

CORRESPONDENCES, FUNCTIONS AND ASSIGNATION RULES

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In this paper we put forward a theoretical position that, in cognitive terms, a differentiation should be made between a correspondence and a function. Important in understanding this difference is the role of an assignation rule; the correspondence acts as a way to identify a rule in context, whilst the function accommodates the rule in a more formal framework providing a secure base for argumentation. This perspective is used to interpret some students' behavior in a task where the identification of a particular relationship is crucial for its solution.

Introduction

When the word 'correspondence' is invoked in mathematics education literature, it is usually done in contrast with what is often termed 'covariation'. This 'duality' in the notion of function can be said to have its roots historically; mathematicians up to the 19th century usually handled perceived relationships as the co-ordination of variables, whereas latter mathematicians tended to insist on ordered pairs. Papers dealing with the development of the concept historically include (Malik, 1980) and (Kleiner, 1989). This issue has been influential on curriculum decisions about how functions should be taught; this was particularly significant during the so-called 'new math' era (see e.g., Eisenberg, 1991). This theme is rather incidental to this paper, but it is important to mention it because in the duality of covariation and correspondence, the correspondence is closely linked with the 'modern' definition of function. In contrast, though, in this paper we wish to stress differences between correspondences and functions.

We shall argue then that although a function can always be constructed to 'express' any given correspondence, the correspondence and the function are essentially different things. In particular the correspondence is always understood within the context of a task environment, so we avoid having to think about the mapping explicitly in the form of ordered pairs (that is a problematic structure for many students if they are exposed to it). The correspondence involves an assignation rule that maps one family of objects to another in a systematic way. The function is a framework that allows the expression of this assignation rule in terms of specific sets. As such, we regard the function as a formal counterpart of the correspondence that is taken as having an intuitive character.

In this paper we shall develop the issue of the previous paragraph, and shall discuss its educational significance. In particular, now we are in the position of comparing an intuitive construct with one that is more formal, a ripe situation for

balancing flexible thought with tight argumentation. We shall illustrate this by describing a particular episode from a fieldwork that displays how the central idea opening up a general direction for solving a specific task comes in the form of a correspondence, but this correspondence is handled securely only when it is ‘converted’ into a function.

Correspondences, functions and assignation rules

Perhaps some channels of communicating mathematics, especially textbooks, would not explicitly discriminate between the terms ‘function’ and ‘correspondence’. This is because they may not aim to provide means for overt discussion between the intuitive and formal levels of expression; evidently, though, the two words carry different connotations.

A correspondence marks the outcome of a mental activity that led to the identification of some association in a system. The ‘human’ component here should be stressed. We should also qualify what we mean by ‘association’ in this context. What governs the character of the association is a natural rule (i.e., the rule is determined by the perceived structure of the system) that provides a particular reason to make a mental linkage between any object of one kind with a (unique) one of another kind. We shall call such a rule an assignation rule. The assignation rule has the role of indicating how any particular relevant single object would be assigned or related to some other object. The associated correspondence expresses the ‘objectification’ of the process implied by the rule, in the sense that one can conceive the rule acting on all relevant objects simultaneously. We stress here that although there is a consciousness of a systematic pairing of objects, a correspondence in itself does not imply an analysis of the exact range, or extent, of the objects for which it makes sense to say that the rule applies. The assignation rule is the prime focus for the correspondence; a common occurrence, though, is that a student understands the rule but tries to apply it where this is not suitable. (The basic rule may be constrained by certain conditions due to the specific aims of the mathematics being done.)

In this paper we choose to consider the role of a function as a counterpart to a correspondence. (In more abstract situations, e.g., when a certain function is known to exist but is not explicitly constructed, there may be no sense of an accompanying correspondence.) The formal definition of a function is strictly given in terms of a (formal) relation, but what is usually used is the medium of the sets domain, co-domain and a means to associate every element of the domain with a (unique) element of the co-domain. A function differs from a correspondence in two ways. First, the function insists on the specification of two sets before any association is to be introduced. This contrasts with the focus that a correspondence has, where an assignation rule is comprehended first contextually, and only after can there be reflection as to the exact range of the objects involved. Second, although a function must express a means to make an assignation, this does not necessarily have to be explicit or to take the form of anything that would be recognized as a rule. How can

these differences be explained, and what possible cognitive problems might accrue for the student?

