THE ROLE OF NUMBER IN PROPORTIONAL REASONING: A PROSPECTIVE TEACHER'S UNDERSTANDING

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We examine a prospective high school teacher's instructional representations of rate of change and right triangle trigonometry to investigate his interpretation and understanding in relation to the development of proportional reasoning. Despite a constant effort by the subject to resort to "real life" examples in order to give meaning to his teaching, as well as a fairly good understanding of how to connect the topic of instruction with the use of proportional reasoning, it occurs to us that the map of his conceptual moves is from ratio, to comparisons, to fractions. Once he enters the abstract world of fractional expressions and quotients, he often displays difficulties in re-connecting his ideas back to ratio.

INTRODUCTION AND FOCUS

Emphasis has recently been put on the importance of proportional reasoning in understanding topics across the high school curriculum. Also, it is necessary for middle grade students to make connections between ratios and fractions and then to use these connections appropriately to solve problems in other proportional reasoning topics (Carraher, 1996; Sowder, Armstrong, et al., 1998). Research shows that not only is proportional reasoning at the core of the mathematics curriculum, but also it is a good indicator of higher mathematical achievement. Although a lot of research has been conducted on students' understanding of proportional reasoning, very few publications actually focus on teachers' conceptual understanding of ratios and proportions, especially at a high school level. However our belief that quality teaching is directly related to subject matter knowledge (Ball, Lubienski, & Mewborn, 2001) strongly justifies the need for studying and enhancing the growth of understanding of proportional reasoning topics among teachers.

In this article we take a look at a prospective teacher's lesson plans on rate of change and right triangle trigonometry in the light of his beliefs of ratios and fractions. We focus on his ability to connect these topics to proportional reasoning (PR) concepts and argue that the mapping of his growth of understanding of ratio and ratio-related topics within the linking model developed by Clark, Berenson and Cavey for evaluating PR (Clark, Berenson, & Cavey, 2003) reflects on his instructional representations. Our initial interest generally focused on a prospective teacher's understanding of proportional reasoning topics and to find patterns in his thinking about teaching PR related topics.

CONCEPTUAL FRAMEWORK

For this research we rely heavily on the Linking Model of Ratio and Fraction proposed by Clark, Berenson, and Cavey (2003) that we summarize here for the purpose of this paper. The intent of building this model was to understand how people, in particular teachers and middle school students, connect ratios and fractions. During their studies the researchers encountered several models of thinking about the relationship between ratios and fractions. Some people saw all ratios as being fractions, hence consider ratios as a subset of the fraction world; others see all fractions as ratios, inversing the previous conclusion. Others see fractions and ratios as two completely different concepts with no relationship, while there are others who consider they are the same. However none of these four models seemed to explain the complexity of the relationship between ratios and fractions, hence the researchers came up with a fifth model, called the Linking Model of Ratio and Fraction, a representation of which is given below (fig.1). The idea is that not all ratios are fractions and not all fractions are ratios, but the two concepts do share an intersection where ratios and fractions can be treated the same. In the ratio-only sub-construct one would find any ratios and rates in a non-fractional form, as well as expressions such as “1 cup sugar : 2 cups flour”. The same example would be expressed “1 cup sugar/3 cups ingredients” in the intersection sub-construct, and “1/3 cup sugar” in the fraction-only region. In the intersection one also finds probabilities and comparisons part-part, part-whole, while the fraction-only region includes percentages and decimal expressions as well as operations or points on the number line.

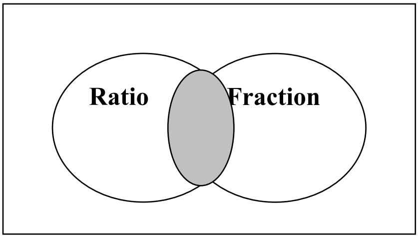

Figure 1. Linking Model of Ratio and Fraction (Clark, Berenson, Cavey, 2003)

