ON LORENTZIAN PARA-SASAKIAN MANIFOLDS

M. TARAFDAR AND A. BHATTACHARYYA

Abstract. The present paper deals with Lorentzian para-Sasakian (briefly \(LP\)-Sasakian) manifolds with conformally flat and quasi conformally flat curvature tensor. It is shown that in both cases, the manifold is locally isometric with a unit sphere \(S^n(1)\). Further it is shown that an \(LP\)-Sasakian manifold with \(R(X,Y).C = 0\) is locally isometric with a unit sphere \(S^n(1)\).

Introduction

In 1989, K. Matsumoto [2] introduced the notion of Lorentzian para Sasakian manifold. I. Mihai and R. Rosca [3] defined the same notion independently and thereafter many authors [4], [5] studied \(LP\)-Sasakian manifolds. In this paper, we investigate \(LP\)-Sasakian manifolds in which

\[(1) \quad C = 0\]

where \(C\) is the Weyl conformal curvature tensor. Then we study \(LP\)-Sasakian manifolds in which

\[(2) \quad \tilde{C} = 0\]

where \(\tilde{C}\) is the quasi conformal curvature tensor. In both the cases, it is shown that an \(LP\)-Sasakian manifold is isometric with a unit sphere \(S^n(1)\). Finally, an \(LP\)-Sasakian manifold with

\[(3) \quad R(X,Y).C = 0\]

has been considered, where \(R(X,Y)\) is considered as a derivation of the tensor algebra at each point of the manifold of tangent vectors, \(X, Y\). It is easy to see that \(R(X,Y).R = 0\) implies \(R(X,Y).C = 0\). So it is meaningful to undertake the study of manifolds satisfying the condition (3). In this paper it is proved that if in a Lorentzian para-Sasakian manifold \((M^n,g)\) \((n > 3)\) the relation (3) holds, then it is locally isometric with a unit sphere \(S^n(1)\). \((n\) has been taken \(> 3\) because it is known that \(C = 0\) when \(n = 3)\).

1. Preliminaries

A differentiable manifold of dimension \(n\) is called Lorentzian para-Sasakian [2], [3] if it admits a \((1,1)\)-tensor field \(\phi\), a contravariant vector field \(\xi\), a covariant
vector field $\eta$ and a Lorentzian metric $g$ which satisfy

\begin{align}
\eta(\xi) &= -1 \\
\phi^2 &= I + \eta(X)\xi \\
g(\phi X, \phi Y) &= g(X, Y) + \eta(X)\eta(Y) \\
g(X, \xi) &= \eta(X), \\
(\nabla_X \phi)Y &= [g(X, Y) + \eta(X)\eta(Y)]\xi + [X + \eta(X)\xi]\eta(Y)
\end{align}

where $\nabla$ denotes the operator of covariant differentiation with respect to the Lorentzian metric $g$.

It can easily be seen that in an LP-Sasakian manifold the following relations hold:

\begin{align}
\phi \xi &= 0 \\
\eta(\phi X) &= 0 \\
\text{rank } \phi &= n - 1.
\end{align}

Also, an LP-Sasakian manifold $M$ is said to be $\eta$-Einstein if its Ricci tensor $S$ is of the form

\begin{align}
S(X, Y) &= a g(X, Y) + b \eta(X)\eta(Y)
\end{align}

for any vector fields $X, Y$ where $a, b$ are functions on $M$.

Further, on such an LP-Sasakian manifold with $(\phi, \eta, \xi, g)$ structure, the following relations hold [4], [5]:

\begin{align}
g(R(X, Y)Z, \xi) &= \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Z) \\
R(\xi, X)Y &= g(X, Y)\xi - \eta(Y)X \\
R(\xi, X)\xi &= X + \eta(X)\xi \\
R(X, Y)\xi &= \eta(Y)X - \eta(X)Y \\
S(X, \xi) &= (n - 1)\eta(X) \\
S(\phi X, \phi Y) &= S(X, Y) + (n - 1)\eta(X)\eta(Y)
\end{align}

for any vector fields $X, Y$ where $R(X, Y)Z$ is the Riemannian curvature tensor.

The above results will be used in the next sections.

