FOSTERING MATHEMATICAL MEANING VIA SCIENTIFIC INQUIRY: A CASE STUDY

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This study addressed the problem of how a prospective mathematics teacher's active engagement in a scientific inquiry can deepen the meaning of her extant mathematical concepts. We used a constructivist framework to analyze a 2-hour interview with a prospective mathematics teacher as she solved an open-ended problem of graphing a 3-D landform. We found two overlapping components in her learning via cycles of action and reflection: interpreting the task and reorganizing extant conceptions to quantify projective, horizontal/vertical distances.

INTRODUCTION

We conducted this study in the context of current reform movement in the teaching and learning of mathematics in the United States (NCTM, 2000). This reform emphasizes students' active investigation of their world as a means to construct meaningful and generalized mathematical concepts. This reform stresses the standard of *Connections*, that is, the recommendation to promote connections between mathematics and other disciplines (e.g., science) as well as among mathematical concepts. Because little is known about how such connections are formed, we set out to examine this sound recommendation empirically. In particular, we attempted to articulate the conceptualization process of a content-specific understanding: how a student can deepen her knowledge to meaningfully and flexibly shift between 2-D and 3-D levels.

CONCEPTUAL FRAMEWORK

This study employed a recent elaboration of the constructivist stance regarding learning as a re-equilibration, or reorganization process (Dewey, 1933; Piaget, 1985; Steffe, 2002; von Glasersfeld, 1995). This elaboration articulates the learning process in a way that provides a teacher with conceptual 'lenses' for analyzing students' extant conceptions and how they might organize those conceptions into desired ones. The core of this elaboration is the mechanism for learning a new conception, namely, reflection on activity-effect relationship (Simon, Tzur, Heinz, & Kinzel, in press; Simon, Tzur, Heinz, Kinzel, & Smith, 1999), which operates as follows.

In a problem situation, a learner sets a goal (e.g., measure projective distances) and executes an activity sequence (e.g., laying a ruler along the contours of a 3-D model) to accomplish that goal. Both the goal and the activity are available through the learner's extant conceptions. While executing the activity sequence the learner may notice that its effects differ from the goal, that is, the learner experiences a perturbation. A common source for such perturbation is the gap between anticipated

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and experiential (or perceptual) results. That is, one aspect of the learner's reflection is the type of comparisons the mental system makes between the goal and the effects of the activity, which leads to sorting activity-effect records as successful or unsuccessful. A second, complementary type consists of comparisons among situations in which such activity-effect records are called upon, which leads to abstracting the activity-effect relationship as a regularity (invariant) in the learner's experiential world. This regularity includes a reorganization of the situation that brought forth the activity in the first place.

From this perspective, a *conception* is considered as a dynamic, mental relationship between an activity and its effects. It consists of the learner's anticipation for effects that necessarily follow an activity. That is, an activity is not just a catalyst to the process of abstracting a new conception or a way to motivate learners. Rather, activity both generates and is a constituent of a conception.

This perspective has two complementary implications for teaching. First, one cannot determine the learning process because setting one's goal(s) in a given situation, initiating activities associated with that goal, noticing effects, and relating effect with activity rest within the learner. By the same token, these four rudimentary components of the learning process do not occur in a vacuum. Rather, they are afforded and constrained by learners' interactions in their environment (e.g., with peers, with a teacher) (Steffe & Tzur, 1994). Consequently, teaching includes: (a) engaging learners in solving challenging tasks that might bring forth their extant conceptions and (b) orienting learners' focus of reflection on activity-effect relationships (Simon, 1995).

