Mathematics is alive and well and thriving in Europe

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This is the text of a lecture given at the first Euroscience Open Forum in Stockholm in August 2004. The text will keep unashamedly the characteristic of the lecture: no previous mathematical knowledge is assumed.

Numbers

This being about mathematics, I may as well start with numbers, and shall write three short formulas.

The **first formula** will have essentially no ingredients, except the positive integers : 1,2,3,4,... Let us write :

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = A.$$

Whenever I stop summing after a finite number of terms, I just have a sum of fractions that can be computed by hand or with a pocket calculator.

But the ... means that I want to continue summing forever, writing the sum of an infinite number of terms. This should not frighten anybody, we know of other examples of such infinite sums yielding a finite number, like

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots = 0,3333... = \frac{1}{3}.$$

In the first formula, the sum also gives a number, which we'll call A.

The **second formula** will have more complicated ingredients, namely the complete list of prime numbers. Recall that a prime number is an integer ≥ 2 which has no divisor except 1 and itself. So 6 is not prime because $6 = 2 \times 3$, and 7 is prime because such a decomposition does not exist. The first prime numbers are $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31...$ Prime numbers have always fascinated mathematicians and Euclid, 24 centuries ago, proved that there is an infinite supply of primes.

Note that his proof is so perfect, short and elegant that it is still the best proof of this result, in fact it can be used as a first illustration of what a proof is.

And now we consider the infinite product

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\left(1-\frac{1}{11^2}\right)...$$

Like the infinite sum above, this yields a (finite) number and we write :

$$B = \frac{1}{\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)...}$$

The **third formula** will use an ingredient from a totally different area, namely classical geometry. As we all know, the length of a circle of radius R is given by $2\pi R$, where π has approximate value

$$\pi = 3.14159265358979323846...$$

Let us write

$$C = \frac{\pi^2}{6}$$

So we have written three numbers, coming from different areas, with no apparent relation. Yet, one can show that $\boldsymbol{A = B = C}$, they are the same number.

It you pause to think about it (do it!) this is unbelievable. It means in particular that some facetious god of mathematics has encoded the length of a circle in the list of prime numbers, totally unrelated a priori.

We draw three first conclusions:

1) Mathematics has beauty and magic.

2) Mathematics is not about specific fields like algebra, geometry, analysis,... but is about the relations between different areas, allowing us to use methods of one to solve problems of the other.

3) Ancient notions and proofs are as fresh today as 24 centuries ago.

Let us push a bit further. In the above formulas, could we replace the squares by cubes? In fact we have

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \ldots = \frac{1}{(1-\frac{1}{2^3})(1-\frac{1}{3^3})(1-\frac{1}{5^3})\ldots}$$

but this is no longer a fraction times π^3.

Likewise, provided x is a real number greater than 1, we have

$$1 + \frac{1}{2^x} + \frac{1}{3^x} + \ldots = \frac{1}{(1-\frac{1}{2^x})(1-\frac{1}{3^x})(1-\frac{1}{5^x})\ldots}$$

This has now become a function of the number x, called the zeta function of Riemann : $\zeta(x)$.

In 1859, in a prodigious paper, Riemann analyses this function and shows its deep relation with number theory, and the distribution of prime numbers in particular.

For that, he shows that ζ can be extended as a function defined also for $x < 1$, and even to the case of $\zeta(z)$, where $z = x + \sqrt{-1}y$ is a complex number. He states his belief that the zeros of zeta, i.e. the values of z for which $\zeta(z) = 0$, beyond the "easy" ones $-2, -4, -6, ...$ are all situated on a line, namely are all of the form $z = \frac{1}{2} + \sqrt{-1}y$.

Unable to prove it, he assumes it as a hypothesis, leaving to others the task of proving it.

The "Riemann hypothesis", still unproven, is considered by many as the most important single open question in mathematics today.