METHODOLOGY

The material available for this study are archived videotapes and transcripts collected during a first methods course offered for pre-service teachers in their sophomore year. The purpose of the course was to increase pre-service teachers’ subject matter knowledge and understanding of proportional reasoning topics through teaching and lesson planning. The subject Brian, a former chemical engineer student with a GPA above 2.5, was interviewed twice early in the semester in order to get an idea of his beliefs and understanding concerning the PR topics. Then the student planned a lesson with the help of textbooks, trying to relate the topic of instruction to ratio and proportion. Finally he explained the lesson to the interviewer. Six weeks after the initial interview and after some group work with other peers on the same topics, Brian was also asked to plan two other lessons for his final grade on both rate of change and right triangle trigonometry. For this article we worked with one specific student's transcripts and followed Maher's methodology suggestions for analyzing videotape data (Powell, Francisco, & Maher, 2003). The main attempt at using this method expressed itself in coding the appearances of numerical representations in Brian's talking and lesson planning. More specifically we focused on where to locate Brian's thinking within the Linking Model of Ratio and Fraction, be it on the right hand side of the model (coded RHS), in the ratio-only sub-construct, on the left hand side of the model (coded LHS), i.e. the fraction-only sub-construct, or in the intersection region of the model (coded IR). The parts of material that had been coded RHS were also studied carefully to emphasize the nature of the numerical representation used, be it a chart, a decimal expression, a number operation, a fractional expression or other representation. Furthermore we looked for critical events in Brian's transcripts, places where he seemed to come to a higher understanding or suddenly make or clarify a connection that appeared to be missing before.

ANALYSIS/RESULTS

Each interview focused on three different points of view: *What do you remember about being taught and the time you were learning this concept?* This question helped us understand how much of the preceding instruction received by the teacher was affecting his own teaching and understanding. *What does this concept mean to you?* We focused on Brian's own interpretation and representations of each concept, a place where personal beliefs and emotional engagement can affect the understanding. *How would you teach this concept?* This question is actually composed of two parts; first Brian was asked how he would teach it to a peer, and then how he would introduce it in a middle grade classroom.

The first interview we consider is rate of change. Before specifically asking questions about rate of change the interviewer tries to reveal Brian's ideas on ratio and proportion. When asked to describe the term ratio in his own terms Brian (B) chooses to define it as "how one quantity is related to another".

B Well, lets keep it simple, so say we're just throwing out a fraction, two-thirds describing roughly sixty-six percent of one. Two-thirds of something. Now at the same time we can take four-sixths. Now they have two different numbers, two and four are obviously different. Three and six are obviously different. But the relationship between two and three is… two over three, we'll just divide that out to see what we get. Three will go into twenty, what's that, about six times? We'll say eighteen, twenty, six feet. Let's come over here. Six will go into forty, four, nine. That's six feet also. Even though these numbers are different we can consider this point six repeating to be the ratios, and match those, they are exactly the same. So different numbers, different sizes, same ratio.

Interpretation: The choice of a fraction for illustrating the concept of ratio belongs to the right hand side of the model we chose for our conceptual framework, and the dexterity with which Brian plays with the fractions indicates a possible comfort zone in this abstract representation of ratio where well trodden techniques give the right results, especially the instant need to divide and find a decimal value for the corresponding fractions. The latter also illustrates the power of equalities in mathematics and the comfort they provide to a problem solver. A closer look at the videotape and of Brian's engagement in the calculations suggests that he feels extremely comfortable with these manipulations. After this first introduction of ratio with two similar fractions, Brian underlines the importance of introducing real life examples for a better understanding of mathematical concepts. Among these examples he chooses to focus on dimensions, be it dimensions of similar triangles, dimensions of a table, another evidence of his familiarity with the right hand side of the Linking Model where measurements belong.

B We're going to have to maintain this three to two ratio with the base and the height. In order to do that we're like, OK, we want to keep a similar triangle and we want it to have a height of fifteen. We'll draw this, not to scale, and we know we want it to be fifteen high. How big of a base do we need? Taking it out to the real world, how much concrete do you need? So let's look at this ratio here, it's a three to two ratio. Let me show you a trick. Going back to this two-thirds and four-sixths, you can take, lets' bring it down here, well let's work left to right, you can factor out two out of both sides. If you notice right here, you have the exact same thing, which is six. Just factor this down to two-thirds and two-thirds. Bring it out on this side, and we've got three over two over here, and then we've got fifteen over something over here.

I What are you going to call the something?