METHOD

We conducted a case study with Kay, a third-year undergraduate student who was enrolled in a college methods course for high school mathematics teachers. Our data consist of videotapes, audiotapes, and artifacts of Kay's work on an open-ended task: generating a graph that corresponds to a 3-D landform—a thin but sturdy plastic molding with several 'hills' and 'valleys' upon which a few points (A-F) were labeled. Conceptually, generating a graph is more challenging and revealing than interpreting a graph because the learner must make sense of the situation to be quantified instead of recognizing (reading) certain pieces of information from the graph (Roth & McGinn, 1997). Kay solved the task during a 2-hour teaching episode in which the second author served as the leading teacher-researcher (TR). The TR provided Kay with several manipulatives (e.g., straws, a flexible measuring tape with English units (referred to as 'flex tape'), rulers, paper strings, and graphing papers), posed follow-up questions to Kay, and probed for further clarifications.

To make inferences into the conceptualization process, we employed an in-depth, micro-analysis of critical events in Kay's solution process (Powell, Francisco, & Maher, in press). First, we coded segments in the episode that appeared as turning points in Kay's behaviors. Then, we used an reviewed each data segment several times to make conjectures about plausible explanations for these behaviors. These conjectures consisted of our interpretations of the goals Kay tried to accomplish, the perturbations she might have experienced, and the nature of her anticipation while executing the activities. As we progressed along the segments, we discerned evidence regarding prior conjectures to obtain a coherent, grounded-in-data story line.

ANALYSIS

Before presenting the task, the TR asked Kay to describe her previous experience with 3-D models. Kay's reaction clearly indicated that: (a) she already encountered such models when using computer simulations in her high school calculus course and (b) she did not like this experience. Kay also shared with the TR that she did not fly in an airplane and had a very limited experience of road trips. An experience of Kay that proved essential in her solution to the problem was that she lives in the foothills.

The TR presented the task by asking Kay to imagine she is driving in the car and looking at the 3-D landscape as it passes by her. The TR asked Kay to construct a graph of this landscape but purposely did not say what type of graph to create. Thus, Kay had to form an anticipation of the graph to be created—what would constitute a sound solution to the task—while considering a 3-D model of a landform that allowed for numerous solutions.

Transcript 1

Kay	What kind of graph do you want?
Teacher	Any kind of graph that you would like to construct.
Kay	Do you want me to pretend like I am driving *through (Touches with her hand some random points on the 3-D model*) … or just what I see? Or what exactly do you want? [A little later]: Do you want just a line graph? (Receiving no response from the TR, she thinks quietly for about 5 seconds): You can do one of those line graphs. (*Her hands first show a cross-section line in the air, then a two-axis system. A little later she talks to herself while tracing with her finger a path along the contours of the 3-D model*.) Say that you started up here at the top of a mountain and then you went [down] to the bottom and it [the slope] would eventually be zero. That would be your minimum right in that area … between *C* and *D* … and then at a steep slope up here along the side of the mountain [where] you are going to put your maximum.

Transcript 1 indicates Kay's formation of her goal. First, she tried to elicit the goal from the TR. Given no indication, Kay detected the 3-D model while calling upon a particular mathematical conception she had available (a line-graph) to resolve the perturbing experience. She reflected on her own notion of a line-graph and explicated, by her hand movements, specific properties of such a graph that would evolve into her set goal: generating on a Cartesian coordinate system a cross-section along a path through some of the red dots. Relative to this goal, she immediately realized that many paths are possible and chose a single one. Kay then grabbed a graphing-paper and used a free-hand motion to trace a line with a minimum and a maximum similar to the one in her final product (Figure 1). Kay explained to the TR that the path may be thought of as one that goes from her grandma's house up on the hill, down to the valley where her house is, and back to the Blue Ridge mountains. Kay added: “That's how I think about things … what's the car going to be doing.”

We considered this as the first turning point, because, 15 minutes into the interview, Kay had resolved her first perturbation (what graph to produce) by linking her 3-D and 2-D experiences. She had established a goal through calling upon extant conceptions from three different domains: her image of the given 3-D model, her image of landforms near her home, and her mathematical conceptions of 2-D graphs. This coordination was indicated by her anticipatory actions as she combined the free-hand drawing of a line with designating locations on the 3-D model as familiar locations surrounding her home.

