

*Univ. Hamburg* **37** (1972) 86–97. Theorem 1.8.1 is from L. Euler, *Solutio problematis ad geometriam situs pertinentis*, *Comment. Acad. Sci. I. Petropolitanae* **8** (1736), 128–140.

Of the large subject of algebraic methods in graph theory, Section 1.9 does not claim to convey an adequate impression. The standard monograph here is N.L. Biggs, *Algebraic Graph Theory*, Cambridge University Press 1974. A more recent and comprehensive account is given by C.D. Godsil & G.F. Royle, *Algebraic Graph Theory*, in preparation. Surveys on the use of algebraic methods can also be found in the *Handbook of Combinatorics* (R.L. Graham, M. Grötschel & L. Lovász, eds.), North-Holland 1995.

*2nd edition: CUP 1993*

*Proof.* Let  $G$  be any  $2k$ -regular graph ( $k \geq 1$ ), without loss of generality connected. By Theorem 1.8.1,  $G$  contains an Euler tour  $v_0e_0 \dots e_{\ell-1}v_\ell$ , with  $v_\ell = v_0$ . We replace every vertex  $v$  by a pair  $(v^-, v^+)$ , and every edge  $e_i = v_iv_{i+1}$  by the edge  $v_i^+v_{i+1}^-$  (Fig. 2.1.5). The resulting bipartite graph  $G'$  is  $k$ -regular, so by Corollary 2.1.4 it has a 1-factor. Collapsing every vertex pair  $(v^-, v^+)$  back into a single vertex  $v$ , we turn this 1-factor of  $G'$  into a 2-factor of  $G$ .  $\square$

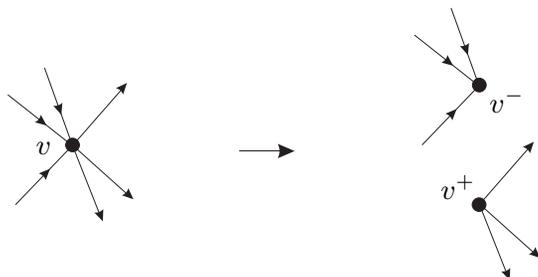


Fig. 2.1.5. Splitting vertices in the proof of Corollary 2.1.5

## 2.2 Matching in general graphs

Given a graph  $G$ , let us denote by  $\mathcal{C}_G$  the set of its components, and by  $q(G)$  the number of its *odd components*, those of odd order. If  $G$  has a 1-factor, then clearly

$$q(G - S) \leq |S| \quad \text{for all } S \subseteq V(G),$$

since every odd component of  $G - S$  will send a factor edge to  $S$ .

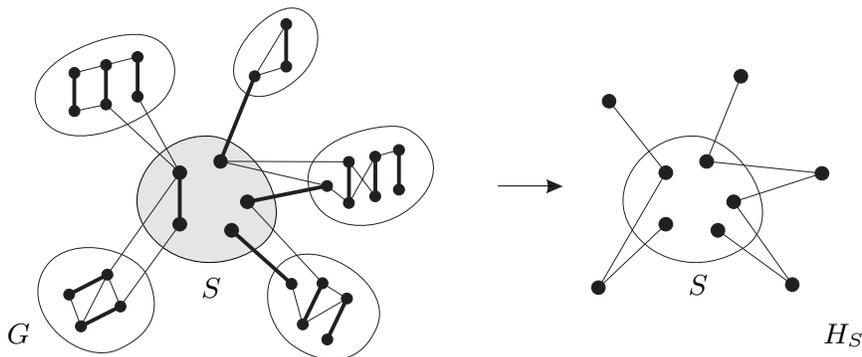
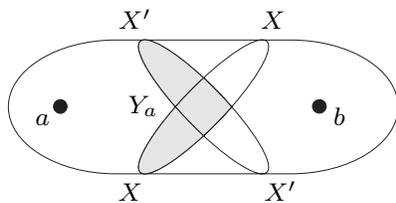


Fig. 2.2.1. Tutte's condition  $q(G - S) \leq |S|$  for  $q = 3$ , and the contracted graph  $H_S$  from Theorem 2.2.3.