B Let's call it x. We'll label this other triangle x. Fifteen, can we make that, we can take a three out of there. So let's take a three out, and we've got five. Now, since we took three out of five… we'll let's do this, this will still be x down here. Wait a minute, just like on this side we factored a common thing out and we ended up with the same thing as this. Let's come back over here, factor the five out of here, and since we did that, this isn't going to be an x anymore. We'll call it a y. Five times y equal x. Factor the two out, two-thirds over here. Factor the five out; we're going to have three-halves right here, just like we did on the other side. So now we know y equals two. Now we know five y equals x, which is the base of our triangle. Let's say five time y, which is five time two, ten. So now we know the base of this triangle is ten. Now we've got similar triangles, completely different sizes, but the ratio is the same.

Interpretation: Once the problem is set about what needs to be done (namely finding the dimension of a similar triangle with a specific height), Brian is very prompt in moving to numerical manipulations again. He shows excitement about giving the interviewer a "trick" to solve and simplify the problem, reducing it to merely finding a common factor for the numerator and denominator, barely mentions the existence of units in the action of measuring, and resorts to the symbolic variables x and y, making the problem more abstract than it was in the first place. Obviously the visual aspect of proportions in this example is lost through the solution given by Brian. Similarly, when asked how to introduce the concept of ratio to middle school students, Brian's first intuition about answering the question is to resort to operations (he mentions division) and numbers without getting into giving a sense of proportions outside of a purely mathematical world. This triggered the final question from the interviewer concerning Brian's beliefs about the connections between ratios and fractions:

I Okay. How about the relationship between ratio and fraction? How would you describe that?

B Well, I guess I would describe fractions as a useful tool to work with ratios. You know, even if you have so many monkeys will fit into so many barrels, you can put it in fraction form. It's going to be so many monkeys over so many barrels. […] I'd just use it as a good tool.

I As a tool. Okay. Well do you see them the same, or do you see them differently?

B Fractions and ratios?

I Uh-huh. If you don't see them differently that's okay.

B Yeah. Well you can put basically anything in a fraction form. Like when you think of fractions, two-thirds is not the same as ten monkeys in two barrels, but we use the same thing, which basically is just a line with one thing over it and one thing under it. When you think of fraction, the name itself implies a fraction of something, a part of something, where as a ratio doesn't have to be like that.

<u>Interpretation:</u> It becomes clear here that Brian's model for thinking about the relationship between ratios and fractions is one where all fractions are ratios. Fractions being a tool, he naturally resorts to them any time he has to deal with ratios, and to a further extent with proportions. Within the "workshop" where he handles fractions, he also has access to a collection of tricks and techniques that he will mention on several occasions throughout the interview, usually becoming very verbal during these occurrences and sometimes expressing the pleasure he gets out of performing these manipulations.

Now that we have an idea of how Brian might perceive proportions, ratios, fractions, and how these concepts relate with each other, let us focus on Brian's understanding of teaching the concept of rate of change. To him rate of change represents "the amount something changes in a given time", or "something per something" such as miles per hour. When asked to show a graph representing velocity he decides to assign speed to the y-axis and does not use the graph to determine which distance he would have gone had he been driving fifty miles per hour for two hours. Instead he goes back to the following numerical manipulation:

B Let's just do this; we'll put our velocity over here, n miles per hour. You can be going ten, twenty, thirty, forty fifty, two billion sixty. We'll call this one hour, and two hours.

Now if we're going fifty miles per hour, steadily as if you have the cruise on going to the beach. Since you have the cruise on, all the time down here that you're driving is fifty miles per hour. So our one, and our half, and hour and a half, you're always going fifty. So constant speed is represented fifty going across here. Now if you want to, say how far have I gone after two hours? Two hours is right here and we've been going fifty… now watch this, velocity on this side and time over here, miles per hours and hours over here. So we know, using a little dimensional analysis, miles per hour and you're going to times that by hours. Your hours are going to cancel out and you're left with miles. So how far have I gone after traveling two hours at fifty miles per hour? Let's say fifty miles per hour times two hours, remember hours cancel out, we'll get a hundred miles.

