25 years proof of the Kneser conjecture
The advent of topological combinatorics

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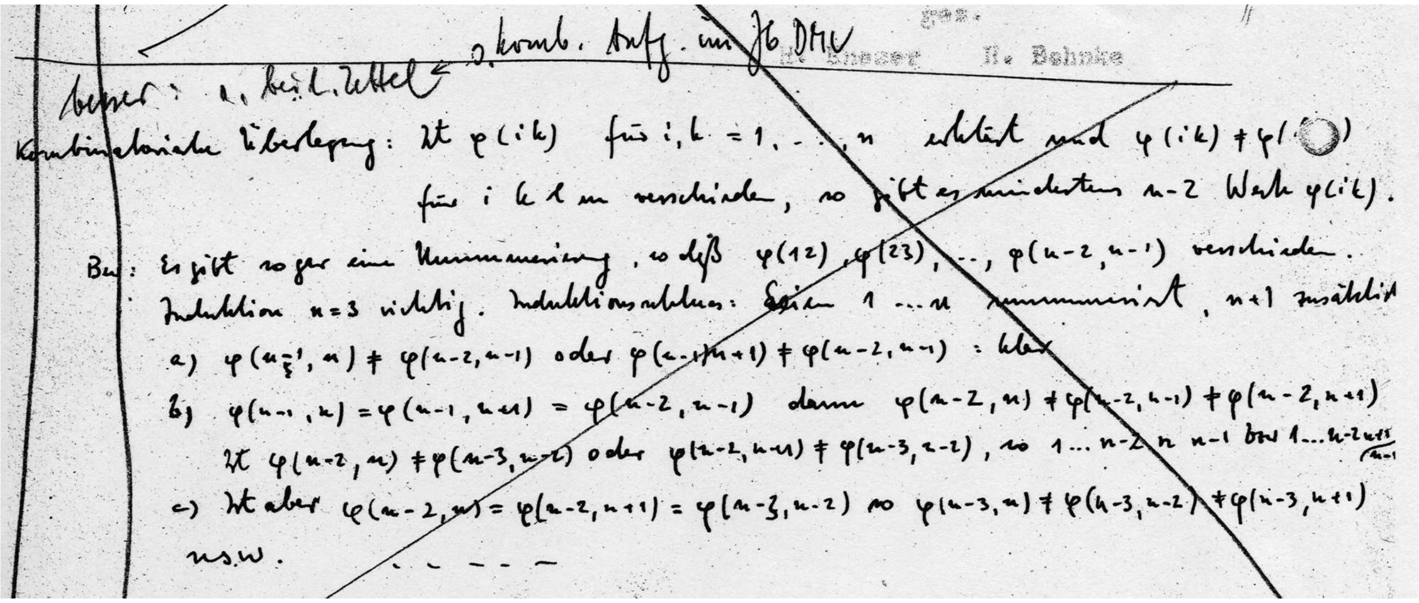

Kombinatorische Überlegung: Ist $\varphi(ik)$ für $i,k = 1, \ldots, n$ erklärt und $\varphi(ik) \neq \varphi(lm)$ für $i\,k\,l\,m$ verschieden, so gibt es mindestens $n-2$ Werte $\varphi(ik)$.

Bew: Es gibt sogar eine Nummerierung, so daß $\varphi(12), \varphi(23), \ldots, \varphi(n-2, n-1)$ verschieden.
Induktion $n=3$ richtig. Induktionsschluss: Seien $1 \ldots n$ nummeriert, $n+1$ zusätzlich

a) $\varphi(n-1, n) \neq \varphi(n-2, n-1)$ oder $\varphi(n-1, n+1) \neq \varphi(n-2, n-1)$: klar

b) $\varphi(n-1, n) = \varphi(n-1, n+1) = \varphi(n-2, n-1)$ dann $\varphi(n-2, n) \neq \varphi(n-2, n-1) \neq \varphi(n-2, n+1)$

Ist $\varphi(n-2, n) \neq \varphi(n-3, n-2)$ oder $\varphi(n-2, n+1) \neq \varphi(n-3, n-2)$, so ...

c) Ist aber $\varphi(n-2, n) = \varphi(n-2, n+1) = \varphi(n-3, n-2)$ so $\varphi(n-3, n) \neq \varphi(n-3, n-2) \neq \varphi(n-3, n+1)$

usw.