What is at the heart of this issue possibly is the following: the genesis of a correspondence is in the mental processing of an observation, whereas the role of a function is more to do with control. Let us expand on this. A correspondence usually results from the identification of an association discovered in terms of the system being studied. Quite likely this discovery will affect the student's cognitive view of the whole system. One thing that can happen is that too much attention is put on the newly realized correspondence, with the outcome that the original aims of the task become confused. Insisting on specifying explicit sets describing the objects to be related would help in controlling the situation. This forces a shift of attention from the correspondence itself (primarily thought about in terms of an assignation rule) to the delineation of the 'arena' for which the correspondence applies. Identifying this 'arena' should provide a balanced perspective about the correspondence within the whole system.

From the discussion above, it is clear from a cognitive point of view that a function formed to 'reflect' a given correspondence does not have the character of a replica in more mathematical terms, nor even of a mathematical model. The correspondence and the function have essential differences that hint that the latter is best thought of as a way of accommodating the former in a controlled mathematical environment. It may be difficult for students to appreciate the two different but related roles taken here. Indeed the educational literature on functions suggests that students do not make a clear distinction between a correspondence and its parallel function. A typical list of students beliefs on functions, as in (Vinner, 1983), hardly would refer to domains and co-domains. Instead it would mostly concern relationships or the usage of given rules, together with the idea of covariation, as well as identification with specialized types of representation such as algebraic formulae and graphs. Covariation provides an alternative intuitive way to process relationships, see the introduction. All of the other items tend to promote the idea of the agency of an assignation rule, and hence would suggest a mentality more allied to a correspondence rather than to a function (from our point of view). Further, certain student behavior, such as over assumption of the property 1:1 in relationships as mentioned in (Dubinsky & Harel, 1992), could be accounted for from the correspondence viewpoint. But what consequences might this over-riding dependence on the correspondence have in real terms?

If, as the evidence indicates, students do not appreciate the role of the definition of a function as an accommodating framework, according to our theoretical position they may well lack the control in refining correspondences. We shall raise a particular point concerning this situation. If a correspondence were realized within a system, would the significance of the correspondence in the general system deflect attention from the specific task aims? Were the respective function formulated, would this

circumstance be better controlled? In the next two sections, we will illustrate this issue by an episode extracted from some fieldwork.

The Fieldwork

The original purpose of the fieldwork was to illustrate a teaching sequence designed to make students become aware of a particular technique (but with the potential of being used for other techniques). The particular technique studied was the construction of a bijection in order to transfer questions about how many elements there are in one set to another. The main instrument in the teaching sequence is a semi-structured discussion between a small group of students with a teacher/researcher prompting its overall direction. The students try to answer some tasks all designed such that they are most conveniently solved via the technique. The prompting is done to help the students to achieve a solution consonant to the technique for each task, but this is not (necessarily) done with the students' being aware of the technique itself. After, further prompting is performed to make the students reflect about the solutions of the tasks and their commonality, in the hope that this would yield a conscious awareness of the technique.

In practice, what the first level of prompting involved was to 'nudge' the students' attention towards a particular relationship understood in terms of the task environment. The second level of prompting largely concerned influencing the students to try to express the relationship as an explicit function. Thus we have the situation where correspondences are first observed in context and then are accommodated as functions, just as in the main theoretical theme presented above. This explains why this fieldwork is pertinent.

As we shall only extract one particular episode from this fieldwork, more functional details are not given here. The study took place at an U.S.A. University in 2001 involving 4 sophomore students all planning to major in disciplines with high mathematical requirements. The discussion was audio and video taped.

