MATHEMATICAL THINKING & HUMAN NATURE: CONSONANCE & CONFLICT

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Human nature had traditionally been the realm of novelists, philosophers, and theologicians, but has recently been studied by cognitive science, neuroscience, research on babies and on animals, anthropology, and evolutionary psychology. In this paper I will show—by surveying relevant research and by analyzing some mathematical "case studies"—how different parts of mathematical thinking can be either enabled or hindered by aspects of human nature. This novel theoretical framework can add an evolutionary and ecological level of interpretation to empirical findings of math education research, as well as illuminate some fundamental classroom issues.

A. INTRODUCTION

This paper deals with the relationship between mathematical thinking and human nature. I take from the young discipline of Evolutionary Psychology (EP) the scientific view of human nature as a collection of universal, reliably-developing, cognitive and behavioral abilities—such as walking on two feet, face recognition or the use of language—that are spontaneously acquired and effortlessly used by all people under normal development (Cosmides & Tooby 1992, 1997, 2000; Pinker 1999, 2002, Ridley, 2003). I also take from EP the evolutionary origins of human nature, hence the frequent mismatch between the ancient ecology to which it is adapted and the demands of modern civilization. To the extent that we do manage to learn many modern skills (such as writing or driving, or some math), this is because of our mind's ability to "co-opt" ancient cognitive mechanisms for new purposes (Bjorklund & Pellegrini, 2002; Geary, 2002). But this is easier for some skills than for others, and nowhere are these differences manifest more than in the learning of mathematics. The ease of learning in such cases is determined by the *accessibility* of the co-opted cognitive mechanisms. I emphasize that what is part of human nature need not be innate: we are not born walking or talking. What seems to be innate is the motivation and the ability to engage the species-typical physical and social environment in such a way that the required skill will develop (Geary, 2002). This is the ubiquitous mechanism that Ridley (2003) have called "Nature *via* Nurture".

These insights have tremendous implications for the theory and practice of Mathematics Education (ME), but to this date they have hardly been noticed by our community (but cf. Tall, 2001; Kaput & Shaffer, 2002). The goal of this paper is to launch an investigation (theoretical at this preliminary stage) of how the insights from EP may bear on the theory and practice of ME. Specifically, the goal is to investigate, in view of the above-mentioned mismatch, how different parts of mathematical thinking can be either enabled or hindered by specific aspects of human nature.

Consider, for example, the well-documented phenomenon that students tend to confuse between a theorem and its converse (e.g., Hazzan & Leron, 1996). As will be shown later, this phenomenon can now be understood as a clash between Mathematical Logic and the Logic of Social Exchange – a fundamental part of human nature.

In the rest of this document, I will describe my synthesis of the existing research on how mathematical thinking is enabled (Section B) or constrained (Section C) by human nature.

B. ORIGINS OF MATHEMATICAL THINKING[1]

In this section I consider the following (admittedly vague) question:

> *Is mathematical thinking a natural extension of common sense, or is it an altogether different kind of thinking?*

The possible answers to this question are of great interest and importance for both theoretical and practical reasons. Theoretically, this is an important special case of the general question of how our mind works. In practice, the answers to this question clearly have important educational implications. I take *common sense* to mean roughly the same as the cognitive part of human nature – the collection of abilities people are spontaneously and naturally "good at" (Cosmides & Tooby, 2000).

Recently, several books and research papers have appeared, which bear on this question, so that the possible answers, though still far from being conclusive, are less of a pure conjecture than they had previously been. The conclusions of the various researchers seem at first almost contradictory: Aspects of mathematical cognition are described as anything from being embodied to being based on general cognitive mechanisms to clashing head-on with what our mind has been "designed" to do by natural selection over millions of years.

However, these seeming contradictions all but fade away once we realize that "mathematics" (and with it "mathematical cognition") may mean different things to different people, sometimes even to the same person on different occasions. In fact, the main goal of this section is to show that all this multifaceted research by different people coming from different disciplines, may be neatly organized into a coherent scheme once we exercise a bit more care with our distinctions and terminology.