*Not all the edges in the components of  $G - S$  are shown.*

Again, this obvious necessary condition for the existence of a 1-factor is also sufficient:



3. Do  $Y_a$  and  $Y_b$  separate  $a$  from  $b$  minimally if  $X$  and  $X'$  do? Are  $|Y_a|$  and  $|Y_b|$  minimum for vertex sets separating  $a$  from  $b$  if  $|X|$  and  $|X'|$  are?
- 4.<sup>+</sup> Suppose that  $X$  and  $X'$  separate  $a$  from  $b$  minimally, and that  $X$  meets at least two components of  $G - X'$ . Show that  $X'$  meets all the components of  $G - X$ , and that  $X$  meets all the components of  $G - X'$ .
- 5.<sup>-</sup> Prove the elementary properties of blocks mentioned at the beginning of Section 3.1.
6. Show that the block graph of any connected graph is a tree.
7. Show, without using Menger's theorem, that any two vertices of a 2-connected graph lie on a common cycle.
8. For edges  $e, e' \in G$  write  $e \sim e'$  if either  $e = e'$  or  $e$  and  $e'$  lie on some common cycle in  $G$ . Show that  $\sim$  is an equivalence relation on  $E(G)$  whose equivalence classes are the edge sets of the non-trivial blocks of  $G$ .
9. Let  $G$  be a 2-connected graph but not a triangle, and let  $e$  be an edge of  $G$ . Show that either  $G - e$  or  $G/e$  is again 2-connected.
10. Let  $G$  be a 3-connected graph, and let  $xy$  be an edge of  $G$ . Show that  $G/xy$  is 3-connected if and only if  $G - \{x, y\}$  is 2-connected.
11. (i) Show that every cubic 3-edge-connected graph is 3-connected.  
 (ii) Show that a graph is cubic and 3-connected if and only if it can be constructed from a  $K^4$  by successive applications of the following operation: subdivide two edges by inserting a new vertex on each of them, and join the two new subdividing vertices by an edge.
- 12.<sup>-</sup> Show that Menger's theorem is equivalent to the following statement. For every graph  $G$  and vertex sets  $A, B \subseteq V(G)$ , there exist a set  $\mathcal{P}$  of disjoint  $A$ - $B$  paths in  $G$  and a set  $X \subseteq V(G)$  separating  $A$  from  $B$  in  $G$  such that  $X$  has the form  $X = \{x_P \mid P \in \mathcal{P}\}$  with  $x_P \in P$  for all  $P \in \mathcal{P}$ .
13. Work out the details of the proof of Corollary 3.3.4 (ii).
14. Let  $k \geq 2$ . Show that every  $k$ -connected graph of order at least  $2k$  contains a cycle of length at least  $2k$ .
15. Let  $k \geq 2$ . Show that in a  $k$ -connected graph any  $k$  vertices lie on a common cycle.

*$X$  and  $X'$  should be minimal as separating vertex sets in  $G$ , not just with respect to two fixed vertices  $a$  and  $b$ .*

contrary to (1). Hence the neighbour of  $v_i$  on  $P$  is its only neighbour in  $C_{i,j}$ , and similarly for  $v_j$ . Thus if  $C_{i,j} \neq P$ , then  $P$  has an inner vertex with three identically coloured neighbours in  $H$ ; let  $u$  be the first such vertex on  $P$  (Fig. 5.2.1). Since at most  $\Delta - 2$  colours are used on the neighbours of  $u$ , we may recolour  $u$ . But this makes  $P\hat{u}$  into a component of  $H_{i,j}$ , contradicting (2).

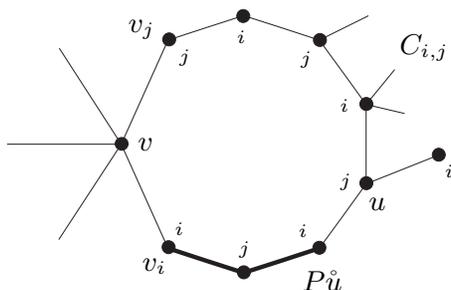


Fig. 5.2.1. The proof of (3) in Brooks's theorem

For distinct  $i, j, k$ , the paths  $C_{i,j}$  and  $C_{i,k}$  meet only in  $v_i$ . (4)

For if  $u \neq v_i \in C_{i,j} \cap C_{i,k}$ , then  $u$  has two neighbours coloured  $j$  and two coloured  $k$ , so we may recolour  $u$ . In the new colouring,  $v_i$  and  $v_j$  lie in different components of  $H_{i,j}$ , contrary to (2).

