ADDING FRACTIONS USING ´HALF´AS AN ANCHOR FOR REASONING

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Previous studies stressed the importance of half as an anchor in performing proportion and probability tasks. Thus, it can be supposed that this reference can help children when adding fractions. This possibility is examined in this investigation, contrasting two situations: one in which half is presented as an anchor during the solution of adding fractions; and another in which other fractional units are offered. The results showed that 8-9-year-old children successfully add fractions when half is offered as an anchor during the solution process, helping in the establishment of equivalencies. In utilising half, children also adopt elaborate strategies expressing equivalence schemes that are relevant to the comprehension of adding fractions.

INTRODUCTION

The difficulty that children display regarding the concept of fractions has long been recognised, especially in the arithmetical learning of fractions. Many of the difficulties stem from the fact that children apply their knowledge of whole numbers to the arithmetic of fractions (e.g., Behr, Wachsmuth, Post & Lesh, 1984; Gelman & Meck, 1992; Kerslake, 1986; Lamon, 1999; Pitkethly & Hunting, 1996; Sophian, Garyantes & Chang, 1997; Streefland, 1991). Furthermore, traditional teaching tends to value the use of symbols, which can often be an obstacle to the comprehension of logic subjacent to algorithmic actions in operating with fractions (e.g., Bezerra, Magina & Spinillo, 2002).

Despite these limitations, some authors have indicated a number of possibilities regarding the thought process of children (Olive, 2003). Zeman (1991), for instance, points out the capacity of children to operate with fractions when using the reference ¼ and the reference *whole* as anchors for adding fractions. Zunino (1995) illustrates how children in the 3rd grade of elementary school successfully work out the adding of fractions in situations that do not require the use of conventional fraction representations (a/b).

It is interesting to note that research studies on complex concepts like proportion and probability have evidenced that 7-year-old children utilise the *half* reference as an anchor to solve tasks involving these concepts (e.g., Spinillo, 1996; 2002; Spinillo & Bryant, 1991, 1999). As verified by Spinillo (1995), children are even capable of learning proportion by way of a specific intervention directed toward the use of estimates and the systematic use of the reference *half*. The intervention offered allows the child to transfer the use of this reference to other analogous, albeit distinct situations; as well as helping to overcome many of the difficulties identified prior to the intervention. It was concluded that children could be taught to make proportional judgements, with *half* being an important reference in helping to deal with quantities and with relations crucial to proportional reasoning. These studies show that children can successfully perform activities that involve complex mathematical concepts when reference points are offered that serve as anchors for reasoning. The results of these studies find support in the perspective defended by Sowder (1995) that the use of anchors helps children to develop a numerical sense in regards to a variety of mathematical concepts. As Nunes and Bryant (1996) have suggested, the initial comprehension of the concept of *half* also favours the establishment of connections between both extensive and intensive aspects of rational numbers, and can be considered an important reference for children to initiate the quantification of fractions. Recently, Singer-Freeman and Goswami (2001), in investigating children's ability to establish equivalence between continuous and discontinuous quantities, observed a greater success with problems involving the fraction *half* than with those involving the fractions ¼ or ¾.

Considering the importance of the reference *half* in performing tasks that involve proportion and probability, it can be supposed that this reference is also used on the part of children in performing operations of adding fractions. This possibility is examined in the present investigation, contrasting two situations in the solution of adding fractions: one in which the reference *half* is presented as an anchor during the solution process; and another in which other references are offered. The main objective of this study was to investigate whether children are able to solve problems of adding fractions by way of *half,* seeking to analyse the strategies adopted by them. The hypothesis is that *half* serves as an anchor for reasoning, helping children to successfully perform the addition of fractions, which is something that would not occur in regards to the other references made available during the solution process.

METHOD

Participants

Forty-two middle-class children from the 2nd grade (mean age: 8yrs 4m) and 3rd grade (mean age: 9yrs 3m) of elementary school in the city of Recife, Brazil. None of the participants had been instructed previously at school in adding fractions.

Material

Four differently coloured circles cut from cards and representing cakes that were sliced in different fashions: the strawberry cake was divided into two equal parts; the lemon cake was divided into three equal parts; the vanilla cake was divided into four equal parts; and the chocolate cake was divided into six equal parts.

Six addition of fractions presented through the slices of cardboard cake so that each slice represented one part of the operation. The additions presented were: (a) ¼ + ¼; (b) 1/3 + 1/6; (c) 1/6 + 1/6 + 1/6; (d) 1/3 + 1/6 + ¼ + ¼; (e) 1/3 + 1/3 + 1/3; e (f) 1/3 + 1/3 + 1/3.

Fractional units (1/2, 1/3, ¼ and 1/6) on cardboard that correspond to the slices of each cake, utilised as anchors in solving the addition problems.