<u>Interpretation:</u> In this sequence Brian is not completely located in the right hand side of the model. Instead he pays attention to keeping in mind the goal of his calculations and in particular the units to help him solve his problem. The interviewer then asks to see the graph of distance over time and this triggers in Brian a sudden reaction where one sees him make the connection between rate of change and ratio and proportion, what we would call a critical event. Once Brian starts playing with the graph and actually looking at its shape, the visual representation of ratio and similarity helps him connect the notion of a linear graph, in other words constant rate of change, to the notion of similar triangles and constant ratio between distance and time. We noted how the visual allowed the occurrence of this critical event in Brian's understanding when he appeared to be fairly enclosed in his numerical manipulations earlier on and lacked the overview needed for him to see an obvious connection between different mathematical concepts. This confirms the idea that using several representations that call for different sensory perceptions helps the understanding of a concept, as well as the intertwining of different concepts within one field. We selected a few critical events related to our own understanding of Brian's thinking this time in the preliminary interview on right triangle trigonometry. Again the interview was following the interviewer's desire to understand what Brian learned about the subject, how he perceived it from his personal belief, and how he would then teach it. From what Brian remembers being taught we find a strong association with the numerical representation of a table of values for the trigonometric functions, with little association to the unit circle or the triangle ratio formulas. However his first association with trigonometry is the unit circle:

B Is the biggest thing that comes to my mind. I remember not really understanding the unit circle. Well, I think I may have. It's hard to remember what I did and what I didn't understand at certain points. It's like we'd have a circle and we'd talk about going, like a point going around the circle, and then you could lay the circle out along the x-axis, say it's wound up a whole bunch.

I Oh, okay. You could take the entire circle and sort of stretch it out along the x-axis.

B Of course it repeats, if you have a big chord of it. Now that I'm actually… I don't know.

I Was that an association that helped you put it all together, the unit circle, or was it a source of confusion at that point?

B I'm not sure it helped much. I eventually did understand. The thing that helped me the most was a chart. We had a chart that I memorized easily. On this side we had the degree, well actually degrees on this side, the most left, then radians, and then like thirty, sixty, ninety. Then we'd have radians and everything that corresponded to it. I had this chart memorized and I think that eventually helped me place things on the unit circle. Like I could see thirty degrees, forty-five, and then know that it keeps on going. Like for ninety, thirty is going to be the same, then in certain areas, and graph like with the sine for instance.

I So once you learned the numbers in the table, sort of by memory, that helped you to understand the concepts that the unit circle was getting at.

I How did you commit all this to memory, that's a lot of numbers, and a lot of things that are close?

B Well it's an easy pattern. Once you get to figuring out that… this is you know radians. I can't really remember radians, the measures (Ahah), but once you figure out that like for sine thirty is going to be the sine of one fifty, and understanding things and what they actually meant.

<u>Interpretation:</u> What seems striking here is that the unit circle is remembered as being stretched out on the real number axis, hence losing its geometrical properties of symmetry, which ties into Brian's inability to remember radians. His approach to trigonometric measures and formulas remains in the counting world and is very sequenced and based on repetition, as opposed to the splitting world that one can associate with the parting of the circle into radians. It then makes sense that mastering the table of values enabled him to get a better picture of the unit circle. Furthermore trigonometric functions are certainly not understood within the context of ratio through this approach, but instead are seen as a list of values, another very different way to represent functions. Brian's entrapment in a number world and the counting scheme is definitely clear here Our analysis of his entrapment into a purely numerical world where sense disappear to leave place to techniques and tools might account for this lack of understanding and visual representation.

CONCLUSION

The comfort zone of number, which Brian seems to heavily rely on, suggests that techniques and number operations are given higher priority than concept clarity. It may also throw students in an abstract and technical world where connections between concepts are harder to make but where the correct answer is found. Brian finds himself resorting to number operations, charts or formulae. He admits seeing fractions as a useful tool to work with ratio hence belongs to model two from Berenson, Clark and Cavey (Clark, Berenson, Cavey, 2003). What is interesting to us is that once he enters the right side of the diagram where fractions belong, he seems trapped there and it becomes very hard for him to step out again. There is the perpetual need for the use of different representations to achieve full understanding of a concept. In our lesson plan context for Rate of Change and Right Triangle Trigonometry the visual representations of graphs or of similar triangles and extended discussion on these seemed to facilitate the learning of a concept more than number-based formulas.

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