PSYCHOLOGICAL ASPECTS OF GENETIC APPROACH TO TEACHING MATHEMATICS

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In this theoretical essay the psycological aspects of genetic approach to teaching mathematics (mainly at universities) are discussed. Analysis of the history and modern state of genetic teaching shows that its psycological aspects may be explained using both Vygotskian and Piagetian frameworks. Experience of practice of mathematical education has been important for the development of genetic approach as well. Furthermore, genetic teaching should be enhanced by stylistic and emotional elements developing students' motivation and interest.

PRINCIPLE OF GENETIC APPROACH

The principle of genetic approach in teaching mathematics requires that the method of teaching a subject should be based, as far as possible, on natural ways and methods of knowledge inherent in the science. The teaching should follow ways of the development of knowledge. That is why we say: “genetic principle”, “genetic method”.

Probably, the first who used the expression “genetic teaching” was prominent German educator F.W.A.Diesterweg (1790-1866) in his published in 1835 “Guide to the education of German teachers”: “...The formal purpose requires genetic teaching of all subjects that admit such teaching because that is the way they have arisen or have entered the consciousness of the human …Though a pupil covers in several years a road that took milleniums for the mankind to travel. However, it is necessary to lead him/her to the target not sightless but sharp-eyed: he/she must perceive truth not as a ready result but discover it...” (Diesterweg, 1962).

Certainly, ideas of genetic principle had been expressed prior to Diesterweg, too. For example, much earlier G.W. Leibnitz (1880) expressed a similar idea: “I tried to write in such way that a learner could always see the inner foundation of things studied, that he could find the source of the discovery and, consequently, understand everything as if he invented that by himself”.

In history and modern state of genetic approach a significant variety of interpretations of the terms “genetic principle”, “genetic method”, “genetic approach to teaching mathematics” is observed... It is clear that today, as noted by Wittenberg (1968, p.127), nobody understands genetic approach as historical, and more appropriate is idea that genetic approach is connected to relevance, which here should be understood as conformity of a method of teaching (and learning) to the most expedient and natural ways of cognition inherent in the given subject (or topic).

Wittenberg is certainly right also in that genetic approach is connected to epistemology, psychology and logic.

Analyzing various interpretations of genetic approach to teaching mathematics in theory and history of mathematics education and taking into account today's experience of teaching undergraduate mathematics and latest achievements of psychology and methods of teaching mathematics, we can reveal the contents and features of genetic approach to teaching mathematical courses in universities.

We will call the teaching of a mathematical discipline *genetic* if it follows natural ways of the origination and application of the mathematical theory. Genetic teaching gives the answer to a question: how the development of the contents of the mathematical theory can be explained?

Genetic teaching of mathematics at universities should have the following properties:

Genetic teaching is based on students' previously acquired knowledge, experience and level of thinking;

For the study of new themes and concepts the problem situations and wide contexts (matching the experience of students) of non-mathematical or mathematical contents are used;

In teaching, various problems and naturally arising questions are widely used, which should be answered by students independently with minimal necessary effective help of the teacher;

Strict and abstract reasonings should be preceded by intuitive or heuristic considerations; construction of theories and concepts of a high level of abstraction can be properly carried out only after accumulation of sufficient (determined by thorough analysis) supply of examples, facts and statements at a lower level of abstraction;

The stimulation of mental and cognitive activity of students should be performed, they should be constantly motivated;

The gradual enrichment of studied mathematical objects by interrelations with other objects, consideration of the studied objects and results from new angles, in new contexts should be carried out.

PSYCHOLOGY OF TEACHING AND GENETIC APPROACH

One of major aspects of genetic approach to teaching mathematics is psychological aspect. As indicated by E.Ch.Wittmann (1992, p. 278), genetic principle should use results of both genetic epistemology of J. Piaget and Soviet psychology based on the concept of activity. Synthesizing not contradicting each other results of two theories concerning construction and development of concepts in the learning process, it is possible to take as a psychological basis of genetic approach to teaching mathematics the following principles of psychology of education:

1) *Principle of problem-oriented teaching*. S.L.Rubinshtein (1989, p. 369) wrote: “The thinking usually starts from a problem or question, from surprise or bewilderment, from a contradiction”. It is similar to Piagetian phenomenon of the violation of balance between assimilation and accommodation. L.S.Vygotsky (1996, p. 168) indicated in 1926 that it is necessary to establish obstacles and difficulties in teaching, at the same time providing students with ways and means for the solution of the tasks posed.

2) *Principle of motivation and development of interest*. L.S.Vygotsky (1996) in “Pedagogical psychology” indicated to the importance of interest and emotional issues in teaching.

3) *Principle of continuity and visual representations*: introducing new contents, it is necessary to maximally use previously generated cognitive structures and visual representations of pupils, familiar contexts. This principle is connected to the Vygotsky's theory of development of scientific concepts (see, e.g., Vygotsky, 1996, p. 86 and 146), and also with his concept of “zone of proximal development”.