2. LP-SASAKIAN MANIFOLDS WITH $C = 0$

The conformal curvature tensor $C$ is defined as

\begin{align}
C(X, Y)Z &= R(X, Y)Z - \frac{1}{n - 2} \{ g(Y, Z)QX - g(X, Z)QY \\
&\quad + S(Y, Z)X - S(X, Z)Y \} + \frac{r}{(n - 1)(n - 2)} \{ g(Y, Z)X - g(X, Z)Y \},
\end{align}

where

\begin{align}
S(X, Y) &= g(QX, Y).
\end{align}
Using (1) we get from (18)
\[
R(X, Y)Z = \frac{1}{n-2}(g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y) - \frac{r}{(n-1)(n-2)}(g(Y, Z)X - g(X, Z)Y).
\]
Taking \(Z = \xi\) in (18) and using (7), (15) and (16), we find
\[
\eta(Y)X - \eta(X)Y = \frac{1}{n-2}\{\eta(Y)QX - \eta(X)QY\} + \frac{n-1}{n-2}\{\eta(Y)X - \eta(X)Y\} - \frac{r}{(n-1)(n-2)}\{\eta(Y)X - \eta(X)Y\}.
\]
Taking \(Y = \xi\) and using (4) we get
\[
QX = \left(\frac{1}{n-1} - 1\right)X + \left(\frac{r}{n-1} - 1\right)\eta(X)\xi.
\]
Thus the manifold is \(\eta\)-Einstein.

Contracting (20) we get
\[
r = n(n-1).
\]
Using (21) in (20) we find
\[
QX = (n-1)X.
\]
Putting (22) in (19) we get after a few steps
\[
R(X, Y)Z = g(Y, Z)X - g(X, Y)Y.
\]
Thus a conformally flat \(LP\)-Sasakian manifold is of constant curvature. The value of this constant is +1. Hence we can state

**Theorem 1.** A conformally flat \(LP\)-Sasakian manifold is locally isometric to a unit sphere \(S^n(1)\).

3. \(LP\)-Sasakian manifolds with \(\tilde{C} = 0\)

The quasi conformal curvature tensor \(\tilde{C}\) is defined as
\[
C(X, Y)Z = aR(X, Y)Z + b(S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY) - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)\{g(Y, Z)X - g(X, Z)Y\}
\]
where \(a, b\) are constants such that \(ab \neq 0\) and
\[
S(Y, Z) = g(QY, Z).
\]
Using (2), we find from (24)
\[
R(X, Y)Z = \frac{b}{a}\{S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY\} + \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)\{g(Y, Z)X - g(X, Z)Y\}.
\]
Taking $Z = \xi$ in (18) and using (7), (15) and (16), we get
\begin{equation}
\eta(Y)X - \eta(X)Y = -\frac{b}{a}\{\eta(Y)QX - \eta(X)QY\} \\
\left\{\frac{r}{an}\left(\frac{a}{n-1} + 2b\right) - \frac{b}{a}(n-1)\right\}\{\eta(Y)X - \eta(X)Y\}.
\end{equation}

Taking $Y = \xi$ and applying (4) we have
\begin{equation}
QX = \left\{\frac{r}{bn}\left(\frac{a}{n-1} + 2b\right) - (n-1) - \frac{a}{b}\right\}X \\
+ \left\{\frac{r}{bn}\left(\frac{a}{n-1} + 2b\right) - \frac{a}{b} - 2(n-1)\right\}\eta(X)\xi.
\end{equation}

Contracting (27), we get after a few steps
\begin{equation}
r = n(n-1).
\end{equation}

Using (28) in (27), we get
\begin{equation}
QX = (n-1)X.
\end{equation}

Finally, using (29), we find from (25)
\begin{equation}
R(X, Y)Z = g(Y, Z)X - g(X, Z)Y.
\end{equation}

Thus we can state

**Theorem 2.** A quasi conformally flat LP-Sasakian manifold is locally isometric with a unit sphere $S^n(1)$.

4. **LP-SASAKIAN MANIFOLDS SATISFYING $R(X, Y).C=0$**

Using (7), (13) and (16) we find from (18)
\begin{equation}
\eta(C(X, Y)Z) = \frac{1}{n-2}\left[\left(\frac{r}{n-1} - 1\right)\{g(Y, Z)\eta(X) \\
- g(X, Z)\eta(Y)\} - \{S(Y, Z)\eta(X) - S(X, Z)\eta(Y)\}\right].
\end{equation}