Figure 1: The origin of the Kneser conjecture in the hand writing of Martin Kneser.

László Lovász's proof of the Kneser conjecture about 25 years ago marked the beginning of the history of topological combinatorics: applications of topological methods and theorems to problems of discrete mathematics which did not (seem to) have any connection to topology.

With this article I want to take the opportunity to sketch the development of topological proofs in discrete mathematics beginning with the proof of the Kneser conjecture about 25 years ago, which eventually led to a new discipline: *topological combinatorics.*

In the beginning of the twentieth century the discipline of *combinatorial topology* already made use of combinatorial concepts in topology, finally leading to the emergence of algebraic topology. Meanwhile, discrete mathematics did not make much use of techniques from (algebraic) topology until the seminal proof of the Kneser conjecture. This situation was going to change in an unexpected and fascinating way.

The essence of topological combinatorics can be characterized by a scheme that many proofs in this field pursue. If we want to solve a combinatorial problem by topological means we carry out the following steps.

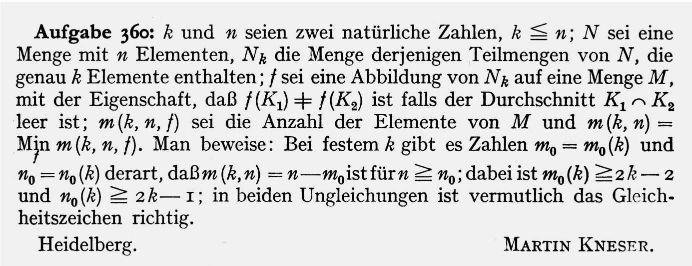
(1) Associate a topological space/continuous map to the given discrete structure such as a graph/graph homomorphism.

(2) Establish a relationship between suitable topological invariants of the space, e.g. dimension, k-connectedness, homology groups, etc. and the desired combinatorial features of the original structure.

(3) Show that the associated space resp. the map has the desired topological properties.

Thus we are concerned with a (functorial) procedure, which is commonly used in mathematics. However, the difference to other fields is that the constructions have to be invented and tailored anew for almost each individual problem. This calls for a certain amount of ingenuity and deeper insight, in particular in steps (1) and (2) of the above procedure.

The first proof of this kind is the proof of the Kneser conjecture by László Lovász. Because of its relevance for the emergence of topological combinatorics, its elegance, and its 25th anniversary, I will sketch this proof and report on the development of topological combinatorics.



Aufgabe 360: k und n seien zwei natürliche Zahlen, $k \leqq n$; N sei eine Menge mit n Elementen, N_k die Menge derjenigen Teilmengen von N, die genau k Elemente enthalten; f sei eine Abbildung von N_k auf eine Menge M, mit der Eigenschaft, daß $f(K_1) \neq f(K_2)$ ist falls der Durchschnitt $K_1 \cap K_2$ leer ist; $m(k, n, f)$ sei die Anzahl der Elemente von M und $m(k, n) = \operatorname{Min}_f m(k, n, f)$. Man beweise: Bei festem k gibt es Zahlen $m_0 = m_0(k)$ und $n_0 = n_0(k)$ derart, daß $m(k, n) = n - m_0$ ist für $n \geqq n_0$; dabei ist $m_0(k) \geqq 2k - 2$ und $n_0(k) \geqq 2k - 1$; in beiden Ungleichungen ist vermutlich das Gleichheitszeichen richtig.

Heidelberg. MARTIN KNESER.

Figure 2: From "Jahresbericht der DMV" 1955.