The way Kay resolved her perturbation of what graph to create demonstrates two claims. First, Kay anticipated the form of the 2-D graph she wanted to produce and the information she needed—a set of value-pairs for projective vertical and horizontal distances. Second, Kay's extant conceptions did <u>not</u> include a way of generating the anticipated value-pairs. Thus, her attempts to create the value-pairs led to a second perturbation—her inability to accurately measure the desired distances.

Kay took a new sheet of graphing-paper, marked the first point for her new graph (*A)*, and tried to measure the distances using a paper string. She laid the string along the contours between two points (e.g., *A-B*), then, holding the two end points, laid the string next to the flex tape. She indicated her growing perturbation by saying: "Even, if you are doing a 3-D model you want [a] slope. If there was a way to just project the points straight down onto the paper." This utterance and actions indicated Kay's perturbation: she needed to create a slope but did not have a way of projecting points straight down, which she anticipated would have solved the problem.

Transcript 2

Kay	But, umm … (*lays the string between A and B but realizes it does not give her a reading of the slope*) … there's no way to really get that angle that you are going down.
Teacher	Hmm … I wonder how you could do that?
Kay	Well, you don't have (pauses for 5 seconds) … You would have to guess what your height was … You could get your slope by measuring your height here and your distance across.
Teacher	Rise over run?
Kay	Yeah … and figure out what your slopes … Actually … I am not sure that this would be the best way to try. You wanna find out exactly how this [cross-section] goes (*turns the 3-D model upside down, then pauses for about 5 seconds while looking at it.*): Put it right there (*holds the ruler inside the model to measure height. Almost immediately her facial expression indicates "I got something" as her work becomes more purposeful*). Okay. This is experimenting. I'm not sure if this will work.

Transcript 2 consists of a critical event in Kay's learning. The change in her action was the first indication that she resolved her perturbation of how to graph slopes by calling up her mathematical conception of slopes between discrete points. This made Kay aware of the lack of accuracy of her previous method, an awareness that brought about the crucial action of turning the 3-D model upside down. Through reflecting on the effect of her action she realized its usefulness, hence the "AHA" moment she seemed to experience.

Kay's attempted to measure the horizontal distance *A-B* with two tools (a piece of folded paper, the flex tape), but both attempts failed. Thus, she picked up a ruler, put it horizontally atop the 3-D model between *A* and *B* as she said: "This isn't perfect because˙… Umm˙ I don't have a way of knowing if it's perpendicular." While struggling to hold both the flex tape and the plastic ruler atop the 3-D model, she abruptly turned the 3-D model on its head again. This abrupt action indicated another realization regarding that change of position. We suggest that Kay reflected on her previous action on the upside down model and coordinated it with her specific goal of measuring the vertical distance from *A* to *B*.

Kay was about to take that measurement with the flex tape underneath the base but then the TR intervened, asking Kay where does the horizontal line goes. After a few seconds of silence, where Kay seemed dissatisfied with the accuracy of her measurements, she explained she was searching for a way to establish a standard reference, a line that is horizontal to the base of the 3-D model. Thus, the TR's interruption proved useful because asking Kay to explain what she tried to accomplish 'sent' Kay into a cycle of reflection and action through which she refined her goal. She began focusing on how to establish a reliable reference. This indicates that Kay's conceptions did <u>not</u> include an anticipation of how to establish such reference. It also indicates the learning that <u>did</u> take place: Kay coordinated her mathematical goal (obtain projective distances) with an evolving scientific understanding—the need to establish a consistent reference. In response to further probing from the TR, Kay identified the reference line with the lowest point on her path, a 'little river' between points *C* and *D,* and said that her goal was to measure the 6 pairs of coordinates. Kay also clarified that she could choose any point as a reference and that she would need to measure at least one more point (between *C* and *D*, which she labeled *C'*) to avoid masking a local minimum. In spite of all these theoretical anticipations, at this time (34 minutes into the interview) Kay began producing a 1-D graph that she called "the distance traveled."

