MOLLY AND EQUATIONS IN A_2: A CASE STUDY OF APPREHENDING STRUCTURE

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This paper explores one student's attempt to apprehend an abstract mathematical structure (similar to Z_{99}). We discuss Karmiloff-Smith's theory of representational redescription as a model for the development of structural understanding and contrast this with existing process-object theories. We use two cycles in Molly's movement from an action conception of the teacher-given aspects of the structure, inherent in the definition, to her conscious and expressible personal ownership of aspects of the structure, to explore how the model helps us account for structural understanding.

BACKGROUND

There is a well discussed difference between understanding, say, a vector as an action of moving from one location to another and a vector as an object. However, to understand a vector space is, we suggest, a quite different matter. In this paper, we use one girl's work to illustrate the notion of *apprehending structure*: developing a conscious and expressible sense of the relationships within, and properties of, an abstract, defined mathematical structure.

THEORETICAL FOUNDATIONS

The idea that a learner's conception of a piece of mathematics changes its meaning as the learner develops is hardly new. There is a wide range of literature which explores how actions metamorphose into objects for learners (Sfard, 1991; Dubinsky, 1991; Gray & Tall, 1994) Implicit in much of this literature is the change of internal representation which takes place. In the case of APOS theory:

> An *action* is any physical or mental transformation of objects to obtain other objects. It occurs as a reaction to stimuli that the individual perceives as external. … When the individual reflects on an action, he or she may begin to establish conscious control over it. We would then say that the action is interiorized, and it becomes a process.
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> A *process* is a transformation of an object (or objects) that has the important characteristic that the individual is in control of the transformation … As the individual reflects on the act of transforming processes, they begin to become objects. (Cottrill et al., 1996, p 171)

In Sfard's theory:

> At the stage of interiorization a learner gets acquainted with the processes which will eventually give rise to a new concept. … The phase of condensation is a period of

> ‘squeezing’ lengthy sequences into more manageable units. … Only when a person becomes capable of conceiving the notion as a fully fledged object, we shall say that the concept has been reified. *Reification,* therefore, is defined as an ontological shift – a sudden ability to see something familiar in a totally new light
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> (Sfard, 1991, pp 18–19)

However, in many areas of mathematics the learner’s appreciation of mathematics as consisting of particular *objects* does not do sufficient justice to the complexity of the situation. Seeing a vector as (say) both the action of moving and an object that can be added to (or even – in differing senses – multiplied with) other vectors is clearly one level of understanding. Apprehending the structure of a vector space is another. APOS theory acknowledges this difficulty by appending the notion of a *schema* to the original action-process-object theory:

> A schema is a coherent collection of actions, processes, objects, and other schemas that are linked in some way and brought to bear on a problem situation. As with processes, an individual can reflect on a schema and transform it. This can result in the schema becoming a new object. (Cottrill et. al., 1996, p 172)

While these theories implicitly acknowledge a structural aspect, they focus on the formation of individual objects. However, switching the figure and ground, to focus on the structure within which these objects lie, enables us to examine different aspects of the learner’s development, particularly in abstract, defined mathematics. The process-object duality tends to model better objects described rather than objects defined (Tall et. al., 2000).

Karmiloff-Smith (1992) provides a theory which seems particularly suited to explaining the development of mathematical structure. Moreover, rather than seeing this as the development of an internal representation that matches a given mathematical structure, ‘representational redescription’ provides a way of explaining how mathematical structure can be the consequence of active mental construction (in the sense of Cobb, Yackel & Wood, 1992) and shows how learners in different phases of redescription can interact with their mathematical environment (and their teachers) differently.

