CONSTRUCTING KNOWLEDGE ABOUT THE BIFURCATION DIAGRAM: EPISTEMIC ACTIONS AND PARALLEL CONSTRUCTIONS

Ivy Kidron — Jerusalem College of Technology
Tommy Dreyfus — Tel Aviv University

The dynamically nested epistemic actions (RBC) model is used to describe the process of constructing knowledge about bifurcations of dynamic processes by a solitary learner. We observe a refinement of the three epistemic actions, Recognizing, Building-with and Constructing that have been identified in previous research based on the RBC model. We also observe that the Constructing actions are not linearly ordered but may go on in parallel. We observe the branching of a new construction from an ongoing construction. We use the term "branching" in order to describe this transition from a single construction to two parallel ones. In the paper we analyse why such branching occurs.

INTRODUCTION

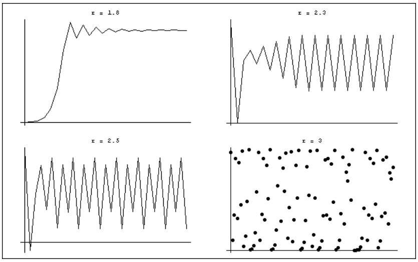
The dynamically nested epistemic actions model called the RBC model is described in Dreyfus, Hershkowitz & Schwarz (2001) and in Hershkowitz, Schwarz & Dreyfus (2001). Stehlikova (2003) demonstrated how it can be applied to introspective data and used for their interpretation. In the present study too, the RBC model is applied to introspective data. The learner in this study, the first author, Ivy, learned alone, with only books, the web and her interaction with Mathematica available as sources of external knowledge. Her detailed notes taken during the learning experience constituted the raw data for a written report she prepared some time after the experience. This report was prepared in collaboration between the two researchers. Ivy wrote what she considered to be a precise and detailed account of her thinking and acting at the time of the learning experience. The second author, Tommy, read this account and challenged the first author on every statement that seemed to reflect later additions, corrections or changes to what happened at the time of the learning experience. The report was then modified so as to answer these challenges. The final version of the report resulted from three rounds of such corrections. Independent research results (Nisbett &Wilson, 1977) show that people can produce accurate reports of their own cognitive activity if salient stimuli are provided. These salient stimuli were the field notes and the challenges.

The Mathematical theme

The learning process described in this article deals with bifurcations of dynamic processes, a complex topic, which occupied the learner during a period of approximately two weeks.

We observe the following iterative process: Given the quadratic function $f(x) = x+rx(1-x)$, where r is a real parameter, we look at the sequence of values $\{x_n\}$

produced from an initial value x_0, $0< x_0 <1$, by successive application of f, that is $x_{n+1} = f(x_n)$, for all $n\geq 0$. Independently of the choice of x_0, $0< x_0 <1$, for r =1.8 the process tends to the final state $x = 1$, which is a fixpoint of f, that is $f(1) = 1$.



However, for r = 2.3 the final state is a periodic oscillation between two values, a 2-period. For r = 2.5 the process approaches a 4-period and for r =3 it does not appear to become periodic at all (See for example Alligood, Sauer and Yorke, 1996, for more details). Ivy used Mathematica to confirm these phenomena numerically and graphically. (Figure 1).

Figure 1: Mathematica time series plots of the process for r=1.8 (top left), r=2.3 (top right), r=2.5 (bottom left) and r=3 (bottom right).

It was intuitively clear to Ivy that the rich and surprising pictures on the computer screen describe a mathematical reality. She felt that it was quite challenging to find the mathematical justification for the above phenomena. In this article, we analyze the epistemic actions in her construction of the transition from the 2-period to the 4-period.

THE STORY OF THE LEARNING EXPERIENCE

We divided the story of Ivy's learning experience, as reflected by the final version of the report into 16 episodes. In this section we present the first seven episodes in some detail and summarize the remaining ones. Each episode forms a cognitively coherent unit. For the purpose of analysis, each episode has been further divided into subunits, denoted by letters a, b, c, ...