4) *Principle of integrity and system approach*: the teaching should aim at the accumulation of integral systems of cognitive structures by the pupil (Itelson, 1972, p. 132). This principle also follows both from the activity approach (Vygotsky, 1996, p. 178-179 and 270; Davydov, 2000, p. 327-328, 400) and from the theory of operator structures of J.Piaget (1994, p. 89-91).

5) *Principle of “enrichment”*: “Accumulation and differentiation of experience of operating by an introduced concept, expansion of possible aspects of understanding of its contents (by inclusion of its various interpretations, increase of number of variables of different degree of essentiality, expanding interconceptual connections, use of alternative contexts of its analysis etc.)” (Kholodnaya, 1996, p. 332). This principle was in various forms repeatedly proposed by psychologists (Rubinshtein, 1958; p. 98-99; Davydov, 2000, p. 429).

6) *Principle of “transformation”*: for revealing essential properties of an object, its essence, “genetically initial general relation” (Davydov, 2000), it is necessary to subject this object to mental transformations, to perform mental experiments, asking questions of the type: “What will happen with the object if? … ”

According to the theory of A.N.Leontyev (1977, p. 208-212), “any substantial activity answers a need materialized in a motive; its main generators are the purposes and corresponding actions… The task is just a purpose given in certain conditions… By operations we mean ways of realization of actions... Actions… are correlated to purposes and operation to conditions… the genesis of an action lays in the exchange of activities and derives from the intrapsychologization of them. Every operation is a result of transformation of an action occurring as a result of its inclusion in another action and its subsequent “mechanization” …Actions are processes subordinate to the conscious purposes… the operations… directly depend on conditions of achievement of a concrete purpose”.

A. N. Leontyev (1981) argued that actions on learning concepts, as well as any actions, consist of operations, which are almost unconscious or completely unconscious. These operations are essentially “contracted” actions with the concepts of the previous level of abstraction. As M. A. Kholodnaya (1997) noted, “a contraction is immediate reorganization of the complete set of all available… knowledge about the given concept and transformation of that set into a generalized cognitive structure”.

Close to the Soviet conceptions of actions and operations as contracted actions in mathematics teaching are the APOS theory of E. Dubinsky (1991), “reification theory” of A. Sfard (1991) and idea of "procept" of Gray and Tall (1994).

Soviet psychologists, basing on the conceptions of L.S.Vygotsky and A.N.Leontyev, have developed the activity approach to teaching. The most important for development of theoretical thinking is the theory of educational activity of V.V.Davydov (1996) who wrote: “The substantial contents of a concept can be revealed only by finding out the conditions of its origination”.

We see that the process of teaching in the theory of V.V.Davydov ultimately uses genetic approach. This theory has shown its efficiency for learning theoretical concepts at the level of elementary school. Here we are discussing the use of genetic approach in teaching mathematics at undergraduate level.

The main difficulty of investigating educational activity during study of mathematical disciplines at universities consists in multilevelness of abstraction, especially in such sections as the theory of algebraic systems, functional analysis etc.

For example, A. A. Stolyar (1986, p. 58-60) has revealed 5 levels of thinking in the field of algebra and has noted, that “the traditional school teaching of algebra does not rise above the third level, and in the logical ordering of properties of operations even this level is not reached completely”. The following is the description of the third, fourth and fifth levels according to A. A. Stolyar (ibid., p. 59):

“On the 3-d level the passage from concrete numbers expressed in digits, to abstract symbolic expressions designating concrete numbers only in determined interpretations of the symbols is carried out. At this level the logical ordering of properties is carried out “locally”.

On the 4-th level the possibility of a deductive construction of the entire algebra in the given concrete interpretation becomes clear. Here the letters designating mathematical objects are used as variable names for numbers from some given set (natural, integer, rational or real numbers), and the operations have a usual sense.

At last, on the 5-th level distraction from the concrete nature of mathematical objects, from the concrete meaning of operations takes place. Algebra is being built as an abstract deductive system independent of any interpretations. At this level, the passage from known concrete models to the abstract theory and further to other models is carried out, the possibility of existence of various algebras derived formally by properties of operations is accomplished".

Thus, to the 5-th level the deductive study of groups, rings, linearly ordered sets etc. corresponds. The highest degree of abstraction here is the study of general algebraic systems with various many-placed operations.

To the 4-th level corresponds, for example, a systematic and deductive study of the sets of natural numbers or integers. Therefore, taking into account that in school teaching even on the 3-rd level is not completely reached, it would be certainly a big mistake to omit in universities the 4-th level (systematic study of an elementary number theory) and immediately pass to the deductive study of groups, rings and even of general universal algebras (as is done in a text-book by L. Ya. Kulikov, 1979). Therefore, systematic study of the elementary number theory can serve as a good sample of the construction of a deductive theory for preparation for the further construction of the axiomatic theories.

