Problem Corner
Contests from Bulgaria Part IV

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Unquestionably, the peak of most sports events culminates in a grand finale. And it's often been this way with intellectual challenges, say, maths contests. Finalists in these meetings have to clear the hurdles of divers selection criteria and/or come through several qualifyings, i.e. local and regional qualifying rounds bearing particularly exotic names, and observe specific rules on the whole. For example, in Bulgaria they are trading under unusual names such as the 'Virgil Krumov' contest or the 'Chernorizets Hrabar Tournament' and so on .

The last three issues of Problem Corner have dealt with the wealth of different competitions. Here, all roads lead to the national highlight for mathematically able youngsters, the National Mathematics Olympiad (NMO). But it would go beyond the scope of this Corner to enumerate the whole plethora of contests that Bulgaria offers to its adolescents in this field. For, in spite of this diversity, the separate maths trials are distinguishing from one another only by nuances. The National Bulgarian Mathematics Olympiad represents a country-wide showdown for native pupils to decorate themselves with the unofficial title of champion of maths, as reported by **Prof. Sava Grozdev**, Institute of Mechanics, Bulgarian Academy of Sciences, Sofia. Here comes his final part of a long story that describes the efforts made in his country towards getting young people really interested in mathematics.

The National Mathematics Olympiad

The Bulgarian National Mathematics Olympiad (BNMO) originates in 1949 and was actually the first competition organized in Bulgaria. Because of a two year interruption during the period 1957-59, the 53^{rd} National Olympiad took place in May, 2004. Initially, only students from grades 8 to 11 were permitted to participate in the Olympiad. At present the contest is open to grades 4 to 12. The BNMO is run in 3 rounds. Naturally, the first round, better known as 'school' round, is conquered by participants. About ten years ago, the number of starters evened out at 150 000 individuals, almost 10 per cent of all participating students. Currently, the initial number is reduced to about 20 000. One of the reasons for this decrease is a reduction in the total number of students due to a strongly abating birth-rate.

During the first round students have to solve three questions within four hours. All problems are compiled and worked out by the Regional Inspectorates of Education while the present teachers are asked to correct the examination papers of their own charges. About 30 per cent of all participants in this round will pass to the second stage, the regional round. Students of different grades are gathered in regional centres and have to solve three problems within four hours again. This time, the responsibility for the set of questions lies in the hands of the National Olympiad Commission and the examination papers are marked under the supervision of regional inspectors. The third round, or the National round, is reserved for students from grades 8 to 12. In former times there was an additional round for 7-graders, and these results were partially used as a permit to enter a Mathematics-, Foreign Language-, or Technical School. Alas, this tradition does not exist any more because of a complete restructuring of the veteran educational system, which is still going on. The third round is a two-day event like the International Olympiad. Students are asked to solve three problems each day within 4 or 5 hours. The National Commission in Mathematics creates the problems and is also responsible for marking the examination papers. The coordination of the results is carried out in the presence of both teachers and students. This procedure is a fully objective, fair and thus democratic element of the construct, called BNMO. Usually, the number of participants in the third round is about 100, with a growing tendency to drop further. For instance, in May 2002 the total number of students in the third round of the National Olympiad reached a minimum of 38 'survivors'. The champions of the final round are awarded the possibility to study Mathematics at a Bulgarian University of their choice, without passing entrance examinations. The universities in Bulgaria (about 40) follow an autonomous policy, which includes such individual tests, but they respect the results achieved at the National Olympiad, and thus are acknowledging high level talents. Besides this, the 12 students with the highest scores in the third round are invited for further selection each year. The selection includes two two-day tests consisting of six problems in all. When the six constituents of the National team have finally been identified, they have to undergo a fortnight preparation for the International Olympiad afterwards.

Bulgaria is one of the founder countries of the International Mathematics Olympiad (IMO) and has participated in all its 42 editions. Only two other countries can look back to a similar constance: Romania and the Czech Republic. Bulgarian students have won 32 gold, 73 silver and 81 bronze medals altogether. During the last 10 years, Bulgaria has ranked among the top ten countries with best performance, and regarding the last 4 years, Bulgaria has even moved into the best five. France, Germany, England, Italy and other countries that are world centres of Mathematics, have much lower rankings. In 1998 in Taiwan, Bulgaria was second, in 1999 and in 2000 - fifth, in 2001 with the participation of exactly 83 countries - third. The International Jury awards exceptional prizes to students with extraordinary achievements. Since 1987, only two prizes have been awarded. Both of these were addressed to Bulgarians: in Australia in 1988 and in Canada in 1985. The recent winner of a special prize was Nikolay Valeriev, who represents a real phenomenon of the IMO: from four participations he has won three gold and one silver medal. During the long history of the IMO, the mother of all maths contests worldwide, there have been only five other participants with comparable achievements. Four of them were before 1974, when the number of participating countries was limited. The fifth was Ride Burton from the USA team, who obtained four gold medals out of four and maintains a world record in this branch until today. Nowadays, the Bulgarian student Alexander Lishkov has the possibility to improve this mark, since he has one silver medal and four more participations to come through.