The proof of the theorem now follows easily. If the neighbours of  $v$  are pairwise adjacent, then each has  $\Delta$  neighbours in  $N(v) \cup \{v\}$  already, so  $G = G[N(v) \cup \{v\}] = K^{\Delta+1}$ . As  $G$  is complete, there is nothing to show. We may thus assume that  $v_1 v_2 \notin G$ , where  $v_1, \dots, v_\Delta$  derive their names from some fixed  $\Delta$ -colouring  $c$  of  $H$ . Let  $u \neq v_2$  be the neighbour of  $v_1$  on the path  $C_{1,2}$ ; then  $c(u) = 2$ . Interchanging the colours 1 and 3 in  $C_{1,3}$ , we obtain a new colouring  $c'$  of  $H$ ; let  $v'_i, H'_{i,j}, C'_{i,j}$  etc. be defined with respect to  $c'$  in the obvious way. As a neighbour of  $v_1 = v'_3$ , our vertex  $u$  now lies in  $C'_{2,3}$ , since  $c'(u) = c(u) = 2$ . By (4) for  $c$ , however, the path  $\hat{v}_1 C_{1,2}$  retained its original colouring, so  $u \in \hat{v}_1 C_{1,2} \subseteq C'_{1,2}$ . Hence  $u \in C'_{2,3} \cap C'_{1,2}$ , contradicting (4) for  $c'$ .  $\square$

As we have seen, a graph  $G$  of large chromatic number must have large maximum degree: at least  $\chi(G) - 1$ . What else can we say about the structure of graphs with large chromatic number?

One obvious possible cause for  $\chi(G) \geq k$  is the presence of a  $K^k$  subgraph. This is a local property of  $G$ , compatible with arbitrary values of global invariants such as  $\varepsilon$  and  $\kappa$ . Hence, the assumption of  $\chi(G) \geq k$  does not tell us anything about those invariants for  $G$  itself. It does, however, imply the existence of a subgraph where those invariants are large: by Corollary 5.2.3,  $G$  has a subgraph  $H$  with  $\delta(H) \geq k - 1$ , and hence by Theorem 1.4.2 a subgraph  $H'$  with  $\kappa(H') \geq \frac{1}{4}(k - 1)$ .

$$\lfloor \frac{1}{4}(k - 1) \rfloor$$

- 12.<sup>+</sup> Let  $1 \leq r \leq n$  be integers. Let  $G$  be a bipartite graph with bipartition  $\{A, B\}$ , where  $|A| = |B| = n$ , and assume that  $K_{r,r} \not\subseteq G$ . Show that

$$\sum_{x \in A} \binom{d(x)}{r} \leq (r-1) \binom{n}{r}.$$

Using the previous exercise, deduce that  $\text{ex}(n, K_{r,r}) \leq cn^{2-1/r}$  for some constant  $c$  depending only on  $r$ .

(Hint. For the displayed inequality, count the pairs  $(x, Y)$  such that  $x \in A$  and  $Y \subseteq B$ , with  $|Y| = r$  and  $x$  adjacent to all of  $Y$ . For the bound on  $\text{ex}(n, K_{r,r})$ , use the estimate  $(s/t)^t \leq \binom{s}{t} \leq s^t$  and the fact that the function  $z \mapsto z^r$  is convex.)

13. The *upper density* of an infinite graph  $G$  is the supremum of  $\|H\| \binom{|H|}{2}^{-1}$ , taken over all non-empty finite subgraphs  $H$  of  $G$ . Show that this number always takes one of the countably many values  $0, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots$  (Hint. Erdős-Stone.)
- 14.<sup>-</sup> In the definition of an  $\epsilon$ -regular pair, what is the purpose of the requirement that  $|X| > \epsilon|A|$  and  $|Y| > \epsilon|B|$ ?
- 15.<sup>-</sup> Show that any  $\epsilon$ -regular pair in  $G$  is also  $\epsilon$ -regular in  $\overline{G}$ .
16. Prove the regularity lemma for sparse graphs, that is, for every sequence  $(G_n)_{n \in \mathbb{N}}$  of graphs such that  $\|G_n\|/n^2 \rightarrow 0$  as  $n \rightarrow \infty$ .

*The upper density of  $G$  should be defined as the infimum of all reals  $\alpha$  such that the finite graphs  $H \subseteq G$  with  $\|H\| \binom{|H|}{2}^{-1} > \alpha$  have bounded order.*

## Notes

The standard reference work for results and open problems in extremal graph theory (in a very broad sense) is still B. Bollobás, *Extremal Graph Theory*, Academic Press 1978. A kind of update on the book is given by its author in his chapter of the *Handbook of Combinatorics* (R.L. Graham, M. Grötschel & L. Lovász, eds.), North-Holland 1995. An instructive survey of extremal graph theory in the narrower sense of our chapter is given by M. Simonovits in (L.W. Beineke & R.J. Wilson, eds.) *Selected Topics in Graph Theory 2*, Academic Press 1983. This paper focuses among other things on the particular role played by the Turán graphs. A more recent survey by the same author can be found in (R.L. Graham & J. Nešetřil, eds.) *The Mathematics of Paul Erdős*, Vol. 2, Springer 1996.