REPRESENTATIONAL REDESCRIPTION IN MATHEMATICS

Representational redescription is a movement from the learner’s conception of a set of atomic behaviours which are externally stimulated, to apprehending structure in which the nature and properties of the structure and the relationships within it are consciously available for communication. It is a phase theory in which

1. Initially information about the structure is encoded as separately stored procedures, with no intra- or inter-domain connections. At this phase, the learner might appear to have a set of actions on pre-existing objects (in the sense of Dubinsky, 1991).
2. These actions become internally redescribed. This redescription is an act of abstraction which retains only some of the aspects of the full procedures (it is

an internal *description* of the procedure, no longer the procedure itself). At this phase, such a description is unconscious, but may manifest itself in the wise choice of objects in the structure with which to work.

3. The learner is able to consciously access the redescription of the procedures, so that they have an appreciation of the relationships within the structure sufficient to guide them in solving structural problems. However, they may not be able to verbalise or symbolically express these relationships. Learners may, for example, note similarities or relationships between objects in the structure, but might not be able to articulate the nature of the similarity or relationship.
4. The final phase is the ability to communicate directly about the relationships and properties of the structure.

This movement is accompanied by different ways in which the learner can engage with the material and the teacher. At the earliest phases, the focus can only be on whether a procedure is being followed correctly while the later phases enable the learner to talk about the structure in it's entirety. A beautiful illustration of such structural development and the nature of the communication about the structure can be seen in Maher and Speiser's (1997) discussion of a learner's development of binomial (and multinomial) structures from its beginnings in investigating building towers from different coloured blocks.

At the earliest phases, there can be no notion of proof, except (perhaps at phases 2 or 3) as generalized calculation. At the last phase, the learner has access to the properties of the apprehended structure, so arguing from those properties is possible. Thus, in terms of Harel and Sowder's proof schemes, there is a movement from empirical to analytic schemes (Harel & Sowder, 1998).

CONTEXT OF THE RESEARCH

We introduced students to what we called a *restricted arithmetic*: a structure $A_2 = (\underline{A}_2, \oplus, \otimes)$ where $\underline{A}_2 = \{1,2,3,...,99\}$ and the binary operators $\oplus$, $\otimes$ are defined in terms of a *reduction* mapping r, r: $\mathbf{N} \to \mathbf{N}$, $r(n) = n - 99 \cdot [n/99]$, (where $[y]$ is the integer part of y). This reduction was introduced as an instruction, illustrated by several concrete examples:

> For a natural number $n < 100$, $r(n) = n$. If $n \geq 100$, we split n into pairs of digits starting from the units digit and add the pairs together. We repeat the procedure until we get an element of $\underline{A}_2$. For example, $r(682) = r(82 + 6) = 88$, $r(7\ 945) = r(45 + 79) = r(124) =$ $= r(24 + 1) = 25$.

Binary operators addition $\oplus$ and multiplication $\otimes$ in $\underline{A}_2$ were defined and illustrated as follows:

$\forall\, x, y \in \underline{A}_2,\ x \oplus y = r(x + y)$ and $x \otimes y = r(x \cdot y)$.

For instance, $72 \oplus 95 = r(167) = 68$, $72 \otimes 95 = r(6\ 840) = 9$.

Many different problems can be solved in the A_2 system, among them additive and multiplicative equations of the type $a \oplus x = b$, $c \otimes x = d$, where elements *a, b, c, d* are parameters and x is the unknown.

Molly, a 20-year-old student training to be a mathematics teacher, along with 11 other students, was introduced to the A_2 structure in the way given above. Unusually, Molly became sufficiently involved in trying to develop her own understanding of A_2 that she quickly developed her own problems in the system and even got to the stage of writing her diploma thesis on the topic (Stehlíková, 2002). In this paper, we will examine two cycles in her attempts to apprehend the structure of A_2. In the first – her work on additive equations – we see a rapid movement through the four phases, while in the second – multiplicative equations – we see the same movement slowed down as she encounters more difficulties and a more intricate structure to apprehend.