Why is this obscure looking question about the zeros of a specific complicated function so important? From

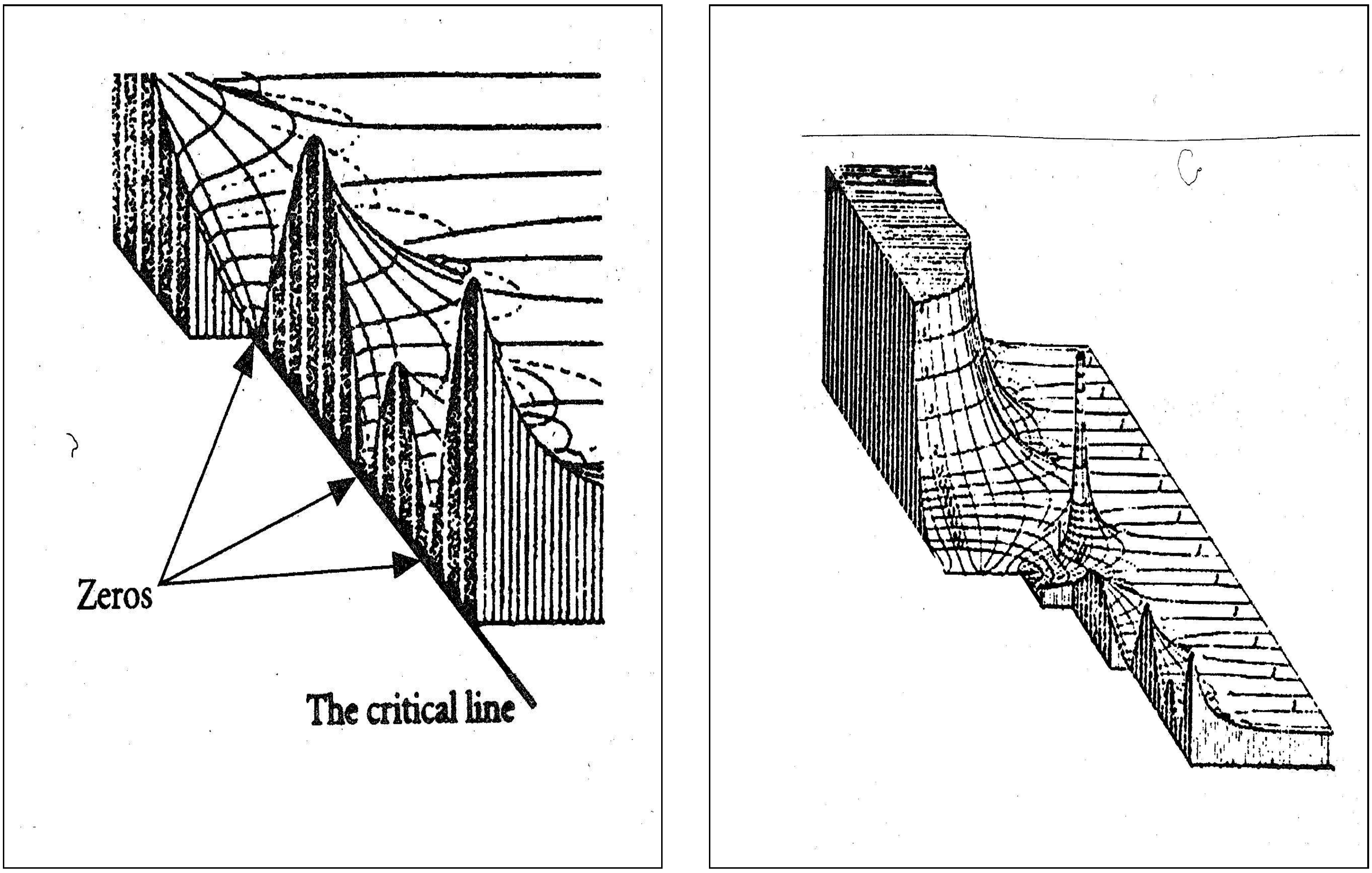

Figure 1: Graphs of the Riemann zeta function

the first formulas, we get a glimpse that it is related to questions of number theory. In fact, hundreds of theorems about prime numbers, or about theoretical computer science, have been proven under the assumption that Riemann's hypothesis is true, and were not proven otherwise. So when - or if - the hypothesis is proven, hundreds of other results will be proven at the same time.