With a view to investigating the transition from 2-period to 4-period, Ivy started in *episode 1* by focusing on r=2.3 (which is in the 2-period region) and confirming that the graphical, numerical and algebraic (quadratic equation) values of x for this r agree. She expected that in the region of the 4-period, there were analogous (though possibly more complicated) algebraic equations to be solved. In *episode 2,* she saw solving these equations as the problem to be tackled. More specifically, she realized that in the case of period 4, there are four values a, b, c and d such that f(a)=b, f(b)=c, f(c)=d and f(d)=a, whence $f^4(a) = f(f(f(f(a)))) = a$, $f^4(b) = b$, $f^4(c) = c$ and $f^4(d) = d$. She thus had to solve the equation $f^4(x) = x$. From a web resource she learned that the roots of $f^2(x) = x$ are also roots of $f^4(x) = x$ and that it was thus sufficient to solve the equation $\frac{f^4(x)-x}{f^2(x)-x} = 0$. This equation is a polynomial equation with parameter r. For simplicity, we will in the sequel denote it by p(x) = 0, or call it "the equation". In *episode 3*, Ivy attempted to solve the equation for a general value of r. This attempt failed and the failure, combined with the use of Mathematica, led her to a numerical way of thinking. Mathematica taught her that in the 4-period region, the equation is of order 12 and cannot be solved in general, but only for specific values of r. In the previous transition, from period 1 to period 2, the solutions were functions of r. In the present transition, this r-dependency became inaccessible since the equation was not solvable. Thus the numeric aspect took over. This led her, in *episode 4*, to focus on using other means to find the r at which the transition occurs.

4a When I got stuck with this polynomial of order 12, I went back to the same website and there I learned that the transition from period 2 to period 4 could be found by setting the discriminant of p(x) equal to zero. There was no explanation why this solves the problem.

In the same web resource Ivy read that the discriminant D of a polynomial is defined as the product of the squares of the differences of the polynomial roots. With this definition, Ivy was not able to progress since she had failed to find the roots. However, in *episode 5*

5a Searching further, I read in another web resource "up to some constant, the discriminant is the 'resultant' of a polynomial and its derivative. That is, the discriminant is the result of computing a certain determinant made from the polynomial coefficients".

5b This was more promising since it contained a hint that the discriminant could be obtained without finding the roots first.

Ivy did not understand how the discriminant D could be obtained, but she was willing to exploit this alley anyway. Mathematica provided D, which was, after simplification, a polynomial of order 40 in r. Ivy needed the zeros of this polynomial. In *episode 6*, she found the one that was important, by asking Mathematica to factor D, yielding $r=\sqrt{6}$ as smallest real root.

Seeing that $r = \sqrt{6}$ is suitably located between 2.3 and 2.5, Ivy interpreted it as the transition point. On the basis of this numerical success, she began in *episode 7* to search for the mathematical reasons behind that success.

7 I was encouraged by the numerical appropriateness of $r=\sqrt{6}$ and

7a I was optimistic that I now had the means to begin the analytical work of connecting between the requirement that D equals zero and the transition from period 2 to period 4.

7b I felt that this undertaking would be helped if I would actually see the solutions of the equation in front of me. This had been my starting point: to solve $f^4(x) = x$.

7c I thus decided to substitute $r=\sqrt{6}$ in the equation $p(x) = 0$ and to obtain the solutions.

7d This was the only option I could think of, which would enable me to observe the structure of the solutions like in the case of the transition from period 1 to period 2.

7e	From the definition of D, I knew that when D equals 0, the equation has multiple roots.
7f	In the resulting of Mathematica output I obtained two pairs of double solutions, each listed twice. … I observed the other solutions. There were four pairs of complex numbers and their conjugates. I had no idea how to interpret the meaning of these complex solutions, not even the fact that they were not real.