The Episode

We shall present an episode extracted from the fieldwork described in the previous section. The participating students will be denoted as S1, S2, S3, S4 and the prompter as L. Material in parenthesis in the transcript is explanatory and not spoken. The task considered is:

A (r,m) -tuple is an ordered string of 0's and 1's where there are r 1's and $(m - r)$ 0's. We denote the set of all (r,m) -tuples by $S_{r,m}$.

Form a bijection between $S_{r-1,m-1} \cup S_{r,m-1}$ and $S_{r,m}$. Explain why this implies:
$${}_{r-1}C_{m-1} + {}_rC_{m-1} = {}_rC_m$$

where ${}_rC_m$ signifies the number of ways of picking r things out of m .

The key in solving the problem is to recognize the natural assignment rule suggested by the action of suppressing the, say, last component of an element of $S_{r,m}$. More specifically, if the last component of an element of $S_{r,m}$ is 0, it is mapped to an element of $S_{r,m-1}$; if it is 1, it is mapped to an element of $S_{r-1,m-1}$. Realizing that the underlying function is a bijection, and that $|S_{r,m}| = {}_r C_m$, this yields the well known numerical identity between binomial coefficients without algebraic manipulation.

The students were not able to proceed on the task on their own. Prompting was done to draw the students' attention to the number of components of the elements in the given sets. This eventually led one student, S2, to state: "Couldn't you just like knock out the last...?" At this point we expected a clear expression of a correspondence to emerge; instead the discussion took another turn:

1. S3: The m term will either be a 1 or a zero. So if it's 1, send it to this set ($S_{r-1,m-1}$) and if it's a zero, send it to the other set ($S_{r,m-1}$).
2. L: Uh-huh. So...
3. S1: ... Well, the only way that would work, mapping the ones with zero in the m th term to the first one ($S_{r,m-1}$), and with 1 in the m th term to the second one ($S_{r-1,m-1}$), the only way that would work is if the two sets have an equal number of elements. That would be the only way that I can see that would be a bijection.
4. L: Can you explain yourself a little bit more on that?
5. S1: Well, the two possibilities are that it has zero in the last term or 1 in the last term. And there are equal number of these terms and these terms... There is an equal number of each... Okay, if there are m terms in this ($S_{r,m}$), there are m over 2 terms of this ($S_{r,m-1}$) and there are m over 2 terms of this ($S_{r-1,m-1}$).
6. S3: They ($S_{r-1,m-1}$ and $S_{r,m-1}$) would have to have the same number of elements as half of that set ($S_{r,m}$)?
7. S1: Well, I was assuming that since we said that a bijection existed, that we're trying to find a bijection, if a bijection did exist. Then this set would have to have the same number of elements as this set anyway. So... I took that on an assumption. But, yes, that's true.
(Some discussion suppressed).
8. S3: I have something. In the case where the m th term is zero, we know there are r 1's. And if you take the $m-1$ elements and you could match them up to $S_{r,m-1}$, that would form half of the bijection. And in the bottom case... in the bottom case you know there are r 1's, and one of them is already the m th term so there are $r-1$ left, and you could map that to that ($S_{r-1,m-1}$) ... and that would be a bijection.
9. L: Okay, so what we had before, I think we had the suggestion that the size of this set ($S_{r-1,m-1}$) and this set ($S_{r,m-1}$) was equal...

10.S1: I'm thinking that's probably not true, no. I was thinking that there were... Okay, no, that's not true at all. I take that back. I'm agreeing with what he's saying now, that that's not important. Okay, I understand.

The students are led to observe a particular action (of 'knocking out' the last component) that induces a way to identify any object of one kind (a tuple with m components) with one of another kind (a tuple with $m - 1$ components). This means the students have produced a natural assignment rule and thus a correspondence.