In 1900, the great German mathematician David Hilbert drew a visionary list of 23 open problems that shaped part of the development of mathematics in the twentieth century.

Riemann's hypothesis is one of them, and the only specific question of the list unsolved after one century.

In 2000, the Clay foundation (established by philanthropist Landon Clay) announced a list of 7 Millennium problems and, times having changed, offered a reward of one million dollars for each. Riemann's hypothesis is - of course - one of them.

The main effect of trying to solve such a difficult problem is to relate it to new results and theories - a very positive result even when it does not solve the problem. Today, the obscure question of Riemann appears to be at the centre of a spider web of mathematical fields and theories.

Physics has also joined the web: Michael Berry related the zeros of zeta to quantum chaos, and Alain Connes to the eigenvalues of an operator, similar to the absorption spectrum of a star.

But is all this confined to the realm of "pure" or fundamental mathematics, highly amusing to mathematicians but unrelated to real life?

G. H. Hardy, who contributed to the Riemann hypothesis in 1914, claimed that number theory's beauty was related to its uselessness, so that it was not related to dull realities or applications.

Well, Hardy was wrong, and all of you use prime numbers regularly, but without knowing it.

Indeed, whenever you use a bank machine, or order anything through the internet, your bank data is of course encrypted. And the only totally safe encrypting code today is based on prime numbers.

Basically, it works as follows:

Anybody who wants to receive encoded messages will choose two very large prime numbers - say of one hundred digits. He will not reveal to anyone the two chosen numbers, the "secret key".

But he will compute the product of the two numbers, obtaining a two hundred digit number that he will advertise freely (the public key).

Using some clever mathematics (available as a software), anybody who wants to send him a message will encrypt it using the 200 digit numbers and can send it without much care. Indeed, the only possibility to decode it is to know the two prime numbers, known only to one person.

This is the principle of RSA cryptography, created in 1977 by Rivest, Shamir and Adleman (and slightly earlier by Ellis, Cocks and Williamson of the British secret service, but they kept it secret).

But, you might say, the two hundred digit number is decomposable in a unique way as the product of the two original prime numbers, and it would be sufficient to find that factorisation to decode all messages that use this number.

The point is that it would take centuries for the larger computer to find the decomposition, because of the size of the numbers.

Could one maybe find by chance one of the prime numbers, hence the other? The odds would be no better than trying to pick up at random a specific particle of the known universe.

So we get two more conclusions about mathematics

4) Old notions still give rise to the most pressing open problems of today and tomorrow.

5)"Pure" mathematics, studied only for the sake of elegance and beauty, suddenly finds crucial applications to science or economic development. To quote the physicist Eugene Wiegner, this is the "unreasonable effectiveness of mathematics applied to natural sciences".

Nobel prizes, Fields medals, Abel prizes and economic development

There are no Nobel prizes in mathematics. The mathematicians managed to keep alive for decades the legend that Alfred Nobel's girlfriend eloped with the Swedish mathematician Gosta Mittag-Leffler, so that Nobel would not create a prize that might go to his rival. However, Nobel's sex life is not so well documented, and the fun went out of the story when geologists and others tried to spread similar stories about their science.

The point is much more probably that Nobel - as an industrialist - was interested in "inventions and discoveries" and didn't see mathematics fitting there.

Anyway, the mathematicians created a prize recognised today as the "Nobel of Mathematics": the Fields medal, first awarded in 1936.

There are three main differences between Nobel prizes and Fields medals.

Compared to Nobel prizes, the medal comes only with a minimal financial prize.

Every four years, 2, 3 or 4 medals are awarded and they are never shared between mathematicians for a joint discovery.

The major difference is an upper age limit of 40 for the award of a medal, illustrating the fact that in most cases mathematical genius can be detected at a young age.

For example, Jean-Pierre Serre was awarded the medal in 1954 at the age of 27 - because he was already an acknowledged master with impressive discoveries.