Ivy was looking for real roots, multiple real roots, and the complex roots came as a surprise diverting her attention from the more relevant real double roots.

The apparition of the complex roots weakened the analogy with the case of the transition to the 2-period, where $D \geq 0$ implies real roots only. In another attempt to "save" the full analogy to the 1-period to 2-period transition, including the direct way of getting the real roots, Ivy refreshed (from a book) her knowledge about complex roots of polynomial equations, but she soon realized that this does not allow her to connect $D=0$ to the existence of real solutions.Numerically the results she had obtained were nice, but they were not satisfactory to her, since she had information on a single r only or on a discrete set of r-values in the best case. Algebraically, she was stuck. The impossibility of algebraic solution resurfaces now explicitly. At this stage of the learning experience, she turned to a graphical mode of thinking. A look at the bifurcation diagram led her to realize that the two (double) real x-values she had obtained for the 4-period at the transition point, were the same values as those of the 2- period as r approaches the transition point, i.e. that a bifurcation of the x-values occurs there. This emerging understanding came in terms of the x-values, together with their dynamic change as r varies. Now the questions were very clear: How are double real solutions, $D=0$, and period-transition connected? A previous search on the web helped to connect D to the coefficients not only of p but also of p', and to also connect multiple solutions of $p=0$ with solutions of $p'=0$. Exploiting the specific structure of p yielded an equation for the derivative that was known to Ivy to be characteristic for fixpoints changing stability. From here on progress was smooth and Ivy constructed her mathematical justification for finding the value of the parameter r corresponding to the transition point from the 2-period to the 4-period.

THE EPISTEMIC ACTIONS

In this section, we discuss Ivy's epistemic actions as we identified them in her report of the learning experience. With respect to Recognizing and Building-with actions, we only present modifications that have not been observed in earlier studies.

Less than Recognizing: R-

As mentioned earlier, Ivy learned alone, with books, the web and her interaction with Mathematica. In such a situation, she was repeatedly offered new information she had not explicitly asked for. In some cases (episodes 4a, 7f), she registered such information, and it later came to play a role in constructions of her knowledge structures. This new information was not a case of re-cognizing, nor did this registration on first encounter constitute a full cognizing of the information. On the other hand, the fact that Ivy was later able to make use of it indicates that some spark of an epistemic action did occur. We will denote such actions by R^- in order to indicate registration of information that, at least in the meantime, constitutes less than Recognizing. As an example of a R^- action, in episode 7f, Ivy had asked Mathematica to solve the polynomial equation. She expected real solutions only. Mathematica provided two pairs of double real solutions but there were also four pairs of complex conjugate numbers. Ivy didn't know how to interpret them but it influenced her further work.

A refinement of Building-with

When analysing Ivy's learning experience for B- actions (Building-with), we noticed a wealth of them. Every one of these B-actions contributes to some building but they do so in very different ways.

We discerned between B-actions of the problem solving type, the kind that had been observed in all previous studies using the RBC model, and B-actions organizing the problem space so as to make its further investigation possible. In fact, Ivy spent a considerable part of her time formulating and reformulating the questions she was asking and the problems she had to solve. This is an activity that is common for learners engaged in investigative activity. Such formulating of questions, tasks and intentions satisfies all criteria of epistemic B-actions: It does not, by itself, produce new mental structures but uses the available ones in order to organize and reorganize horizontally not only the knowledge one has, but also the knowledge one does not have yet and is looking for – the problem space. This distinction led to two quite distinct and different classes of B- actions. Hence we decided to use two different letters to denote them: P for B-actions of the Problem-solving type and V for B-actions supporting the inVestigation of the problem space. An example of a B-action of the V-type is 7a, in which Ivy formulates and thus clarifies to herself her intention to change the type of her thinking from empirical-numerical to analytic-algebraic and declares her aim to connect between D=0 and the transition point. She thus builds the cognitive fundament on which she can attempt further progress in the problem space. The verbalization of the algebraic connections she wants to establish, anticipates the process of generating a mathematical justification of the way in which she obtained the value $r=\sqrt{6}$ at the transition point. This motivation of achieving a justification drove the entire learning process.