To this end, I will distinguish three levels of mathematics, *rudimentary arithmetic*, *informal mathematics* and *formal mathematics*, each with its own different cognitive mechanisms[2]. When interpreted within this framework, the research results show that while certain elements of mathematical thinking are innate and others are easily

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[1] More details on the topics of this section can be found in Leron (2003a).

[2] Strictly speaking, 'formal' and 'informal' ought to refer not to the mathematical subject matter itself but to its presentation, and may in fact describe two facets of the *same* piece of mathematics.

learned, certain more advanced (and, significantly, historically recent) aspects of mathematics—formal language, de-contextualization, abstraction and proof—may be in direct conflict with aspects of human nature.

Following is a brief survey of theses studies and the various suggested cognitive mechanism, organized according to my 3-level framework.

Level 1: Rudimentary arithmetic

Rudimentary arithmetic consists of the simple operations of subitizing, estimating, comparing, adding and subtracting, performed on very small collections of concrete objects. Research on infants and on animals, as well as brain research, indicates that some ability to do mathematics at this level is hard-wired in the brain and is processed by a 'number sense', just as colors are processed by a 'color sense'. Comprehensive syntheses of this research are Dehaene (1997) and Butterworth (1999). The innate character of this ability is evidenced by its existence in infants, its localization in the brain and its vulnerability to specific brain injuries. The likelihood of its evolutionary origins comes from its complex design features, the survival advantage it have likely conferred on our hunter-gatherer ancestors (e.g., in keeping count of possessions, and in estimating amount of food and number of enemies), and its existence in our non-human relatives (such as chimpanzees, rats and pigeons).

Level 2: Informal Mathematics

This is the kind of mathematics, familiar to every experienced teacher of advanced mathematics, which is presented to students in situations when mathematics in its most formal and rigorous form would be inappropriate. It may include topics from all mathematical areas and all age levels, but will consist mainly of "thought experiments" (Lakatos, 1978; Tall, 2001; Reiner & Leron, 2001), carried out with the help of figures, diagrams, analogies from everyday life, generic examples, and students' previous experience.

Some recent research, as well as classroom experience, indicates that informal mathematics *is* an extension of common sense, and is in fact being processed by the same mechanisms that make up our everyday cognition, such as imagery, natural language, thought experiment, social cognition and metaphor. From an evolutionary perspective, it is only to be expected that mathematical thinking has "co-opted" older and more general cognitive mechanisms, taking into account that mathematics in its modern sense has been around for only about 25 centuries – a mere eye blink in evolutionary terms (Bjorklund & Pellegrini, 2002; Geary, 2002).

Two recent books have presented elaborate theories to show how our ability to do mathematics is based on other (more basic and more ancient) mechanisms of human cognition. First, Lakoff & Núñez (2000) show (more convincingly in some places than in others) how mathematical cognition builds on the same mechanisms of our general linguistic and cognitive system; namely, they show how mathematical cognition is first rooted in our body via embodied metaphors, then extended to more abstract realms via "conceptual metaphors", i.e., inference-preserving mappings between a source domain and a target domain[3]. Secondly, Devlin (2000) gives a different account than Lakoff & Núñez, but again one showing how mathematical thinking has co-opted existing cognitive mechanisms. His claim is that the metaphorical "math gene"—our innate ability to learn and do mathematics—comes from the same source as our linguistic ability, namely our ability for "off-line thinking" (basically, performing thought experiments, whose outcome will often be valid in the external world). Devlin in addition gives a detailed evolutionary account of how all these abilities might have evolved.

Significantly for the thesis presented here, both theories mainly seek to explain the thinking processes involved in Level 2 mathematics, so that their conclusions need not apply to Level 3 mathematical thinking[4]. In fact, as I explain in the next section, they generally don't. Devlin's account, in particular, fits well situations in which people do mathematics by constructing mental structures and then navigate within those structures[5], but not situations where such structures are not available to the learner. For example, it is hard to imagine any "concrete" structure that will form an honest model of a uniformly continuous function or a compact topological space.