Putting $Z = \xi$ in (30) and using (7), (16) we get
\begin{equation}
\eta(C(X, Y)\xi) = 0.
\end{equation}

Again, taking $X = \xi$ in (30), we get
\begin{equation}
\eta(C(\xi, Y)Z) = \frac{1}{n-2}\left[\{S(Y, Z) + (n-1)\eta(Y)\eta(Z)\} \\
- \left(\frac{r}{n-1} - 1\right)\{g(Y, Z) + \eta(Y)\eta(Z)\}\right].
\end{equation}
Now
\[(33)\quad (R(X, Y)C(U, V)W = R(X, Y)C(U, V)W - C(R(X, Y)U, V)W \]
\[-C(U, R(X, Y)V)W - C(U, V)R(X, Y)W:
\]
Using (3), we find from above
\[g[R(\xi, Y)C(U, V)W, \xi] - g[C(R(\xi, Y)U, V)W, \xi]
- g[C(U, R(\xi, Y)V)W, \xi] - g[C(U, V)R(\xi, Y)W, \xi] = 0.
\]
Using (7) and (13) we get
\[(34)\quad -\eta(C(U, V, W, Y) = g(C(U, V)W, Y).
\]
Putting \(U = Y\) in (34) we find
\[(35)\quad -\eta(C(U, V, W, U) - \eta(U)\eta(C(U, V)W) + \eta(U)\eta(C(U, V)W)
+ \eta(V)\eta(C(U, U)W) + \eta(W)\eta(C(U, V)U) - g(U, U)\eta(C(U, V)W)
- g(U, W)\eta(C(U, U)W) - g(U, W)\eta(C(U, V)U) = 0.
\]
Let \(\{e_i : i = 1, \ldots, n\}\) be an orthonormal basis of the tangent space at any point, then the sum for \(1 \leq i \leq n\) of the relations (35) for \(U = e_i\) gives
\[(36)\quad \eta(C(\xi, V)W) = 0\quad \text{as } n > 3.
\]
Using (31) and (36), (34) takes the form
\[(37)\quad -\eta(C(U, V, W, Y) - \eta(Y)\eta(C(U, V)W) + \eta(U)\eta(C(Y, V)W)
+ \eta(V)\eta(C(U, U)W) + \eta(W)\eta(C(U, V)Y) = 0.
\]
Using (30) in (37) we get
\[(38)\quad -\eta(C(U, V, W, Y) + \eta(W)\left[\frac{1}{n-2} \left(\frac{r}{n-1} - 1\right) \{\eta(U)g(V, Y)
- \eta(V)g(U, Y)\} - \{\eta(U)S(V, Y) - \eta(V)S(U, Y)\}\right] = 0.
\]
In virtue of (36), (32) reduces to
\[(39)\quad S(Y, Z) = \left(\frac{r}{n-1}\right) g(Y, Z) + \left(\frac{r}{n-1} - n\right) \eta(Y)\eta(Z).
\]
Using (39), (37) reduces to
\[(40)\quad -\eta(C(U, V, W, Y) = 0,
\]
i.e.

\[(41) \quad C(U, V)W = 0.\]

Hence the manifold is conformally flat. Using Theorem 1, we state

**Theorem 3.** If in an \(LP\)-Sasakian manifold \(M^n (n > 3)\) the relation \(R(X, Y), C = 0\) holds, then it is locally isometric with a unit sphere \(S^n(1)\).

For a conformally symmetric Riemannian manifold [1], we have \(\nabla C = 0\). Hence for such a manifold \(R(X, Y), C = 0\) holds. Thus we have the following corollary of the above theorem:

**Corollary 1.** A conformally symmetric \(LP\)-Sasakian manifold \(M^n (n > 3)\) is locally isometric with a unit sphere \(S^n(1)\).

**References**


M. TARAFAHDR AND A. BHATTACHARYYA, DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF CALCUTTA, 35, BALLYGUNGE CIRCULAR ROAD, CALCUTTA: 700019, INDIA

E-mail address: manjusha@cubmb.ernet.in