Most of the Building-with actions of the V-type address the big issues Ivy dealt with and are thus intimately related to the constructing actions, which we will describe in the next subsection.

Constructions

The methodology we followed for identifying constructing actions closely followed the one that was used in Hershkowitz, Schwarz & Dreyfus (2001). Through detailed analysis of the report, we identified instances of vertical reorganization of knowledge structures, of added depth, and of integration of structures. Novelty of the resulting structure for the learner was used as a central criterion in the identification of constructions, and so was its verbal expression. The use of such verbal expression for further explanation was taken to be a definite sign of construction. In this section, we give a very short description of some of the constructions that form Ivy's overall construction C.

C_1 *Constructing the solutions of the polynomial equation:* We denote by C_1 the process of constructing the solution of the polynomial equation p(x) = 0 in order to find the 4-periodic points. The solution process is considered algebraically and numerically. Construction C_1 appears in episodes 2, 3 and 7 and in the later episodes.

C_2 *Constructing algebraic connections:* We denote by C_2 the process of constructing the algebraic connections between the existence of multiple (double) real solutions of the equation p(x)=0, the zeros of D and the transition point from period 2 to period 4. Construction C_2 appears in episodes 4-7 and in many later episodes. C_2 is different from the other constructions in the sense that it was very clear what the knowledge was that had to be constructed, but Ivy got stuck at several points in the process. The algebraic mode did not yield the results Ivy was looking for. As a consequence, the need for the other constructions arose, especially C_3.

C_3 *Constructing the link between the discriminant and the derivative:* We denote by C_3 the process of constructing the links between the derivative of a polynomial and the zeros of its discriminant. Construction C_3 appears in episode 5 and in the episodes toward the end of the story. The link between D=0 and the derivative first appeared in episode 5 but has been identified as the beginning of a construction only a posteriori, on the basis of the epistemic actions in the later episodes, where the role of the derivatives of p and related functions became central to Ivy's construction of knowledge.

C *Constructing the justification:* We denote by C the process of constructing the mathematical justification for finding the value of the parameter r corresponding to the transition point from period 2 to period 4. This process extends over all 16 episodes. The motivation of achieving a justification drove the entire learning process. This desire to generate a justification motivates C and thus indirectly motivates the other constructions, each of which is nested in and contributes to the overall construction C.

INTERACTING PARALLEL CONSTRUCTIONS

Ivy's construction of a justification for the transition from the 2-period to the 4-period constitutes a complex learning process. The complexity of the construction expresses itself in the fact it consists of several interweaved constructing actions that go on in parallel. In the previous section, these constructing actions were described separately. In the present section, we analyze the manner in which they interact with each other. The constructions are closely interrelated, and should therefore ideally develop in parallel. On the other hand, they are too substantial to go on simultaneously. As a consequence, the analysis of the learning experience produced interesting patterns by which constructions arise and interweave. These appear to be patterns that have not been observed in previous research. The following characteristic patterns have been observed: A new construction branches off from an ongoing construction, several constructions go on in parallel and then combine, one of several parallel constructions is interrupted and later resumed. Because of space limitations, we focus in this paper on branching. This phenomenon appears in episode 5 (C_3 branching off from C_2) and in episode 7 (C_1 branching off from C_2). Examples for combining and interrupting will be analyzed in a bigger paper in which we will describe in detail the later episodes of the learning experience.

Branching

In this subsection we describe the branching off of a new construction from an ongoing construction; such branching leads to a transition from a single construction to two parallel ones. We will provide an explanation why such branching occurs. We first deal with the branching point itself and then with the parallel development of the two constructions after branching.

Why C_1 branches off from C_2 at the beginning of episode 7?