Turán's theorem is not merely one extremal result among others: it is the result that sparked off the entire line of research. Our proof of Turán's theorem is essentially the original one; the proof indicated in Exercise 10 is due to Zykov.

Our version of the Erdős-Stone theorem is a slight simplification of the original. A direct proof, not using the regularity lemma, is given in L. Lovász, *Combinatorial Problems and Exercises* (2nd edn.), North-Holland 1993. Its most fundamental application, Corollary 7.1.3, was only found 20 years after the theorem, by Erdős and Simonovits (1966).

The regularity lemma is proved in E. Szemerédi, Regular partitions of graphs, *Colloques Internationaux CNRS 260—Problèmes Combinatoires et*

Proc. Colloq. Math. Soc. János Bolyai (1996). The theorem marks a breakthrough towards a conjecture of Burr and Erdős (1975), which asserts that the Ramsey numbers of graphs with bounded average degree are linear: for every  $d \in \mathbb{N}$ , the conjecture says, there exists a constant  $c$  such that  $R(H) \leq c|H|$  for all graphs  $H$  with  $d(H) \leq d$ . This conjecture has been verified also for the class of planar graphs (Chen & Schelp 1993) and, more generally, for the class of graphs not containing  $K^r$  (for any fixed  $r$ ) as a topological minor (Rödl & Thomas 1996). See Nešetřil's Handbook chapter for references.

Our first proof of Theorem 9.3.1 is based on W. Deuber, A generalization of Ramsey's theorem, in (A. Hajnal, R. Rado & V.T. Sós, eds.) *Infinite and finite sets*, North-Holland 1975. The same volume contains the alternative proof of this theorem by Erdős, Hajnal and Pósa. Rödl proved the same result in his MSc thesis at the Charles University, Prague, in 1973. Our second proof of Theorem 9.3.1, which preserves the clique number of  $H$  for  $G$ , is due to J. Nešetřil & V. Rödl, A short proof of the existence of restricted Ramsey graphs by means of a partite construction, *Combinatorica* **1** (1981), 199–202.

The two theorems in Section 9.4 are due to B. Oporowski, J. Oxley & R. Thomas, Typical subgraphs of 3- and 4-connected graphs, *J. Combin. Theory B* **57** (1993), 239–257.

... with bdd. average degree  
in every subgraph are linear  
 ... all graphs  $H$  with  
 $d(H') \leq d$  for every  $H' \subseteq H$

- 16.<sup>-</sup> For every  $k \geq 1$ , find a threshold function for  $\{G \mid \Delta(G) \geq k\}$ .  
(Hint. This is a result from the text in disguise.)
- 17.<sup>-</sup> Given  $d \in \mathbb{N}$ , is there a threshold function for the property of containing a  $d$ -dimensional cube (see Ex. 2, Ch. 1)? If so, which; if not, why not?
- 18. Show that  $t(n) = n^{-1}$  is also a threshold function for the property of containing *any* cycle.
- 19. Does the property of containing any tree of order  $k$  (for  $k \geq 2$  fixed) have a threshold function? If so, which?
- 20.<sup>+</sup> Given a graph  $H$ , let  $\mathcal{P}$  be the property of containing an induced copy of  $H$ . If  $H$  is complete then, by Corollary 11.4.6,  $\mathcal{P}$  has a threshold function. Show that  $\mathcal{P}$  has no threshold function if  $H$  is not complete.  
(Hint. Show first that no such threshold function  $t = t(n)$  can tend to zero as  $n \rightarrow \infty$ . Then use Exercise 12.)
- 21.<sup>+</sup> Prove the following version of Theorem 11.4.3 for unbalanced subgraphs. Let  $H$  be any graph with at least one edge, and put  $\varepsilon'(H) := \max \{\varepsilon(F) \mid \emptyset \neq F \subseteq H\}$ . Then the threshold function for  $\mathcal{P}_H$  is  $t(n) = n^{-1/\varepsilon'(H)}$ .