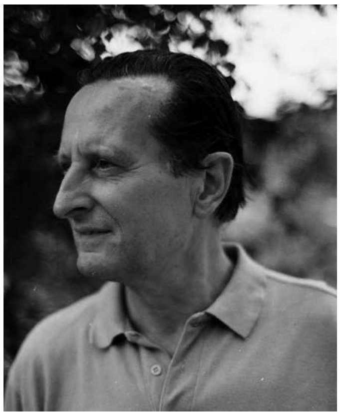


Figure 3: Martin Kneser.

Figure 4: László Lovász.

The Kneser conjecture and its proof

The occupation with an article by Irving Kaplansky on quadratic forms, from 1953, led Martin Kneser to question the behaviour of partitions of the family of k-subsets of an n-set:

> Consider the family of all k-subsets of an n-set. It is easy to partition this family into $n - 2k + 2$ classes $C_1 \cup \cdots \cup C_{n-2k+2}$, such that no pair of k-sets within one class is disjoint. Is it possible to partition the family into $n - 2k + 1$ classes with the same property?

Kneser conjectured that this is not possible! He presented his conjecture in the "Jahresbericht der Deutschen Mathematiker–Vereinigung" in 1955 in the form of an exercise [6] (cf. Figure 2).

We will translate Kneser's conjecture into graph theory language. For that purpose we define the Kneser graph $KG_{n,k}$ in a suggestive manner: the vertices are the k-subsets of an n-set and the edges are given by pairs of disjoint k-sets. Figure 5 shows this graph for the parameters $n = 5$ and $k = 2$. For example, there is an edge between the sets $\{1, 2\}$ and $\{3, 5\}$ because they have empty intersection.

The afore mentioned partitions of the k-subsets of an n-set now correspond to partitions of the vertex set of the graph in so called *colour classes*. The property that no partition set contains a pair of disjoint k-sets now translates into the property that no colour class contains two vertices that are adjacent via an edge in the graph. Such a partition is called a *graph colouring*. The *chromatic number of a graph* is the smallest number of colour classes in a graph colouring. As mentioned above it is easy to define a graph colouring of $KG_{n,k}$ with $n-2k+2$ colour classes. The following example shows $5-2\cdot2+2 = 3$ possible colour classes C_1, C_2, C_3, which define a colouring of $KG_{5,2}$.

$$
\begin{aligned}
C_1 &= \{\{1,2\},\{1,3\},\{1,4\},\{1,5\}\} \\
C_2 &= \{\{2,3\},\{2,4\},\{2,5\}\} \\
C_3 &= \{\{3,4\},\{3,5\},\{4,5\}\}
\end{aligned}
$$

(The example already suggests the general principle for a partition into $n - 2k + 2$ colour classes!) The Kneser conjecture now states that it is impossible to colour the graph with fewer colour classes, i.e., a *lower bound of* $n-2k+2$ *for the chromatic number of the graph* $KG_{n,k}$.

Twenty three years after Kneser posed his problem László Lovász initiated the development of topological combinatorics with the publication of his proof of the Kneser conjecture [7]. In the following sketch of his proof we will make use of some topological notions that will not be further explained, but which can be found in almost any textbook on topology, such as [11].

Lovász' proof of the Kneser conjecture pursues the scheme that we mentioned in the introduction. Step (1) is based on the invention of a simplicial complex associated with any graph G, the so called neighbourhood complex $\mathcal{N}(G)$. Simplices in the neighbourhood complex are defined by sets of vertices that have a common neighbour in the graph.

As an example we consider the graph G in Figure 6. It defines the neighbourhood complex $\mathcal{N}(G)$:

$$
\begin{aligned}
\mathcal{N}(G) = \{&\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \\
&\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \\
&\{2,3\}, \{2,5\}, \{3,4\}, \\
&\{1,2,5\}, \{1,3,4\}\}
\end{aligned}
$$

The inclusion maximal simplices in $\mathcal{N}(G)$ are given by $\{1,2,5\}$, $\{1,3,4\}$ and $\{2,3\}$. In the graph G they correspond to the neighbourhoods of the vertices 3, 2, 1 respectively. Figure 7 shows the topological space that realizes the simplicial complex geometrically. The simplices $\{1,2,5\}$ and $\{1,3,4\}$ correspond to the shaded triangles, the simplex $\{2,3\}$ corresponds to the line segment from 2 to 3.