C_1 has been interrupted at the end of episode 3 when the information provided by Mathematica was that the equation p(x) = 0 could be solved only numerically, and thus solving the equation would not yield the desired value of r at the transition point. At the end of episode 6, Ivy knew this value ($r = \sqrt{6}$). Thus this was an appropriate time for C_1 to be resumed. Moreover, C_2 couldn't go on in episode 7 without C_1, for the following reason: Ivy's aim was to connect D=0 and the transition point (7a). She used D=0 in order to obtain r. She had no idea how the discriminant was computed and at that stage, she wasn't ready to find out (which would have meant to resume C_3). She preferred to go back to the previous definition (4b) that connects the discriminant with the polynomial roots and to observe the structure of the solutions, and this was possible only with a return to C_1.

In both examples of branching, C_3 branching off from C_2 in episode 5, and in the present example of C_1 branching off from C_2 there is an ongoing construction, namely C_2, and another construction branching off from it. In the first case, the construction that branches off from the ongoing one is a new construction, C_3, and in the other case, the construction that branches off from the ongoing one consists of a return to a construction that had previously been active, namely C_1. But in both cases, the branching was essential for the ongoing construction to continue. The special character of C_2 contributes to the branching. As mentioned earlier, C_2 is different from the other constructions in the sense that it was very clear what the knowledge was that had to be constructed, but the algebraic mode did not yield the results Ivy was looking for. As a consequence, the need for the other constructions arose.

The parallel development of the two constructions after branching

Branching leads to two constructions being active simultaneously. This is a high demand on any learner. As a consequence, one of the constructions was interrupted at the end of the episodes that started with branching (C_3 at the end of episode 5 and C_1 in episode 7; both interruptions are caused by diverting attention to C_2). Nevertheless, the constructions develop in parallel during one episode after the branching point before the interruption. This may lead to the establishment of connections between the two knowledge structures being constructed. One of the two constructions provides the motivation for the other one. The new construction that branches off from the ongoing construction allows the influence of additional ideas to flow into the process. This positive influence will find its full expression only in the later episodes but it begins in the earlier stage in which the two constructions seem only to disturb each other.

We conclude with an observation about the relationship between the epistemic action R- and the parallel development of two constructions after branching. Sometimes new, unexpected R- information (for example, the information about the complex roots) obtained within one construction (C_1), at the moment when another construction (C_2), calls for the learner's attention, constitutes the immediate cause for the interruption of the first construction (C_1). However, at a later stage when the learner has assembled more knowledge, the same R- action can lead to the resumption of the interrupted construction, the resumption being based on the same information that was received without being requested.

In summary, we found, that in some contexts, such as a solitary learner dealing with an advanced mathematical topic, epistemic actions may be more varied and construction processes more intricate than observed in previous studies based on the dynamically nested RBC model of abstraction.

References

Alligood, K. T., Sauer, T. D., & Yorke, J. A. (1996). *Chaos. An introduction to dynamical systems.* Springer-Verlag, New York.

Dreyfus, T., Hershkowitz R. & Schwarz B. B. (2001). The construction of abstract knowledge in interaction. In M. van den Heuvel (Ed.), *Proceedings of the 25th International Conference for the Psychology of Mathematics Education*, Vol. 2 (pp. 377-384). Utrecht, The Netherlands: OW&OC.

Hershkowitz, R., Schwarz, B. B. & Dreyfus, T. (2001). Abstraction in context: Epistemic actions. *Journal for Research in Mathematics Education 32,* 195-222.

Nisbett, R. & Wilson, T. (1977). Telling more than we can know: Verbal reports on mental processes. *Psychological Review 84*(3), 231-259.

Stehlikova, N. (2003). Emergence of mathematical knowledge structures: Introspection. In N. A. Pateman, B. J. Dougherty & J. T. Zilliox (Eds.), *Proceedings of the 27th International Conference for the Psychology of Mathematics Education*, Vol. 4 (pp. 251-258). Honolulu, HI, USA: University of Hawai'i.