A. A. Stolyar built his classification of levels from the point of view of teaching school algebra. In our view, development of algebra as a science in the last decades (after the World War II, under the influence of works of S. Eilenberg and S. MacLane, 1945, and A. I. Maltsev, 1973) allows to distinguish one more higher, the 6-th level of algebraic thinking - we will name it the *level of algebraic categories*. At this level the entire classes of algebraic systems together with homomorphisms of these systems - varieties of universal algebras, categories of algebraic and other structures (for example, topological spaces, sets and other objects) are considered. Thus, the abstraction from concrete operations in these structures and from the nature of homomorphisms and generally of mappings takes place; morphisms between objects of categories are considered simply as arrows subordinate to axioms of categories – e.g., to the associativitiy law for the composition. Moreover, the functors between categories – certain mappings compatible with the laws of the composition of morphisms, and natural transformations of functors are considered.

Note that J. Piaget in the last years of his life was interested in the theory of categories as the highest level of abstraction in the development of algebra (Piaget and Garcia, 1989).

Essential in teaching algebra and number theory in universities are the 4-th and 5-th levels in the classification of A. A. Stolyar. First of all, the 4-th level (which is already beyond the school curricula) should be reached. Therefore, during the first introduction of the definition of a group in the beginning of the algebra course, one should not immediately begin the full deductive treatment of the axiomatic theory of groups. Only after the experience of the study at the 4-th level of thinking in the field of algebra, namely of the study of the elements of number theory, it is possible to consider a deductive system of the most simple constructions and statements of the group theory, and the systematic account of complicated sections of the theory should be postponed to a later time, after studying at the 4-th level such themes as complex numbers and arithmetical vector spaces.

J. Piaget who developed the classification of levels for thinking in the fields of geometry and algebra (“intra”, “inter” and “trans”), noted that it is possible to distinguish sublevels inside each level (Piaget and Garcia, 1989).

DEVELOPMENT OF MOTIVATION OF LEARNING WITH THE HELP OF STYLE AND EMOTIONAL ELEMENTS

The great significance for the development of motivation of learning belongs to the individual style of a lecturer or author of the textbook.

Many prominent educators seriously recognized the importance of the individual style in teaching. Diesterweg demanded to make learning interesting by means of 1) diversity, 2) liveliness of the teacher 3) by the whole personality of the teacher.

E.Wittmann (1997, p. 175-178) devoted a special epilogue in his fundamental guidebook to the human (i.e. emotional) factor in teaching mathematics.

The style and psychological effect on students (including readers of textbooks) play the extremely important role in teaching. In our view, it is possible to use many elements of artistic technique (from the areas of theatre, literature and music). Fruitful for refreshing the attention of students are the stylistic elements causing the violation of inertia of perception, e.g. elements making the different levels of the discourse conflict with each other, making the discourse strange. For example, one can very thoroughly and meticulous speak about elementary things, and, on the other hand, soften the discussion of very difficult, complicated and abstract things by humor, unexpected comparisons (as it did such classics of a science as D. Hilbert and A. Einstein). It is possible to study the skill to soften serious and hard things from the playwrights W. Shakespeare and A. Chekhov. Requires additional study the role in teaching of such aesthetic category, as catharsis (see Vygotsky, 1987).

Consider in more detail such important means of emotional influence on an audience as unexpectedness.

In our view, awakening and maintaining interest of students in a mathematical discipline should be carried out through various channels of perception. It is natural to take advantage of the experience of art.

N.N.Luzin (1948, p. 5) in the foreword to his textbook mentions with gratitude his teacher B.K.Mlodzeevsky who "always put forward the strong requirements to the artistic side of scientific discourse".

Certainly, mathematics has something in common with art. Both in mathematics and in art the important criterion of value is the economy of efforts: in science it is economy of thought, as indicated by E.Mach and A. Poincare (1990, p. 383), in art it is economy of art means (Masel, 1991, p. 182). Such economy gives also grace and ease of perception both to scientific results and to works of art.

Poincare especially noted that the grace is reached "by unexpectedness of rapproachement of such things that we have not used to pull together … Important is not a pattern in general but a pattern unexpected " (Poincare, 1908).

It is discovered by art researchers that the element of a paradoxical contradictoriness is inherent to the nature of art (Masel, 1991, p. 223). Therefore, those using means of art in teaching should also use elements of surprise.

The elements of surprise can be used in different aspects of teaching - both in the contents and in the form, and also at different levels of the discourse (in a lecture or a textbook).

As well as in art, the surprises are more effective when they are well prepared. Any concept intended to be considered in a new, unexpected context, should be in anticipation imprinted on the minds of the students, so that they really could recall it in a new situation. Lecturers might imitate authors of detective stories: the keys to the disclosure of a crime or mystery usually are distributed in different parts of a story, so that the reader, even after overlooking these moments, at once recollects them in a final scene. In mathematics it means that the lecturer, introducing any concept, should make it unusual, connect it with an interesting example, method or application.

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