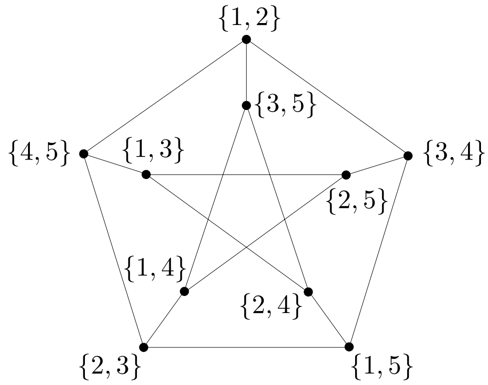


Figure 5: The familiar Petersen graph in the guise of the Kneser graph $K_{5,2}$.

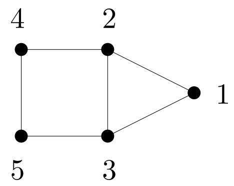


Figure 6: A graph G.

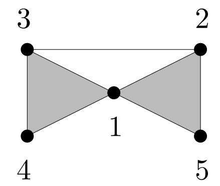


Figure 7: The neighbourhood complex $\mathcal{N}(G)$ associated to G.

In step (2) Lovász applies the *Borsuk–Ulam theorem* from topology, about which Gian–Carlo Rota, in an essay about Stan Ulam, once wrote:

> "While chatting at the Scottish Café with Borsuk, an outstanding Warsaw topologist, he [Ulam] saw in a flash the truth of what is now called the Borsuk–Ulam theorem. Borsuk had to commandeer all his technical resources to prove it."

One of the commonly used versions of this theorem reads:

> If there exists an antipodal continuous map $f : \mathbb{S}^n \to \mathbb{S}^m$ from the n-sphere to the m-sphere, i.e. a continuous map that satisfies $f(-x) = -f(x)$ for all $x \in \mathbb{S}^n$, then $m \geq n$.

Lovász now shows: if $\mathcal{N}(G)$ is topologically m-connected, then G is not $(m+2)$-colourable, i.e., there is no colouring of G with $m+2$ colour classes. In the example of Figure 7, the neighbourhood complex is 0-connected, i.e. connected, but not 1-connected since for example the loop 1→2→3→1 cannot be contracted to a point. This reflects the fact that G is not 2-colourable but 3-colourable.

We want to sketch this essential part of the proof from a slightly more modern point of view (see e.g. [2] and also quite current is [10]). An $(m+2)$-colouring of a graph G induces a graph homomorphism $G \to K_{m+2}$ from G to the complete graph K_{m+2} on $m+2$ vertices in which all pairs of vertices are connected by an edge. Such a graph homomorphism induces a continuous map $\mathcal{N}(G) \to \mathcal{N}(K_{m+2})$ of topological spaces. It is easy to verify that $\mathcal{N}(K_{m+2})$ is an m-dimensional sphere. Now, if $\mathcal{N}(G)$ is n-connected, then with the help of the map $\mathcal{N}(G) \to \mathcal{N}(K_{m+2})$, one can construct an antipodal continuous map $f : \mathbb{S}^{n+1} \to \mathbb{S}^m$. Hence by the Borsuk–Ulam theorem $m \geq n+1$, resp. $n \leq m-1$ as desired.

In step (3) Lovász completes his proof by verifying that $\mathcal{N}(KG_{n,k})$ is indeed $(n-2k-1)$-connected.

With his proof Lovász had identified the topological core of the problem: the Borsuk–Ulam theorem. In the same year, 1978, a much shorter proof by Imre Bárány followed, which employs the Borsuk–Ulam theorem in a more direct fashion. As recently as 2002, another substantial simplification has been found by the American mathematics student Joshua Greene [4]. For his proof, which can be considered as "Proof from the Book", Greene has been awarded the 2003 AMS-MAA-SIAM Morgan prize.