Data were collected from numerous sources: transcripts of clinical interviews (in the sense of Ginsburg, 1981), Molly's written work produced both for the interviews and independently, from field notes and from Molly's development of a journal describing A_2. These data were analysed using methods adapted from grounded theory (Glaser & Strauss, 1967) and, within the area of additive and multiplicative equations, four categories (seen later as phases of apprehending structure) emerged.

MOLLY'S APPREHENDING OF EQUATIONS IN A_2

Molly's investigation of additive and multiplication equations was mainly triggered by two experimenter's interventions: first, Molly was presented with a list of several additive and multiplicative equations which she could solve in any order and second, she was asked to classify these equations according to their solubility. As this classification is of a different character and difficulty for additive and multiplicative equations, we will examine them separately.

The investigation of additive equations was apparently straightforward for Molly and the phases below followed on quickly from each another.

1. An initial procedure. Molly's first attempts involved treating the elements of $\underline{A}_2$ as if they were elements of **N**. This worked well for a small sub-class of problems, such as $x \oplus 6 = 92$ ($x = 92 - 6 = 86$). However, when she noted this procedure failing, after a short pause, she came up with a strategy (which we call the *strategy of inverse reduction* or *SIR*) based on her procedural understanding of the reduction function: $61 \oplus x = 4 = r(202) = r(103)$, hence $x = 42$

2. Development of a procedure in A_2. The SIR procedure enabled her to solve any additive equation, though she was able to move between the procedure adapted from **N** and using SIR fluently and appropriately.

3. Properties of A_2. It became clear, after a while, that (though she was unable to enunciate it) she was implicitly using the idea that each additive equation has just one root (in A_2).

4. Justifying and communicating structure. After some time, Molly was able to justify the unique root claim by referring to inverse reductions of b and their structure and much later when she had defined subtraction in A_2, she was able to justify her result in these terms.

These same four phases can be seen in more detail in Molly's work on multiplicative equations, which was much more involved and which took more cognitive effort on her part.

1. An initial procedure. Again, Molly initially adapted a procedure from her previous experience by treating elements of A_2 as if they were elements of **N** (or **Q**). As we can see in figure 1, she did this for almost all of the multiplicative equations which she was given, which resulted in some solutions which are incorrect in A_2 (such as $x \otimes 8 = 92$, hence $x = 23/2$) and in incomplete solutions (such as $3 \otimes x = 45$, hence $x = 15$). However, when she considered the equation $2 \otimes x = 99$ she did not use this adapted strategy but rather used her knowledge of properties of 99 within the A_2 structure (which she had discovered previously) and concluded that $x = 99$.

$$3 \odot x = 45 \qquad x = 15$$
$$x \odot 8 = 92 \qquad x = \frac{92}{8} = \frac{46}{4} = \frac{23}{2}$$
$$14 \odot x = 91 \qquad x = \frac{91}{14}$$
$$2 \odot x = 99 \qquad \cancel{x} = 99$$

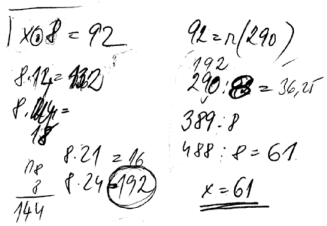
M: So, the first task, so *x* is 15.
Obviously then 2 times *x* is 99, that will be that 99, that is the case of the zero.
Well, now *x* is 92 eights. It could be cancelled 46 quarters… (pause)…it will go on, that is 23 halfs.

Figure 1: Molly's use of procedures from **N**

She later appeared to notice the discrepancy between equations which led to elements of A_2 (or **N**) and those which led to fractions. She used trial-and-error methods, substituting numbers of varying size for x and exploring how this affected the size of numbers on the right hand side.

In fig. 2, we can see that, after a prompt to consider how she had worked with additive equations, she used the SIR procedure for an equation which previously resulted in a fractional answer and succeeded in finding a root in A_2. After this, she went back to her previous fractional solutions and re-solved the problems with a solution in A_2. However, she always stopped after finding the first root, which suggests that her belief that these equations must have at most one root was strong. For instance, when solving the equation $33 \otimes x = 66$, she just gave the answer $x = 2$.