Kay's utterances as she produced the new, 1-D graph indicated that she did not consider this to satisfactorily solve the task. For example, she said: "It sounds like we don't get much of a graph." Thus, once she completed the 1-D graph (about 10 minutes), Kay turned back to her original goal of generating a 2-D graph. She had a clear theoretical anticipation of the measurement she needed, but not yet a method to <u>actually</u> measure projective distances. He perturbation intensified as she moved from an inaccurate measurement of the horizontal distance *A-F* to measuring the vertical boundaries (the range) for her graph. No matter how hard she tried holding the rulers perpendicular or parallel to the desk (even with the TR's help), she was continually obtaining effects of her measuring actions that did not meet her goal. In one of those attempts, however, Kay explicitly talked about her inability to 'drop' a vertical line from the dots on the top of the 3-D model to the desk or to measure vertical distances between points far apart. These two reflections led to a change in her actions—she again turned the 3-D model upside down. In the context of these specific reflections, this time she had available mental images that could be related anew—substituting one measuring action (to the desk as a reference) by another (to the base of the 3-D from beneath). Note that we do <u>not</u> claim that such a relationship was a necessary result of the mental 'items' Kay focused on; only that this focus on her unsuccessful actions allowed for such adjustment of means to ends.

Kay continued by gauging several vertical distances between points. For example, she said she was going to find *E-F* 'real quick' but this took longer than she planned (over 20 seconds) and was unsatisfactory, as she commented: "This is so unscientific." The TR asked Kay which tools could help and Kay explained she would like to have something that goes through the sturdy plastic. The TR asked Kay how might earth scientists make up a graph without cutting through the landform. She replied, "They set up reference points that are going to be horizontal and vertical to whatever they choose as their [reference]. Then, Kay turned to measure the tiny distances for *D-C'* and *C'-C* in clear anticipation of the difficulty ahead.

Transcript 3

Kay It is going to be almost impossible to … (*Abruptly, as if having another "AHA" moment, turns the 3-D model upside down, using the flex tape to measure the vertical distance between C and C'*): It's almost like [the height of] C-prime. I know you can't turn the earth upside down (laughs), but I have a model so it's happening.

In response to this action, the TR probed Kay to which horizontal line in the 2-D rough sketch consisting of vertical and horizontal lines through the minimum and maximum points does the ruler held across the base correspond. Kay responded "to any of them," indicating that this was a critical but limited event. It was critical in that for the first time Kay coordinated the turning of the model with a particular measuring action (*D-C',* vertically). It was limited in that, initially, Kay did not generalize it to measuring projective distances for <u>any point</u>. When the TR probed, "Any of them?" Kay thought for a few seconds, then responded that actually it was the horizontal line through *C'* in her rough sketch. As Kay reflected on this particular correspondence and on her next action (measuring *C'-C*), she finally related the effect of accurately measuring a projective distance with the action (ruler across the base) that invariably allowed for such measurement. She excitedly said: "I should have done this [technique] the whole way." Thus, Kay grabbed another graphing-paper, systematically measured and recorded all vertical distances, then all horizontal distances, and finally completed a new graph of the cross-section *A-F* (see Figure 1).