(Hint. Imitate the proof of Theorem 11.4.3. Instead of the sets  $\mathcal{H}_i$ , consider the sets  $\mathcal{H}_F^2 := \{(H, H'') \in \mathcal{H}^2 \mid H' \cap H'' = F\}$ . Replace the distinction between the cases of  $i = 0$  and  $i > 0$  by the distinction between the cases of  $\|F\| = 0$  and  $\|F\| > 0$ .)

$(H', H'')$

## Notes

There are a number of monographs and texts on the subject of random graphs. The most comprehensive of these is B. Bollobás, *Random Graphs*, Academic Press 1985. Another advanced but very readable monograph is S. Janson, T. Łuczak & A. Ruciński, *Topics in Random Graphs*, in preparation; this concentrates on areas developed since *Random Graphs* was published. E.M. Palmer, *Graphical Evolution*, Wiley 1985, covers material similar to parts of *Random Graphs* but is written in a more elementary way. Compact introductions going beyond what is covered in this chapter are given by B. Bollobás, *Graph Theory*, Springer GTM 63, 1979, and by M. Karoński, *Handbook of Combinatorics* (R.L. Graham, M. Grötschel & L. Lovász, eds.), North-Holland 1995.

A stimulating advanced introduction to the use of random techniques in discrete mathematics more generally is given by N. Alon & J.H. Spencer, *The Probabilistic Method*, Wiley 1992. One of the attractions of this book lies in the way it shows probabilistic methods to be relevant in proofs of entirely deterministic theorems, where nobody would suspect it. Another example for this phenomenon is Alon's proof of Theorem 5.4.1; see the notes for Chapter 5.

The probabilistic method had its first origins in the 1940s, one of its earliest results being Erdős's probabilistic lower bound for Ramsey numbers (Theorem 11.1.3). Lemma 11.3.2 about the properties  $\mathcal{P}_{i,j}$  is taken from Bollobás's Springer text cited above. A very readable rendering of the proof that,

algorithm exists (no matter how slow) that decides for any given graph whether or not that graph is knotless. To this day, no such algorithm is known. The property of knotlessness, however, is easily ‘seen’ to be hereditary: contracting an edge of a graph embedded in 3-space will not create a knot where none had been before. Hence, by the minor theorem, there *exists* an algorithm that decides knotlessness—even in polynomial (cubic) time!

However spectacular such unexpected solutions to long-standing problems may be, viewing the graph minor theorem merely in terms of its corollaries will not do it justice. At least as important are the techniques developed for its proof, the various ways in which minors are handled or constructed. Most of these have not even been touched upon here, yet they seem set to influence the development of graph theory for many years to come.

### Exercises

1. Let  $\leq$  be a quasi-ordering on a set  $X$ . Call two elements  $x, y \in X$  *equivalent* if both  $x \leq y$  and  $y \leq x$ . Show that this is indeed an equivalence relation on  $X$ , and that  $\leq$  induces a partial ordering on the set of equivalence classes.
2. Let  $(A, \leq)$  be a quasi-ordering, and assume that every descending chain  $a_0 > a_1 > \dots$  in  $A$  is finite. For subsets  $X \subseteq A$  let

$$\text{Forb}_{\leq}(X) := \{ a \in A \mid a \not\leq x \text{ for all } x \in X \}.$$

Show that  $A$  is a well-quasi-ordering if and only if every subset  $B \subseteq A$  closed under  $\geq$  (i.e. such that  $x \leq y \in B \Rightarrow x \in B$ ) can be written as  $B = \text{Forb}_{\leq}(X)$  with some finite  $X \subseteq A$ .

3. Find a quasi-ordering  $(A, \leq)$ , without an infinite antichain, such that *not* every subset  $B \subseteq A$  closed under  $\geq$  has the form  $B = \text{Forb}_{\leq}(X)$ . ... with finite  $X$   
(Compare the previous exercise.)
4. Prove Proposition 12.1.1 and Corollary 12.1.2 directly, without using Ramsey’s theorem.
5. Given a quasi-ordering  $(X, \leq)$  and subsets  $A, B \subseteq X$ , write  $A \leq' B$  if there exists an *order preserving* injection  $f: A \rightarrow B$  with  $a \leq f(a)$  for all  $a \in A$ . Does Lemma 12.1.3 still hold if the quasi-ordering considered for  $[X]^{<\omega}$  is  $\leq'$ ?
6. Show that the relation  $\leq$  between rooted trees defined in the text is indeed a quasi-ordering.
7. Show that the finite trees are not well-quasi-ordered by the subgraph relation.