Again, it was only a prompt ("Is that all?") which motivated Molly to try to use the SIR procedure to demonstrate the uniqueness of roots and which finally led to her discovery that there might be multiple roots. She re-examined the previous equations to see if she had found all the roots.

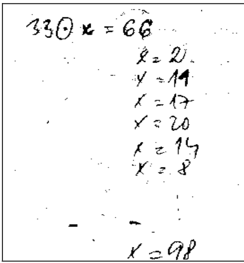


E: Consider the way you solved the equation $61 \oplus x = 4$ last time.

M: So 92 could be written as 290 … (short pause) … so 290 divided by 92, no 8 is … [used calculator] … the next one 389 … also no, 488, it should work, it is 61, which means that the x should be 61.

Figure 2: Developing a different procedure for A_2

2. Development of a procedure in A_2. At this phase, she was able to solve all the multiplicative equations she worked with and could apparently see that there were equations with zero, one or multiple roots. She made some observations such as multiple roots of an equation make an arithmetic sequence and number 3, 9 and 11 "cause problems" (fig. 3) but at this phase she did not explain more than this.



M: So 33 times 20, so 20 is the next root…(short pause) … now 14 … (laughs) (pause) … 8 … (pause) … 5 … (pause) … so it will always differ by 3.
It could be calculated using arithmetic sequence.

Figure 3: Recognising properties of A_2

3. Properties of A_2. As she attempted to classify multiplicative equations, Molly solved many different equations which she posed for herself. While at the beginning she picked them apparently at random, later she decided to explore systematically the equation $c \otimes x = d$ for c up to 16. This led to an important discovery which became the basis of a new solving strategy: if σ is a difference between roots and p is the number of roots, then $\sigma \cdot p = 99$.

She noted that this also holds for equations with one root (then $\sigma = 99$, $p = 1$). She also made some observations such as "We cannot divide the terms of the equations by multiples of 3, 9, 11, 33… by multiples of 3 and 11". While at the beginning she substituted individual elements of A_2 for c, later her investigations involved her implicitly working with sets of numbers at once. For example, without expressing it like this, she began to work with the number 3 as if it was the representative of all multiples of 3.

4. Justifying and communicating structure. Eventually, Molly felt able to summarise all of her knowledge about multiplicative equations in a table (a part of which is reproduced in fig. 4)

p	conditions	example
1	$a \neq 3 \otimes k \wedge a \neq 11 \otimes l$, where $k, l \in \mathbf{A_2}$	$8 \otimes x = 9$
3	$a = 3 \otimes k \wedge a \neq 9 \otimes l \wedge a \neq 11 \otimes m \wedge b = 3 \otimes n$, where $k, l, m, n \in \mathbf{A_2}$	$3 \otimes x = 9$
0	$a = 3 \otimes k \wedge a \neq 9 \otimes l \wedge a \neq 11 \otimes m \wedge b \neq 3 \otimes n$, where $k, l, m, n \in \mathbf{A_2}$	$3 \otimes x = 5$
9	$a = 9 \otimes k \wedge a \neq 11 \otimes l \wedge b = 9 \otimes m$, where $k, l, m \in \mathbf{A_2}$	$18 \otimes x = 9$

Figure 4: Molly's knowledge of multiplicative equations in A_2

She could now articulate the fact that the multiples of 3 and 11 represent, in fact, a set of zero divisors (even though she was not first able to give this set its name, later she called them divisors of 99). She could justify the non-existence of solution by reference to the digit sum ("if d is not divisible by 3 and 11, neither are its inverse reductions $d + k99$, $k \in \mathbf{N}$") and the existence of multiple solutions by reference to divisibility tests she had developed for the A_2 structure.