Procedure

The children were individually asked to solve the same six addition of fractions problems under two conditions: Condition 1 – with the reference *half* (1/2); and Condition 2 – without this reference, using other references (1/3, ¼ and 1/6). The examiner provided the following story context:

"Pedro and Arthur (show the figures of the two boys) are brothers and they like cake very much. One day, their mother made a strawberry cake, a chocolate cake, a vanilla cake, and a lemon cake. She sliced each cake in a different way, telling them that if they wanted to eat the cakes, they would have to eat the slices in the sizes that she had cut. The cakes were cut in this way: the strawberry cake was cut in two equal parts; the vanilla cake was cut in four equal parts; the lemon cake was cut in three equal parts; and the chocolate cake was cut in six equal parts (show each cake and how it was sliced). Arthur ate pieces of cake of different flavours at the same time, while Pedro ate only one flavour at a time, but Pedro always wanted to eat the same amount of cake that Arthur ate."

At each item, the examiner asked the children to solve the addition problem using the reference fraction that was given him/her (½ , 1/3, ¼ or 1/6).

Example of *Condition 1* (with the *half* reference), Item: 1/3 + 1/6:

"One day, Arthur ate a slice of lemon cake (1/3) and a slice of chocolate cake (1/6). Pedro wanted to each the same amount of cake, but he only wanted to eat strawberry cake (1/2). In order to eat the same amount of cake that Arthur ate, how many slices of strawberry cake does Pedro have to eat? Tell me how you solved the problem".

RESULTS

The Wilcoxon Test revealed that the performance under Condition 1 (with *half*) was significantly better that that under Condition 2 (without *half*) ($Z= - 4.8982$, $p= .0000$), as displayed in Table 1.

Grade	Condition 1 (with half)	Condition 2 (without half)	Total
2nd	74.6	42.8	58.7
3rd	92	52.4	72.2
Total	83.3	47.6	65.5

Table 1: Percentage of correct responses per grade under each condition.

This same pattern of results is observed in relation to each grade (Wilcoxon: 2nd grade: $Z= -3.2900$, $p= .0010$; and 3rd grade: $Z= -3.6214$, $p= .0003$). However, the performance between grades does not differ under Condition 1 (with half) ($Z= -1.9505$, $p= .0511$) nor under Condition 2 (without half) ($Z= -1.0345$, $p= .3009$). This occurred because under Condition 1 the children in both grades presented equally good performances, while under Condition 2 the performance was equally low.

Strategies

From the analysis of the protocols of each child, five types of strategies were identified. They are described and exemplified below:

Strategy I (size comparison): the child compares the size of the slices that constitute the portions of the operation and the reference slice to perform the additions. Examples:

Item 1/6 + 1/6 + 1/6 (chocolate cake) Fractional unit of reference: ¼ (vanilla cake)

Child: (Place ¼ over 1/6) Two.

Interviewer: Why?

Child: Because the chocolate slice is smaller than the vanilla one.

Strategy II (absolute number of slices): the child reasons in absolute terms, giving the same number of slices present in the operation as an answer. Examples:

Item 1/3 + 1/3 + 1/3 (lemon cake) Fractional unit of reference: ½ (strawberry cake)

Child: Three.

Interviewer: Why?

Child: Because he ate three here (1/3 + 1/3 + 1/3) and three more here (1/2 + 1/2 + 1/2).

Interviewer: And is that the same amount of cake?

Child: Yes, three and three.

Strategy III (inadequate composition and decomposition): compositions and decompositions based on juxtapositions of the reference slices on the operation slices. The child becomes confused and cannot establish all the necessary equivalencies by way of simple compensations. Examples:

Item ¼ + ¼ + ¼ + ¼ (vanilla cake) Fractional unit of reference: 1/3 (lemon cake)

Child: (Put together 1/3 + 1/3. Only two. Because this one (1/3) is the same as these (¼ + ¼) and this one (referring to the other 1/3 slice) is the same as these (¼ + ¼).

Strategy IV (global): the child, in a global manner, seeks to determine if the slices present in the operation form either an entire cake or half of a cake. Upon determining this, the child tries to form either an entire cake or half of a cake with the reference slices. Examples:

Item 1/3 + 1/6 + ¼ + ¼ (slices from diverse cakes) Fractional unit of reference: 1/2 (strawberry cake)

Child: This would be a whole one (referring to the slices from diverse cakes). It would be two slices of strawberry to make a whole cake too.

