given a particular output (5) after completing the processes x3, –7. Only one student gave an incorrect solution. Solution methods of the students who obtained correct answers were almost equally distributed between the formation of an algebraic equation or an inverse operation. Differences in the students' understanding after the course may be seen through the analysis of two questions. The first assesses students' understanding of the notion of 'undoing' and the property of 'one-to-one and onto' whilst the second investigates their ability to deal with the concept of inverse function in a graphical situation.

The First question asked the students to:

Consider two non-empty sets, A = {a, b, c, d} and B = {e, f, g}. Is it possible to define a function from A to B, say f, that has an inverse function, say f^1? Give your answer with the underlying reasons.

Within this question there is neither an explicit recipe nor a visual figure to facilitate the students' movement between the sets of elements. They had no choice other than to construct a process in the situation without losing the meaning of inverse function and the related properties. Five different responses were produced (see table 2).

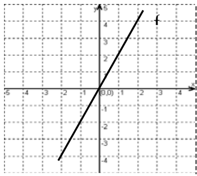
	Class A		Class B	
	n	%	n	%
Incorrect (verbal explanation)	5	18	1	4
Incorrect (verbal explanation & corresponding figure)	1	4	9	33
No response	3	11	3	11
Correct (verbal explanation)	10	36	5	19
Correct (verbal explanation & corresponding figure)	9	32	9	33
Total (N)	28		27	

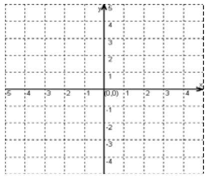
Table 2: Distribution of the answers by methods used and correctness.

Incorrect verbal explanations did not make sense or articulated an idea that illustrated a misunderstanding the concept of inverse function — *"...we cannot define such a function, because the sets A and B do not have a common element."* The common error in the second type of answers is about the univalence condition. Although students who made this error flexibly shifted to visual figures, mainly Venn diagrams, they either constructed a 'one-to-one' relation from A to B and then claimed that it has an inverse function, or defined a proper function from A to B ignoring the univalence condition on the way back. One third of students in Class B (Mehmet's class) provided incorrect explanations though they worked on a visual figure. Only one in Class A (Ahmet's class) did so. Approximately one quarter of the total number of students appear to have a cognitive control over the processes in both ways. These students explained verbally why the construction of such a function is not possible with a clear articulation that *'an inverse function undoes what a function does'* with a particular emphasis upon *'one-to-one and onto'* condition. They did not use a visual figure to justify their thoughts. However, again class differences appear. For each student who displays this characteristic in Class B there are two students in Class A. The last group of answers also indicates the recognition of what an inverse function does and the property of 'one-to-one and onto'. However, though it is difficult to make a decision about the mode of students' thinking on the basis of written responses, it is inferred, from the evidence presented, that these students were dependent upon a visual figure to think about the problem.

The second question that we will consider was presented in graphical form.

The graph of function f is given as follows. Sketch the graph of inverse function, f^{-1}, in the Cartesian space below, and give the reasons for your answers.





Excluding those who gave no response this question produced three types of answers (see table 3)

	Class A		Class B	
	n	%	n	%
Incorrect	7	25	15	57
No response	0	0	2	7
Correct (point-wise approach)	14	50	10	37
Correct (global approach)	7	25	0	0
Total (N)	28		27	

Table 3: Distribution of answers by methods used and correctness.

Incorrect responses involved several types of misunderstandings, such as sketching a line passing through the points (2, 0) and (0, 1) on the x and y-axes respectively, sketching the graph given as the graph of an inverse or reflecting the graph of the function given in the y-axis. Note that almost two thirds of class B gave an incorrect response or no response. Correct responses involved two qualitatively different approaches. The first group of students displayed a point-wise approach either by marking certain points, such as (2, 1), (4, 2), (-2, -1), in the Cartesian space and then drawing a straight line through them or using the algebraic form of the function for transition from the graph given to that required. The second group of students, all of whom are in class A, sketched the graph of inverse function at once without any attempt to deal with the graph point by point. The common method is reflecting the graph given in the line of $y = x$.

Conclusion

The impact of teaching practices on students' learning is a fruitful but at the same time a controversial research topic. Whereas educational sociologists emphasise the complexity of the social environment, within which there are several other variables that would profoundly affect the students' learning (Peaker, 1971), educational psychologists argue that the individual's cognitive growth is the most determinant factor in his/her acquisition of knowledge (Inhelder & Sinclair, 1969). We are fully aware that the impossibility of eliminating all the internal and external factors does not allow us to explain the influence of teaching practices on students' learning in the sense of cause-and-effect relationships. However, our findings suggest that teaching practices that differ in a qualitative way are apt to produce qualitatively different learning outcomes. The epistemology of the inverse function was the basic criterion in our examination of the students' learning, the teacher's teaching practices, and the interaction between the two. We conclude, primarily, that students would have difficulty in attaining a meaningful understanding of inverse function without experiencing it through conceptually focused and cognitively challenging tasks using a variety of representations. Making use of students' previous knowledge (the knowledge of inverse operation) or providing analogies from real life situation might be productive for the construction of a foundation, but it is not adequate enough to promote the students' conceptual understanding of inverse function. We suggest that what determines the quality of teaching, and would subsequently enhance the students' meaningful learning, is making use of a variety of appropriate representational systems, examining the concept through conceptually focused and cognitively challenging tasks, linking the inverse function to the concept of 'one-to-one and onto' function as well as to the concept of function itself, and ensuring active involvement of the students within the process of knowledge construction.

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