The task environment refers to certain sets (of tuples), whose appearance (we supposed) would help the students to make a smooth transfer from the correspondence to a suitable function. As we see from the protocol above, this transfer in fact took some time to be effected. In 1, we see that the correspondence was mentally reprocessed in a coarser way to how it was first conceived. Instead of associating a tuple T (in $S_{r,m}$) with a particular tuple with $m-1$ components, the rule is re-read only to register to which out of the two sets, $S_{r-1,m-1}$ or $S_{r,m-1}$, T will be 'designated' to. In this way the focus is turned away from the natural 1:1 matching that the correspondence suggests. It seems that this image dominates until we get to 8. The correspondence is being used to gain a particular perspective on the general system, but it is not being developed as a function. From this situation and as the task environment explicitly asks for a bijection, the students actually look elsewhere to find a bijective function. One student claims that the sets $S_{r-1,m-1}$ and $S_{r,m-1}$ have the same number of elements in 3, a proposition that is not (in general) correct. There seemed to be two reasons why the student makes this proposition. First in 5 he expressed the (false) belief that the number of elements of $S_{r,m}$ 'ending' with 0 equals the number of elements 'ending' with 1. (We conjecture that the source of this belief is based on an under criticized sense of symmetry.) The understood designation of sending elements of $S_{r,m}$ into either the set $S_{r-1,m-1}$ or the set $S_{r,m-1}$ then leads him to his claim. Second, in 3 and 7 it comes clear that he is also influenced by the appearance of the word 'bijection' in the question; he 'knows' that he must fit one somehow in the system and the only way he can 'see' one is through the supposition that $|S_{r-1,m-1}| = |S_{r,m-1}|$ that would ensure that a bijection exists between these two sets. (Notice in taking this stance means that no bijection has been explicitly constructed.) Only in 8, that occurred a significant time after the start of this discussion (some material has been omitted in the protocol), did another student identify a bijection with the original correspondence. This led the student S1 to immediately renounce his claim in 10. Notice how this student was not interested in explaining explicitly why his claim was false, but he dismissed it as if there was no longer any reason to believe that it should be true.

We find this episode interesting because it illustrates how a correspondence observed in a system may not be readily converted into a function, but can be an influence to throw a new intuitive perspective of the system. However doing this led to an unhelpful line of thought. Only when finally the correspondence was fully integrated with the sets being talked about (so that it could be recognized as a

function) did this confusion dissipate. Hence the forming of the function acted as a device of control in the soundness of argument made about the correspondence.

Concluding Remarks

This paper has discussed the process of forming functions from the basis of observing correspondences drawn from context. In this respect we make a differentiation between the character of a correspondence and that of a function. We believe that this perspective is not well represented in the extant literature on functions. We agree with the argument given in the review article (Thompson, 1994) that the current trend of concentrating on the so-called representations of functions (usually graphs, algebraic expressions, tables) and their co-ordination is being over emphasized. This tendency reflects the current teaching practices that has severely restricted the students' image of what functions are in general, as this image is dominated by certain paradigms, see (Bakar & Tall, 1991). Further, the representation perspective tends to neglect the question how, and indeed why, functions come to be constructed. There are studies that go some distance in this direction, where within the task environment a certain relationship is pointed out, and the task itself is to express it by a suitable real function (sometimes where only qualitative description in terms of properties is feasible, see e.g., Monk, 1992 Thompson, 1994). What often seems to occur in such studies is that students lose control concerning what families of objects should accommodate the intuitive sense of the relationship. In this paper, we go a little further in that even the obtaining of the relationship is part of the solving procedure. It is perhaps in the latter circumstance that the drawing of a distinction between a correspondence and a function is at its most compelling. The correspondence is how a suitable relation observed in a system is first thought of, and the allied function has the role to ensure that the subsequent mental argumentation of the correspondence is grounded on an explicit mathematical framework that should remove vagueness and arbitrary interpretations. A crucial part of this is that the function accommodates the assignation rule understood for the correspondence in an unequivocal way. This issue was well illustrated by the episode described in this paper. We believe that the perspective that we have laid down on correspondences, functions and assignation rules should provide not only a good way to explain students' behavior whilst constructing functions, but should be taken in account in how functions are taught. We plan to expand on these themes in subsequent papers.

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