DISCUSSION

We can see broad similarities in Molly's phases of development in working with additive and multiplicative equations. First she works with elements of A_2 as if they were elements of **N** (using familiar number procedures). She is then able to use her understanding of a fundamental part of the A_2 structure (inverse reduction – which was itself developing alongside the work discussed here) to solve equations which previously were beyond her. At this phase she begins to get a sense of the structure of these equations – the number and nature of the roots – which manifests itself in the choices she makes in developing problems to investigate. Finally she is able to articulate findings about the system and begin to justify them in terms of the properties of the structure.

Both investigations are accompanied by a 'U shape' observable in her fluency in solving these equations. First she seems fluent as she adapts known procedures from **N**, then she realizes that these procedures do not work and becomes much slower as she has to develop the SIR procedure to the situation. As she does so, and as it gives her a sense of the structure of the equations and their roots, she again becomes fluent. However, as Karmiloff-Smith notes, the down-curve of the 'U-shape' "is deterioration at the behavioural level, not at the representational level" (Karmiloff-Smith, 1992). Indeed, the loss of fluency seems to accompany her change in perception of the structure.

We do not see the development of her understanding of these aspects of A_2 as a movement from a process to an object conception – rather, apparently familiar objects (numbers) have taken on new properties and relationships to form a new structure for her. Thus, we suggest that these two example investigations are small scale cycles in representational redescription as Molly moves to apprehending the A_2 structure. She moves from working with procedures adapted from an old structure, to implicitly recognizing some aspects of the new structure, to having (but being unable to articulate) a sense of how the equations work, to – finally – being able to express her findings and begin to justify them in terms of properties.

While beyond the scope of this paper, Molly's perseverance with A_2 shows us the same process of apprehending structure writ large: with a change of perspective, her involvement with additive and multiplicative equations become steps to her apprehending A_2 as a new and fascinating mathematical structure of her own.

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References

Cobb, P., Yackel, E. & Wood, T. (1992). A constructivist alternative to the representational view of mind in mathematics education, *Journal for Research in Mathematics Education*, 23(1), 2–33.

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K. & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a co-ordinated process schema, *Journal of Mathematical Behavior*, 15, 167–192.

Dubinsky, E., (1991), Reflective abstraction in mathematical thinking, in D. Tall (ed.), *Advanced Mathematical Thinking*, Dordrecht: Kluwer.

Ginsburg, H. (1981). The clinical interview in psychological research on Mathematical thinking, *For the Learning of Mathematics*, 1, 4–11.

Glaser, B. & Strauss, A. (1967), *The discovery of grounded theory*: *Strategies of Qualitative Research.* London: Weidenfeld & Nicolson.

Gray, E. & Tall, D. (1994). Duality, Ambiguity and Flexibility: A Proceptual View of Simple Arithmetic, *Journal for Research in Mathematics Education*, 26 (2), 115–141.

Harel, G. & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies, In A. H. Schoenfeld, J. Kaput & E. Dubinsky (Eds.), *Research in collegiate mathematics education III*. Providence, RI: AMS

Karmiloff-Smith, A. (1992). *Beyond Modularity,* Cambridge, MA: MIT Press.

Maher, C. A. & Speiser, M., (1997). How Far Can You Go With Block Towers? Stephanie's Development, *Journal of Mathematical Behavior*, 16(2), 125–132,

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin, *Educational Studies in Mathematics*, 22, 1–36.

Stehlíková, N. (2002). "A case study of a university student's work analysed at three different levels", A.D. Cockburn & E. Nardi (Eds.), *PME 26 proceedings* (vol. 4, pp 241–248), Norwich, UK: University of East Anglia.

Tall, D.O., Davis, G., Thomas, M., Gray, E., & Simpson, A. (2000). "What is the object of the encapsulation of a process", *Journal of Mathematical Behavior*, 18, 2, pp 1–19.