What is expected from those who have worked their way from a large field of starters up to the finale can best be learnt by an examination of the problems posed in my recent Corner. The new set is a bundle of questions which were used while preparing the Bulgarian team for participating in the International Mathematical Olympiad, held in Glasgow, Scotland, July 17 - 30, 2002. All problems are original and are worked out by members of the Team for Extra Curricula Research in cooperation with the Union of Bulgarian mathematicians.

164 Strange animals live in a building with $n \geq 3$ floors. The roof is considered the (n+1)st floor. Exactly one animal is living on the first floor, while one animal at most is living on each of the other floors. Within a month curious things will happen: Exactly once a month each animal gives birth to a new animal. Immediately after his birth the new animal moves to the nearest upper floor and remains there if the floor is unoccupied by another animal. If not, the newborn animal eats the previous inhabitant and walks on to the next upper floor repeating the same procedure. But a giant is dwelling on the roof and if an animal enters the roof, it will be eaten by the giant. The same things will happen the next month and so on. It is known that all the animals give birth to new animals simultaneously. What is the smallest number of months, after which the building will be settled the same way as it was at the beginning?

165 Consider a regular polygon with 2002 vertices and all its sides and diagonals. How many different ways can you choose some of them, such that they form the longest continuous (= connected) route?

166 Given the sequence: $x_1 = \frac{a}{2}, x_{n+1} = x_n^2 + x_n + 1, \quad (n \geq 1)$, where a is a positive integer. Prove that $\sqrt{x_n}$ is irrational for every $n \geq 3$.

167 All points on the sides of an acute triangle ABC are coloured white, green or red. Prove that there exists 3 points of the same colour, which form the vertices of a right triangle, or rather there exists 3 points of different colours, which are vertices of a triangle similar to the given one.

168 Given is a sequence of polynomials: $P_1(x) = x$, $P_2(x) = 4x^3+3x$, …, $P_{n+2}(x)= (4x^2+2)\cdot P_{n+1}(x) - P_n(x)$, $n \geq 1$. Prove that there are no positive integers k, l and m, such that $P_k(m) = P_l(m+4)$.

169 Find all continuous functions f: $\mathbf{R}\to\mathbf{R}$, such that f satisfies the functional equation: $f(x + f(x)) = f(x)$ for all real x.

It remains for me to present solutions to questions 152 to 157, published in Issue 49 of the Corner. All problems come from Hungarian sources, which stand for high quality of course.

152 The first four terms of an arithmetic progression of integers are a_1, a_2, a_3, a_4.
Show that $1\cdot a_1^2 + 2\cdot a_2^2 + 3\cdot a_3^2 + 4\cdot a_4^2$ can be expressed as the sum of two perfect squares.

*Solution by **J.N. Lillington, Wareham, UK**.*

Let $a_1 = a$, $a_2 = a+d$, $a_3 = a+2d$, $a_4 = a+3d$, where a,d are integers. Then $1\cdot a_1^2 + 2\cdot(a+d)^2 + 3\cdot(a+2d)^2 + 4\cdot(a+3d)^2 = a^2 +2a^2 +4ad+2d^2+3a^2+12ad+12d^2+4a^2+24ad+36d^2 = 10a^2+40ad+50d^2 = a^2+(3a)^2+40ad+d^2+(7d)^2 = (3a+7d)^2 + (a-d)^2$.
Also solved by Niels Bejlegaard, Copenhagen, Denmark; Pierre Bornsztein, Maisons-Laffitte, France; Erich N. Gulliver, Schwäbisch-Hall, Germany; Gerald A. Heuer, Concordia College, Moorhead, MN, USA, and Dr Z Reut, London, UK.

153 Is it possible to get equal results if $\sqrt{10^{2n} - 10^n}$ and $\sqrt{10^{2n} - 10^n + 1}$ are rounded to the nearest interger? (n is a positive interger).

*Solution by **Gerald A. Heuer**, **Concordia College, Moorhead, MN, USA**.*

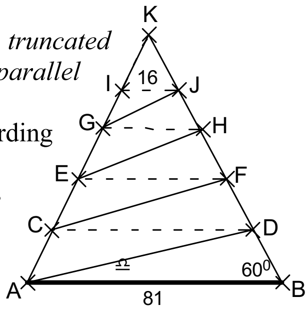
No. From the fact that $10^{2n} - 10^n < 10^{2n} - 10^n + \frac{1}{4} < 10^{2n} - 10^n + 1$ it follows that

$\sqrt{10^{2n} - 10^n} < 10^n - \frac{1}{2} < \sqrt{10^{2n} - 10^n + 1}$. Therefore the nearest integer to $\sqrt{10^{2n} - 10^n}$ is at most 10^n-1,

while the nearest integer to $\sqrt{10^{2n} - 10^n + 1}$ is 10^n.
Also solved by Niels Bejlegaard, Pierre Bornsztein, E.N. Gulliver, J.N. Lillington, and Dr Z Reut, London.