On the other hand, Andrew Wiles, who achieved eternal (and even newspaper) fame by proving Fermat's last theorem (a question open since 1637), did not get a Fields medal because he was 41 when his proof was completed.

Now, many political and economical studies comparing the effectiveness of scientific research in various continents refer to Nobel prizes as an indicator and draw a negative conclusion about Europe.

Indeed, between 1980 and 2003, the Nobel prizes in biology and medicine, physics and chemistry give :

68 for Europe

154 for the USA

with the gap growing with time.

For mathematics, the Fields medals give a much better picture for Europe.

In the same period, Fields medals were awarded to :

9 Europeans (one working in the USA)

5 U.S. citizens

4 Russians (one working in Paris, two in the US)

1 Japanese

1 New Zealander (working in the US)

All together, including Russia in Europe where it should be, we get

10 working in Europe

9 working in the USA

a success story for Europe.

Fields medals are awarded for recent developments, so we see no growing gap at all in mathematics.

More recently, to celebrate the two hundredth anniversary of the birth of Niels Henrik Abel, the Norwegian government created a prize for mathematics similar to the Nobel prize, fittingly called the Abel Prize.

After two awards, the laureates are Jean-Pierre Serre (49 years after his Fields medal), Michael Atiyah and Isadore Singer - two Europeans and one US citizen.

To sum up, mathematics in Europe is at top level, quite cheap to run and extremely efficient.

It must be acknowledged that the usefulness of fundamental mathematics will not always appear quickly (even if 24 centuries from Euclid to internet is an extreme example).

But as stated by Timothy Gowers in his millennium address at the Clay foundation: "If you were to work out what mathematical research has cost the world in the last hundred years, then work out what the world has gained in crude economic terms, you will discover that the world has received an extraordinary return on a very small investment".

Since return is not immediate, it falls outside the scope of an industrial company that needs to meet its objectives within a few years. Needless to say, it is outside the aims of a financial group buying a company with the sole aim to sell it after five years, after raising its stock exchange value and nothing else.

Thus, funding for fundamental curiosity driven mathematics must come from public money, for everybody's long term interest.

This should be managed by programmes better suited for mathematics at the level of the European Commission and the national policies.

To include this better in the overall scientific planning of the E.U., a minimum step would be to include a mathematician in EURAB, the 45 member advisory committee of the European Commission on science (so far, 67 people have been members of EURAB - none of them mathematicians).

It is sometimes said that mathematicians work by themselves in their office, with a pen and a piece of paper. First one should not forget the waste paper basket, much used in mathematical research.

And then there are the serious needs for mathematical work.

First, of course, we need positions, either in universities or research institutes. Mathematics is done by people - and this is the priority.

In Europe, we face the same paradox as in other sciences. Indeed, we educate high level mathematicians up to the doctoral level, then have some post-doctoral positions, but usually do not provide the bridge between these positions and the tenured ones. At that stage, the USA step in and offer tenure-track jobs to scientists fully educated in Europe.

We should strongly promote the creation of timely tenure-track positions in our universities.

In this respect, I believe the European Commission is making a mistake in putting the accent on "training" of doctoral students (up to four years research experience) and not more experienced ones (up to 10 years) in all its Marie Curie activities. Doesn't this simply educate more scientists to be swallowed by the American system ?

A special consideration should be given to Central and Eastern Europe. The extremely high level of the mathematical tradition there explains why for instance we have four Russian Fields medallists out of twenty. But the dramatic economic situation explains that only one remained in Russia.

A smallish investment to preserve that tradition would be in everyone's interest.

Secondly, mathematicians need regular contacts with other researchers, the world over. This happens during conferences, and by short or long term visits.

These provide an immense acceleration of research. In one or two hours, a specialist can explain the basics and recent trends in his field, whereas it would take months to get that information from the literature. Also mathematicians are not working together in very large centres, so they regularly need to see specialists in their domain, often rather thinly spread.

Thirdly, easy access to the literature is necessary. We have seen that older articles keep all their value, and access is needed to all good level literature present and past.

No university library today can keep up with the rising cost of journals, but electronic access offers new unprecedented opportunities.

Most new papers are typed in the software TEX, and can be made accessible to all.