Level 3: Formal Mathematics

The term "formal mathematics" refers here not to the contents but to the form of advanced mathematical presentations in classroom lectures and in college-level textbooks, with their full apparatus of abstraction, formal language, de-contextualization, rigor and deduction. The fact that understanding formal mathematics is hard for most students is well-known, but my question goes farther: is it an extension (no matter how elaborate) of common sense or an altogether different kind of thinking? Put differently, is it a "biologically secondary ability" (Geary, 2002), or an altogether a new kind of thinking that ought perhaps to be termed "biologically tertiary ability"? This issue will be our focus in the next section. The mathematical case studies, as well as the persistent failure of many bright college students to master formal mathematics, suggest that the thinking involved in formal mathematics is *not* an extension of common sense; that it either can't find suitable abilities to co-opt, or it can even clash head-on with what for all people "comes naturally".

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[3] However, as Tall (2001) points out, we need also to take into account a process going in the reverse direction. Some of the results of the formal axiomatic theory (called "structure theorems") may feed back to develop more refined intuitions, or embodiments, of the concepts involved.

[4] The authors are not always explicit on the scope of mathematics they discuss, but see, e.g., "I am not talking about becoming a great mathematician or venturing into the heady heights of advanced mathematics. I am speaking solely about being able to cope with the mathematics found in most high school curricula." (Devlin, 2000, p. 271); and "Our enterprise here is to study everyday mathematical understanding of this automatic unconscious sort." (Lakoff & Núñez, 2000, p. 28).

[5] See in this connection his "mathematical house" metaphor on p. 125.

C. MATHEMATICAL THINKING VS. HUMAN NATURE: SOME MATHEMATICAL CASE STUDIES

The theoretical framework outlined above will now be applied to a sample of three mathematical "case studies" which, except for that framework, would have largely remained an unexplained paradox. Each of the case studies deals with a well-defined mathematical topic or task, which on the one hand seems rather simple and elementary, but on the other hand has been shown to cause serious difficulty for many people (i.e. most people fail on it).

Memorizing the Tables

Dehaene (1997) discusses the ample research evidence by psychologists, that many people find memorizing the multiplication table extremely hard: they take a long time to answer and they make many errors. There is also much evidence that all children demonstrate prodigious learning and memory capabilities in, say, learning the vocabulary and use of their mother tongue. How is it then that many have such difficulty remembering a few (less than 20 if you count carefully) multiplication facts? How can human memory, which in other contexts performs astonishing feats, fail on such a simple-looking task?

This example demonstrates in a nutshell—and elementary setting—the typical paradox that is at the core of this paper: People fail this task not because of a weakness in their mental apparatus, but because of its *strength*! Trouble is, what may have been adaptive in the ancient ecology of our stone-age ancestors, and is still adaptive today under similar conditions, may often be maladaptive in modern contexts. Returning to the present example, the particular strength of our memory that gets in the way of memorizing the tables is its insuppressible *associative* character: The *forms* of the number facts cannot be separated out and remain hopelessly entangled with each other. To quote Dehaene (1997): "Arithmetic facts are not arbitrary and independent of each other. On the contrary, they are closely intertwined and teeming with false regularities, misleading rhymes and confusing puns."

Mathematical Logic vs. the Logic of Social Exchange

Cosmides and Tooby (1992, 1997) have used the Wason card selection task, which tests people's understanding of "if P then Q" statements, to uncover what they refer to as people's evolved reasoning algorithms. They presented their subjects with many versions of the task, all having the same logical form "if P then Q", but varying widely in the contents of P and Q and in the background story. While the classical results of the Wason Task show that most people perform very poorly on it, Cosmides and Tooby demonstrated that their subjects performed relatively well on tasks involving conditions of *social exchange*. In social exchange situations, the individual receives some benefit and is expected to pay some cost. In the Wason experiment they are represented by statements of the form "if you get the benefit, then you pay the cost" (e.g., if you give me your watch, then I give you $20). A *cheater* is someone who takes the benefit but do not pay the cost. Cosmides and

Tooby explain that when the Wason task concerns social exchange, a correct answer amounts to detecting a cheater. Since subjects performed correctly and effortlessly in such situations, and since evolutionary theory clearly shows that cooperation cannot evolve if cheaters are not detected, Cosmides and Tooby have concluded that our mind contains evolved "cheater detection algorithms".