Topological combinatorics

The role of the Borsuk–Ulam theorem is not restricted to the proof of the Kneser conjecture. General bounds for the chromatic number of a graph, partition results, complexity bounds for algorithmic problems, and much more, have all been established with the help of Borsuk–Ulam's theorem and its generalizations. Maybe one of the most important generalizations is a theorem presented by Albrecht Dold [3] in 1983:

> Let G be a non-trivial finite group which acts freely on (well behaved) spaces X and Y. Suppose X is $(n-1)$-connected and Y has dimension m. If there exists a G-equivariant map from X to Y, then $m \geq n$.

For $X = \mathbb{S}^n$, $Y = \mathbb{S}^m$, and G the two–element group acting via the antipodal map, we re-obtain the Borsuk–Ulam theorem. As a matter of fact, there is such a variety of applications of this theorem and its generalizations that Jiří Matoušek dedicated an entire (and entirely wonderful) book to them with the title "Using the Borsuk–Ulam Theorem" [8].

While Borsuk–Ulam's theorem so far plays the most prominent role in topological combinatorics, most standard tools from algebraic topology have now found their applications in combinatorics, from homology- and cohomology computations, characteristic classes up to spectral sequences. A recent example is the article "Complexes of graph homomorphisms" by Eric Babson and Dmitry Kozlov [1]. Even some methods from differential topology have found a combinatorial analog, e.g. in the invention of *discrete Morse theory* by Robin Forman.

I want to mention a few applications of these methods. Most notably there are *graph colouring problems*. By now a multitude of bounds for the chromatic number of graphs and hypergraphs have been established with topological methods. *Partition results* of different kinds were solved, such as the *necklace problem*, which was solved in its full generality in 1987 by Noga Alon. Moreover, *complexity problems,* such as the complexity of linear decision tree algorithms and the complexity of monotone graph properties in connection with the *evasiveness conjecture* have been addressed with techniques from topological combinatorics. Another huge topic is the *topology of partially ordered sets.* In the early 80s the Swedish mathematician Anders Björner introduced the concept of shellability of a partially ordered set. With a partially ordered set one can associate a simplicial complex and thus a topological space. Shellability of a partial order as defined by Björner implies that the associated topological space is a bouquet of spheres. This combinatorial concept along with its topological consequences found numerous applications, such as in the theory of Bruhat orders and questions in the area of algebraic combinatorics. It should be remarked that using the concept of shellability, it is easy to see that the neighbourhood complex $\mathcal{N}(KG_{n,k})$, as it appears in Lovász' proof of the Kneser conjecture, is indeed $(n-2k-1)$-connected.

Back to combinatorics

Ever since combinatorial theorems were proved with topological methods, the natural question followed as to whether the topological argument could be replaced by a combinatorial one. A first breakthrough in this direction was made by Jiří Matoušek in the year 2000 with a combinatorial proof of the Kneser conjecture [9]. His proof relies on a special case of a combinatorial lemma by A.W. Tucker which is essentially "equivalent" to the Borsuk–Ulam theorem. Combinatorial relatives of the Borsuk–Ulam theorem have also been applied in the construction of approximation algorithms in connection with *fair division problems* . Moreover, the concrete questions in discrete mathematics pointed out the need for explicit methods for the computation of homology groups and other invariants of simplicial complexes which led to efficient programs for the computation of these invariants.

As mentioned in the introduction, combinatorial topology led to the formation of algebraic topology. The term "algebraic topology" was apparently first used in a public lecture by Solomon Lefschetz at Duke University in 1936 (cf. [5]):

> "The assertion is often made of late that all mathematics is composed of algebra and topology. It is not so widely realized that the two subjects interpenetrate so that we have an algebraic topology as well as a topological algebra."

The latter has now also become true for combinatorics and topology.

Addendum: Prof. Dr. Martin Kneser died on February 16, 2004 in Göttingen.

The EMS-Newsletter is grateful to the editors of the Newsletter of Deutsche Mathematiker–Vereinigung for their kind permission to publish the author's English translation of this article which originally appeared in Mitteilungen der DMV, 4/2003 *under the title "25 Jahre Beweis der Kneservermutung".*

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