Another objective is to digitise the whole literature - an operation estimated at around 50 million euros. Good co-ordination is needed, so that all digitised papers are accessible with the same standard. Also, the ownership of the database should not be left to purely profit-making organisations, and this again requires public funding now.

Note that this database could be made available at minimal cost in less developed countries, where high level mathematics is present but faces major economic difficulties.

Finally, note that mathematical research is accomplished both in universities and in a string of high level research centres, with regular movement of researchers from one to the other, so that all are necessary.

Differential equations

Since the invention of calculus by Newton and Leibnitz, differential and partial differential equations have been the central tool in mathematics applicable to science - first astronomy and physics, then chemistry, biology and now economy and finance.

It is a huge subject, both in fundamental mathematics and its applications.

To take but a specific example, I'll consider the Navier-Stokes equations. They are a system of partial differential equations that model the movements in fluid dynamics. They were written by Navier in 1822, then justified more precisely by Stokes a few years later.

They are both extremely useful in applications and extremely hard to study with mathematical rigour.

After almost two centuries, we have no formula giving the solutions (this is usual for partial differential equations), and we don't even have decent results on their existence and properties.

In fact, their mathematical study is the object of one of the seven Millennium problems of the Clay foundation.

But still they are used in applications like shaping cars and planes, modelling the flow of blood in the cardiovascular system and many others.

I shall briefly describe an application developed at the Fraunhofer Institute of Applied Mathematics in Kaiserlautern, Germany: the conception of airbags for cars.

For many years, it was an inaccessible idea. Indeed, in case of an accident, the bag has to be inflated fully in 1/20 of a second or it is too late.

Everything will count: the way the gas is injected, the shape of the bag, the way it is folded.

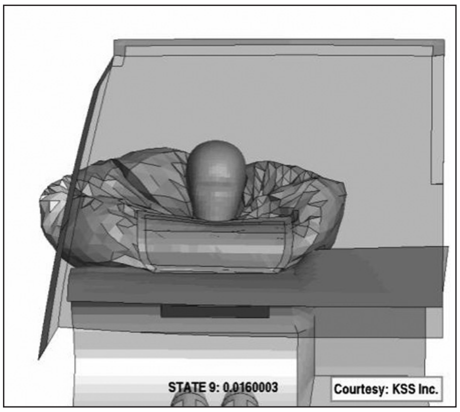

Figure 2. Airbag simulation

The most efficient (and cost efficient) way to find the best shape is by computer modelling. The researchers in Kaiserslautern developed a computer software that allows them to model any shape and gas speed, and look for an efficient one. They can vary the shape and folding many times and observe the blowing up of the bag.

Comparing the "films" of their mathematical bags with the film of concrete realisation of the bags built afterwards shows that they obtain a remarkable precision.

But this achievement required the development of new mathematics.

The standard way to model partial differential equations on a computer is to choose a mesh on the domain of the equation, then approximate derivatives by differences of the values at the different vertices.

Here the domain (the airbag) varies quickly, because of the varying gas pressure. Therefore, "mesh free" methods have to be developed.

We see here the same scheme appearing in an endless series of examples. There is first a mathematical theory, maybe fairly old. It is still the object of difficult theoretical questions. It also gives rise to unexpected applications which in turn give rise to new mathematical questions. Their study will provide new theories which again will motivate new problems and give rise to new unexpected applications.

Talking in Stockholm on partial differential equations, I should mention a grand master of the subject: Lars Hörmander.

His early work was so exceptional that he was appointed full professor in Stockholm at the age of 25.

At the age of 31 he got a Fields medal, but moved to the Institute for

Advanced Study in Princeton.

This brain drain provoked a strong reaction in Sweden, and the parliament voted the "Lex Hörmander", a law allowing the creation of personal chairs in exceptional case.

Hörmander came back to Lund and the law also allowed them to attract back Lennard Carlesson, another major figure.

Together, they had 36 PhD students in Sweden, who in turn had PhD students, so that the total number of their mathematical descendants in Sweden is now over 180.