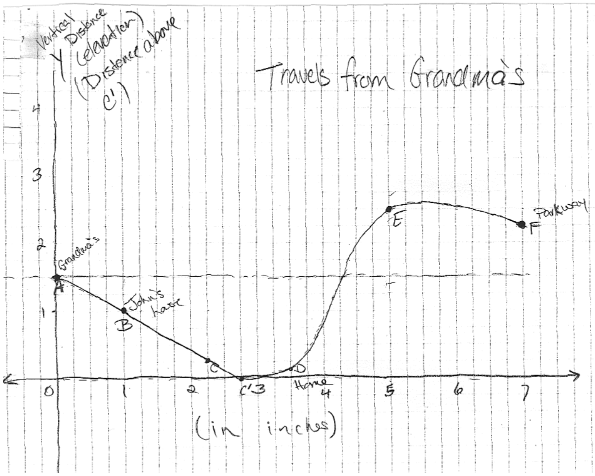

Figure 1. Kay's Final 2-D Cross-Section Graph

DISCUSSION

This study demonstrated how a problem-based, scientific inquiry of a 3-D model might foster a meaningful conceptualization of 2-D graphs. In particular, it demonstrated the conceptual link that a high school mathematics prospective teacher did not have available in order to shift between the two dimensions. This demonstration is important for two reasons. First, quite often the *Connections* standard (NCTM, 2000) is thought of in terms of introducing new ideas to students. However, the present study indicates that fostering connections can enhance meaningful understandings much later, when a person reorganizes her or his rather limited, textbook-like notions (e.g., 'rise-over-run'). Thus, the present study can contribute to teacher educators' identification of weak areas in teachers' mathematics and of ways to foster a more meaningful understanding via scientific activities.

Second, the present study demonstrates how the mechanism of reflection on activity-effect relationship (Simon et al., in press) helps in analyzing and designing activities that are likely to foster a desired, meaningful understanding. For example, this mechanism provided the conceptual lenses needed to articulate the generative power of two activities that Kay used: turning the 3-D model upside down and laying the ruler across its base. These activities became constituents of a new conception because Kay could notice the effect of both activities, first in a local manner (a reference point for measuring *D-C'*), then in a general, invariant manner (a reference point for measuring any vertical or horizontal distance). Thus, she was able not only to accomplish the complex enough task of generating a graph (Roth & McGinn, 1997), but also to form a quantified image of any chosen path along the contours of a landscape. Having formed this new conception allowed her to meaningfully carry out the mental and practical back-and-forth shifts between a graphed cross-section and changes of slope in actuality. This kind of understanding is necessary, for example, to make sense of computer simulations because a user does not have access to these back-and-forth shifts as they are carried out by the computer—a plausible reason for Kay's disengaging experience with 3-D simulations.

References

Dewey, J. (1933). *How We Think: A Restatement of the Relation of Reflective Thinking to the Educative Process*. Lexington, MA: D.C. Heath.

NCTM. (2000). *Principles and Standards for School Mathematics*. Reston, VA: Author.

Piaget, J. (1985). *The Equilibration of Cognitive Structures: The Central Problem of Intellectual Development* (T. Brown & K. J. Thampy, Trans.). Chicago: The University of Chicago.

Powell, A. B., Francisco, J. M., & Maher, C. A. (in press). An analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data. *Journal of Mathematical Behavior*.

Roth, W.-M., & McGinn, M. K. (1997). Graphing: Cognitive ability or practice? *Science Education, 81*, 91-106.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114-145.

Simon, M. A., Tzur, R., Heinz, K., & Kinzel, M. (in press). Explicating a mechanism for conceptual learning: Elaborating the construct of reflective abstraction. *Journal for Research in Mathematics Education*.

Simon, M. A., Tzur, R., Heinz, K., Kinzel, M., & Smith, M. S. (1999). On formulating the teacher's role in promoting mathematics learning. In O. Zaslavsky (Ed.), *Proceedings of the 23rd Annual Meeting of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 201-208). Haifa, Israel.

Steffe, L. P. (2002). A new hypothesis concerning children's fractional knowledge. *Journal of Mathematical Behavior, 20*, 1-41.

Steffe, L. P., & Tzur, R. (1994). Interaction and children's mathematics. *Journal of Research in Childhood and Education, 8*(2), 99-116.

von Glasersfeld, E. (1995). *Radical Constructivism: A Way of Knowing and Learning*. Washington, D.C.: Falmer.