154 An Aztec pyramid is a square-based right truncated pyramid. The length of the base edges is 81 m, the top edges are 16 m and the lateral edges 65 m long. A tourist access is designed to start at a base vertex and to rise at a uniform rate along all four lateral faces, ending at a corner of the top square. At what points should the path cross the lateral edges?

*Solution by **J.N.Lillington.***



(*Ed. We refer to the opposite figure. It can easily be seen that the lateral faces of the truncated pyramid form trapezoids with 60^0 base angles. Line segments CD, EF, GH, IJ are drawn parallel to base AB.)*

First, we show that triangle ABK is equilateral. Triangles ABK and IJK are similar according to the preface, thus $\frac{KJ}{16} = \frac{KJ+65}{81}$ (because the lateral edges measure 65 m) or KJ = 16, which gives the result.

Let the tourist walk be the path ADCFEHGJI crossing the right lateral edge at D,F,H.

Applying the sine rule on triangles ABD, and ACD gives: $\frac{81}{\sin\left(120^0-\delta\right)} = \frac{AD}{\sin 60^0}$ and

$\frac{AD}{\sin 120^0} = \frac{CD}{\sin(60^0-\delta)}$ or (because of $\sin 120^0 = \sin 60^0$ and $\sin(120^0\text{-}\delta) = \sin[180^0\text{-}(120^0\text{-}\delta) = \sin(60^0+\delta)$)

this leads to $\frac{81}{\sin\left(60^0+\delta\right)} = \frac{CD}{\sin\left(60^0-\delta\right)}$ or $81 = CD\cdot\frac{\sin\left(60^0+\delta\right)}{\sin\left(60^0-\delta\right)}$.

Then applying successively to triangles CDF, CEF, EFH, EGH, GHJ and GIJ we get $81 = 16\cdot\left(\frac{\sin\left(60^0+\delta\right)}{\sin\left(60^0-\delta\right)}\right)^4$

or $\frac{3}{2} = \frac{\sqrt{3}+\tan\delta}{\sqrt{3}-\tan\delta}$ which finally simplifies to $5\cdot\tan\delta = \sqrt{3}$.

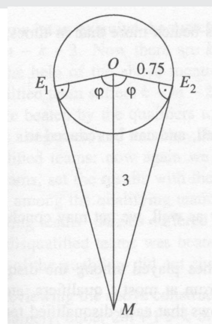
Applying the sine rule on triangle ABD gives $\frac{DB}{\sin\delta} = \frac{81}{\sin\left(60^0+\delta\right)}$ or $DB = \frac{81}{\frac{\sqrt{3}}{2}\cdot\cot\delta+\frac{1}{2}}$. Hence DB = 27

and applying successively we yield DF = 18, FH = 12, and HJ = 8.

Also solved by Niels Bejlegaard.

155 The billposters of the Mathematician's Party observed that people read the posters standing 3 meters away from the centres of the cylindrical advertising pillars that have a 1.5 m diameter. The Party wants to achieve that, after sticking the posters around a pillar, a whole poster will be visible from any direction. How wide should the posters be?

*Solution by **Niels Bejlegaard**, **Copenhagen**,* slightly revised by the editor.



(Ed. A person standing at M, a distance of 3 meters away from the centre, is able to see the part of the cylindrical advertising pillar that is bounded by two tangent planes drawn to it. Clearly, it will be sufficient to view a circle as a cross-section of the pillar). According to the figure a simple calculation shows:

$\tan\varphi = \frac{\sqrt{3^2-.75^2}}{.75} = \sqrt{15}$ or $\varphi \approx 75.52^o$. The length of a bill along the cylinder is s = arctan $\sqrt{15}$ ≈ 1.98 m (*or* $s \approx .75\cdot 2\pi\cdot\frac{2\cdot 75.52^o}{360^0}$*).*

Obviously two equal bills are not sufficient to be viewed completely because the visual angle would be 0^o then.

If we denote the width of the posters by w, then we must have $w \le \frac{s}{2}$. Since we are looking for the maximal width, we may assume the posters to tile the pillar, i.e. the circumference of the pillar is an integer multiple of the poster width. If we have, say, n posters, then $\frac{.75\cdot 2\pi}{5} \le \frac{s}{2}$ must hold, from which we get n > 4.767 … . So the smallest possible value of n is 5, and the poster width must be $w = \frac{.75\cdot 2\pi}{5} \approx .942$ m.