Thus, their presence in Sweden helped not only the general reputation of the country, but more concretely the local development of science and industry.

Let us have a Lex Hörmander at the European level!

Pour l'honneur de l'esprit humain

So far, this lecture has concentrated on the far reaching economic benefits following the development of fundamental mathematics. I described only two examples, for lack of space, but could go on endlessly in most if not all fields of human activity. Fast developing examples include medical imaging, image compression for storage or transmission, epidemiology, biostatistic, mathematical genomics, finance, control theory for plane safety or energy saving ...

In this presentation, I followed the present day trend of having to prove the economic value of all activities, including art, mathematics, and culture.

But this is a sad evolution of our society, and mathematics should also be pursued "for the honour of the human mind", to coin the phrase of Carl Gustav Jacobi.

So I want to point out now that mathematics, like art, philosophy and science, are essential parts of civilisation.

The obvious example is the Greek civilisation, which left us as heritage (with contributions of the Arabic civilisation) art, mathematics, philosophy and the beginnings of science. It is these four aspects, and not the value of their stock exchange or whatever they had, that founded the rebirth of our civilisation after the Renaissance.

Civilisation as we live it is not defined in economic terms, and is our most precious asset.

The princes of Medicis, wealthy as they were, will be remembered forever for triggering that Renaissance.

Likewise, how should we remember Carl Wilhelm Ferdinant, Duke of Brunswick?

In my dictionary, he is described as a duke soldier who was beaten by the French in Valmy, then again in Iena.

But I must say I looked only in a French dictionary. Still, an uninspiring notice.

But one day, he got a report from a school teacher that a young boy seemed remarkably gifted in mathematics. The boy was the son of a poor gardener and bricklayer, so his future should have been rather bleak.

But the Duke liked mathematics, saw the boy and was convinced by his obvious talent (if not by his good manners). Thus he supported his studies and career throughout his life.

The boy's name was Carl Friedrich Gauss, and we owe to him (and the Duke) the Gauss law of prime numbers, the Gauss distribution in probability, the Gauss laws of electromagnetism, most of non-Euclidean geometry, and the Gauss approximation in optics.

Obviously, we need more Dukes of Brunswick in our governments.

More to the point, we need a European Research Council to cater for fundamental research.

Going back to the idea of civilisation, it seems that abstract mathematics - disjoint from practical applications - appears early in the development of the human mind.

The first mathematical ideas were practical: counting objects (like cattle), computing the area of a field, the volume of a pyramid.

But then why did Euclid care about prime numbers, or abstract proofs of geometric theorems? Why did the Babylonians, 1000 years before Pythagoras, engrave his theorem on their clay tablets? Why did the Egyptians develop a sophisticated and quite useless system of fractions?

On a bone, found by anthropologist Jean de Heinzelin in Ishango, Africa, a series of scratches provides the beginning of a multiplication table and a short list of prime numbers. The bone is dated between 7000 and 20000 years BC and the scratches could be a coincidence - or one of the first appearances of abstract mathematics.

I believe that the human mind, early in its evolution, has the need to think in abstract terms, and that mathematics unavoidably appears at this stage.

Finally, why do mathematicians do mathematics?

Why did Michelangelo paint and sculpt, why did Beethoven compose?

All for the same reason: because they must. As David Hilbert put it in 1930: Wir müssen wissen, wir werden wissen (we must know, we shall know).

Asked why he insisted in trying to climb Mount Everest, the famous mountaineer George Mallory answered: "Because it is there".

Likewise, mathematicians attack their own Everest (like Riemann's hypothesis) because it is there.

But when they reach the top, they have changed the scenery by their achievement.

Looking backward, they see the hard path that they have followed, but also some much easier and simplified paths now open to others.

Looking forward, they see higher mountains which were invisible or maybe did not exist before. Mountains that must now be climbed.

Because they are there.

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A former chairman of the Belgian Mathematical Society, he has been associated with the European Mathematical Society since its creation in 1990, being a member of the Council from 1990 to 1997, a member of the group on relations with European Institutions since 1990, Liaison Officer with the European Union since 1993, and Vice President since 1999.