Strategy V (adequate composition and decomposition): the child is able to establish the necessary equivalencies, making appropriate compositions and decompositions from groupings of the operation slices and groupings of the reference slices. Examples:

Item 1/3 + 1/6 + ¼ + ¼ (slices from diverse cakes) Fractional unit of reference: 1/2 (strawberry cake)

Child: One whole cake. Two slices. (Why?) Because this one (referring to one of these slices of the unit of reference: 1/2) is the same as two of these (referring to the slices of vanilla cake: ¼ + ¼). Plus this one (referring to the slice of chocolate cake: 1/6) completes this other one (referring to the 1/3 slice, which together with the 1/6 slice forms ½).

Item 1/3 + 1/6 (slices of lemon and chocolate cake, respectively) Fractional unit of reference: ¼ (vanilla cake)

Child: Three. Because this one (referring to the slice of chocolate cake: 1/6) is the same as this one here (referring to the slice of the unit of reference: ¼). And this other one (referring to the slice of lemon cake: 1/3) needs two.

These strategies express increasing levels of sophistication going from Strategy I to Strategy IV and Strategy V. The distribution of strategies is presented in Table 2.

Strategies	**2nd Grade**	**3rd Grade**
I (n=51)	51	49
II (n=50)	100	0
III (n=108)	47.2	52.8
IV (n=117)	42	58
V (178)	42.7	57.3

Table 2: Percentage of strategies per grade.

Strategy I: size comparison; Strategy II: absolute quantity of slices; Strategy III: inadequate attempts at composition and decomposition; Strategy IV: global; Strategy V: adequate composition and decomposition.

The Kolmogorov-Smirnov Test just detected significant differences between grades in relation to the use of Strategy II ($Z = 1.697$, $p = .006$), which was only used by the children in the 3rd grade.

The relations between the conditions and the types of strategies adopted were examined by way of the Wilcoxon Test (Table 3).

Strategies	**Condition 1 (with half)**	**Condition 2 (without half)**
I (n=51)	39.2	60.8
II (n=50)	38	62
III (n=108)	10.2	89.8
IV (n=117)	65	35

V (n=178)	70.8	29.2

Table 3: Percentage of strategies per condition.

Significant differences were detected in relation to Strategy III ($Z = -4.7821$, $p = .0000$), Strategy IV ($Z = -3.0239$, $p = .0025$) and Strategy V ($Z = -4.2231$, $p = .0000$). These differences occurred because Strategy III was more frequently adopted under Condition 2 (without half) than under Condition 1 (with half), whereas Strategies IV and V were more frequent under Condition 1 than Condition 2.

It was further observed that the distribution of the number of correct responses varied according to the type of strategy (Table 4).

Strategies	Correct response	Incorrect response
I (n=51)	56.9	43.1
II (n=50)	6	94
III (n=108)	18.5	81.5
IV (n=117)	89.7	10.3
V (n=178)	97.2	2.8

Table 4: Percentage of correct and incorrect responses per type of strategy.

The Wilcoxon Test identified significant differences between correct and incorrect responses in relation to all the strategies (with the exception of Strategy 1). It was verified that Strategies II and III were more frequent in items answered incorrectly than in items answered correctly. The opposite was observed in relation to Strategies IV and V, which were more frequent in items answered correctly than incorrectly. It seems that the more elaborate strategies were associated to items that were answered correctly.

DISCUSSION AND CONCLUSIONS

The data of this study show that children who have not yet been instructed in addition at school successfully add fractions when the reference *half* is offered as an anchor during the solution process, helping in the establishment of equivalencies. In utilising this reference, children not only exhibit better performances, but also adopt more elaborate strategies expressing equivalency schemas that are relevant to the comprehension of adding fractions. These schemas are related to the ability to synthesise units in order to generate a further unit, as for example, identifying that the sum of ¼ + ¼ equals half. The compositions and decompositions established (Strategies IV and V) indicate a comprehension, albeit intuitive, concerning adding fractions, with the reference *half* playing an important role in this comprehension.

The main point to be underlined here is that beyond the limits of children's thinking there are possibilities that need to be explored as much in regards to the psychology of cognitive development as in regards to classroom practices. The use of anchors in mathematical reasoning in general, and the comprehension of fraction arithmetic in particular, is an example of this.

The use of the reference *half* seems to play a facilitating role in the comprehension of adding fractions. This result is important in terms of mathematics education on the elementary school level. As such, it would be relevant to conduct an intervention study in which children were encouraged to use the reference *half* in solving problems of adding fractions. Besides exploring the use of this reference, children could be asked to reflect on the different ways to reason and add fractions. An interesting classroom situation would be to ask students to evaluate the inadequacy of their answers and mistakes made when solving addition problems by way of numeric calculations with paper and pencil (adding numerators and adding denominators), contrasting this form of problem solving with that which they adopt when solving the same addition presented in Condition 1 of this study.

The reference *half* opens new possibilities to be considered in terms of what other types of anchors could facilitate children's comprehension of adding fractions. We may suppose the reference *whole* is also a powerful anchor. This possibility should be examined in future research studies.

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