Also solved by J.N. Lillington.

156 Find a simpler expression for the sum $S = 3 + 2\cdot 3^2 + 3\cdot 3^3 + \ldots + 100\cdot 3^{100}$.

Solution by ***Dr Z Reut, London****.*

The sum can be written as $S = \sum_{n=1}^{100} n\cdot 3^n = \sum_{n=1}^{100}(n-1+1)\cdot 3^n = \sum_{n=1}^{100}(n-1)\cdot 3^n + \sum_{n=1}^{100} 3^n$. The first term can now be written as follows: $\sum_{n=1}^{100}(n-1)\cdot 3^n = 3\cdot\sum_{n=1}^{100}(n-1)\cdot 3^{n-1} = 3\cdot\sum_{n=1}^{99} n\cdot 3^n = 3\cdot(\sum_{n=1}^{100} n\cdot 3^n - 100\cdot 3^{100})$ $= 3\cdot(S - 100\cdot 3^{100})$. The second term is a geometric series, which is reduced as follows: $\sum_{n=1}^{100} 3^n = 3\cdot\frac{3^{100}-1}{3-1} = \frac{3}{2}\cdot\left(3^{100}-1\right)$. The first equation becomes $S = 3\cdot\left(S - 100\cdot 3^{100}\right) + \frac{3}{2}\cdot\left(3^{100}-1\right)$. Solving for S gives the result $S = \left(150 - \frac{3}{4}\right)\cdot 3^{100} + \frac{3}{4} = \frac{3}{4}\cdot\left(199\cdot 3^{100}+1\right)$.

Also solved by Pierre Bornsztein, Niels Bejlegaard, Erich N. Gulliver, Gerald A. Heuer, and J.N. Lillington.

157 Given are two non-negative numbers x,y, that satisfy the inequality $x^3 + y^4 \le x^2 + y^3$. Prove that $x^3 + y^3 \le 2$.

Solution by ***Pierre Bornsztein****.*

First, the result is trivially true for $x = y = 0$. Thus, we assume $(x,y) \ne (0,0)$.
Suppose, for a contradiction, that $x^3 + y^3 > 2$.

The inequality between means of order 2 and order 3 gives $\sqrt{\frac{x^2+y^2}{2}} \le \sqrt[3]{\frac{x^3+y^3}{2}}$.

Thus, $x^2+y^2 \le \left(x^3+y^3\right)^{\frac{2}{3}}\cdot 2^{1-\frac{2}{3}} = \left(x^3+y^3\right)^{\frac{2}{3}}\cdot 2^{\frac{1}{3}} < \left(x^3+y^3\right)^{\frac{2}{3}}\cdot\left(x^3+y^3\right)^{\frac{1}{3}} = x^3+y^3$.
It follows that $x^2 - x^3 < y^3 - y^2$.
On the other hand we have $y^3-y^2 \le y^4-y^3 \Leftrightarrow 0 \le y^2\cdot(y-1)^2$, and this inequality is generally true.
It follows that $x^2-x^3 < y^3-y^2 \le y^4-y^3$, which contradicts the statement of the problem.
Thus, $x^3 + y^3 \le 2$, as desired.
Also solved by Niels Bejlegaard and J.N. Lillington.

(*Ed.* ***Marcel G. de Bruin****, Faculty of Electrical Engineering, Mathematics and Computer Science, Department of Applied Mathematics, Delft, The Netherlands, has proposed an improvement of the outcome of problem 147, given in the NewsletterNo. 51. The question reads as follows:*

Given 100 positive integers $x_1, x_2, \ldots, x_{100}$, such that $\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \ldots + \frac{1}{\sqrt{x_{100}}} = 20$, prove that at least two of the integers must be equal.

A simple sharpening of the integral-test allows one to establish a better estimation, for the following inequality can be easily shown: $\frac{1}{\sqrt{n}} < \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{dx}{\sqrt{x}}$, $n \in \mathbb{N}\setminus\{0\}$. *Assuming $x_1, \ldots, x_m$ to be distinct positive integers, we find* $\sum_{n=1}^{m}\frac{1}{\sqrt{x_n}} \le \sum_{k=1}^{m}\frac{1}{\sqrt{n}} < \int_{\frac{1}{2}}^{m+\frac{1}{2}}\frac{dx}{\sqrt{x}} = 2\sqrt{m+\frac{1}{2}} - \sqrt{2}$. *The 'best' integer m follows easily from* $2\sqrt{m+\frac{1}{2}} - \sqrt{2} \le 20$, *i.e. m = 114.)*

That completes this issue of the Corner. Send me your nice solutions and generalizations.