Significantly for mathematics education, Cosmides and Tooby have also tested their subjects on the "switched social contract" (mathematically, the converse statement "if Q then P"), in which the correct answer by the logic of social exchange is different from that of mathematical logic (Cosmides and Tooby, 1992, pp. 187-193). As predicted, their subjects overwhelmingly chose the former over the latter. When conflict arises, the logic of social exchange overrides mathematical logic. This theory adds a new level of support, prediction and explanation to the many findings (e.g. Hazzan & Leron, 1996) that students are prone to confusing between mathematical propositions and their converse. Again, they fail not because of a human cognitive weakness, but because of its *strength*: the ability to negotiate social exchange and to detect cheaters. Unfortunately for mathematics education, this otherwise adaptive ability, clashes with the requirements of modern mathematical thinking.

Do functions make a difference?

The phenomenon reported here came up in the context of research on learning computer science (specifically, functional programming), but has turned out to be really an observation on mathematical thinking. Interestingly, it is hard to see how this phenomenon could have been revealed through a purely mathematical task. The empirical research reported here is taken from Tamar Paz's (2003) dissertation, carried out under the supervision of this author. I only have space here for a very brief outline of this study. Cf. Leron (2003b) for more details.

In functional programming, functions are mainly viewed as a process, starting with an input value, performing some operations on it, and returning an output value. This image is nicely captured by the function machine metaphor. I have called this the *algebraic image* of functions, as opposed to the *analytic image*[6]. For example, the function `Rest L` takes a list `L` as input (for example, the value of `L` could be the 4-element list `[A B C D]`) and returns the list without its first element (in this example, `[B C D]`). Paz (2003) has found that many of the students' programming errors could be attributed to their assumption that the function actually changes the input variable (so that after the operation, L assumes the new value `[B C D]`). But in functional programming, as in its parent discipline mathematics, functions do *not* change their inputs, they merely map one value to another.

I propose to view this empirical finding as an example of the clash between the modern mathematical view of functions, and their origin in human nature. To do this, we need to look for the roots (mainly cognitive and developmental, but also

[6] We have two *concept images* of function, sharing one *concept definition* (Vinner & Tall, 1981).

historical) of the function concept – a synthesis of Freudenthal's (1983) "didactical phenomenology" and Geary's (2002) "biologically primary abilities": What in the child's natural experience during development, may have given rise to the basic intuitions on which the function concept is built? (Fruedenthal, 1983; Kleiner, 1989; Lakoff & Núñez, 2000.)

According to the algebraic image of functions, an operation is *acting on an object*. The agent performing the operation takes an object and does something to it. For example, a child playing with a toy may move it, squeeze it, or color it. The object before the action is the input and the object after the action is the output. The operation is thus transforming the input into the output. The proposed origin of the algebraic image of functions is the child's experience of acting on objects in the physical world. This is part of the basic mechanism by which the child comes to know the world around it, and is most likely part of what I have called universal human nature. Part of this mechanism is perceiving the world via objects, categories and operations on them, as described by Piaget, Rosch, and others. Inherent to this image is the experience that an operation *changes its input* – after all, that's why we engage in it in the first place: you move something to change its place, squeeze it to change its shape, color it to change its look.

But this is not what happens in modern mathematics or in functional programming. In the modern formalism of functions, nothing really changes! The function is a "mapping between two fixed sets" or even, in its most extreme form, a set of ordered pairs. As is the universal trend in modern mathematics, an algebraic formalism has been adopted that completely suppresses the images of process, time and change[7].

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[7] Professional mathematicians are able to maintain these images *despite* the formalism, but for novices the connection is hard to come by.

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