**CM-Selectors for Pairs of Oppositely Semicontinuous Multifunctions and Some Applications to Strongly Nonlinear Inclusions**

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**Abstract.** We present a new approximate joint selection theorem which unifies Michael’s theorem (1956) on continuous selections and Cellina’s theorem (1969) on continuous $\varepsilon$-approximate selections. More precisely, we show that, given a convex-valued $H$-upper semicontinuous multifunction $F$ and a convex-closed-valued lower semicontinuous multifunction $G$ with $F(x) \cap G(x) \neq \emptyset$, one can find a continuous function $f$ which is both a selection of $G$ and an $\varepsilon$-approximate selection of $F$. We also prove a parametric version of this theorem for multifunctions $F$ and $G$ of two variables $(s, u) \in \Omega \times X$ where $\Omega$ is a measure space. Using this selection theorem, we obtain an existence result for elliptic systems involving a vector Laplacian and a strongly nonlinear multi-valued right-hand side, subject to Dirichlet boundary conditions.

**Keywords:** Joint, continuous and $\varepsilon$-approximate selectors, $H$-upper and lower semicontinuous multifunctions, multifunctions satisfying one-side estimates, Dirichlet elliptic inclusions, multi-valued elliptic systems, problems with strong non-linearities, with lack of compactness and with critical exponents

**AMS subject classification:** Primary 54C65, 54C60, 28B20, 35R70, secondary 47H04, 34A60

**1. Introduction**

The first purpose of this paper is to present a new continuous joint selection theorem (Theorem 2.1) which unifies two known theorems due to E. A. Michael [13] in 1956 and to A. Cellina [7] in 1969. More precisely, we prove that if $F$ is an $H$-upper semicontinuous convex-valued multifunction from $X$ to $2^Y$, $G$ is a lower semicontinuous convex-closed-valued multifunction from $X$ to $2^Y$, and $F(x) \cap G(x) \neq \emptyset$ for all $x \in X$, then there exists a $CM$-selector for the pair $(F, G)$, i.e. there exists a continuous function which is both a selector for $G$ (as in Michael’s theorem) and an $\varepsilon$-approximate selector for $F$ (as in Cellina’s theorem). In the case $G(x) = Y$ Theorem 2.1 reduces to the Cellina

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theorem for $F$. In the case $F(x) \equiv Y$ it reduces to the Michael theorem for $G$. In the general case our theorem can be interpreted as an "intermediate" theorem between the Michael and Cellina theorems.

The notion of $CM$-selectors and the problem of their existence find motivation in our research on the existence of solutions of strongly nonlinear multi-valued problems: nonlinear Hammerstein multi-valued equations (inclusions) and elliptic boundary value problems with strongly nonlinear multi-valued right-hand sides $F$ satisfying some one-sided estimates (the sign condition, the generalized sign condition, the Hammerstein one-sided estimate, etc.). We observed that in such a case each one-sided estimate generates in the multi-valued setting some pair $(F,G)$, where the multifunction $G$ is lower semicontinuous (see Theorems 2.2 and 3.1). The strong nonlinearity of $F(s,x)$ means that we have to consider problems involving $F$ in the cases of the lack of compactness and of critical exponents in the exact non-compact Sobolev embedding theorems (Sobolev's and Pokhozaev-Trudinger's).

The second part of the present paper (Sections 4 and 5) is therefore devoted to some applications of $\varepsilon$-approximate $CM$-selectors as well as to the study of the simplest multi-valued strongly nonlinear problem (inclusion). To this end, we prove a simple parametric version of Theorem 2.1 (see Theorem 3.1) for multifunctions $F$ and $G$ of two variables $(s,x) \in \Omega \times X$ where $\Omega$ is a measure space. Next we apply the result to constructing a sequence of single-valued strongly nonlinear Dirichlet problems $-\Delta u(s) = f_n(s,u(s))$ approximating the original multi-valued strongly nonlinear Dirichlet problem $-\Delta u(s) \in F(s,u(s))$ in an "appropriate" sense such that the functions $f_n(s,x)$ satisfy the same one-sided estimate as the multifunction $F(s,x)$ (see our construction of $CM$-relaxations $f_n(s,x)$ in (5.1) - (5.2) of Step 1 in Section 5; cf. with usual truncated relaxations $f_n(s,x)$ in the proof of [4: Theorem 2] and [15: Formula (28)]).

Finally we formulate and prove an existence theorem (Theorem 4.1) for the above multi-valued strongly nonlinear problem (the simplest inclusion with lack of compactness), emphasizing seven main steps characteristic of our weak convergence analysis via the use of the above $CM$-relaxations (see Steps 1 - 7 in Section 5).

By the way, it is interesting to notice that in the proof of a recent result of Hu and Papageorgiou [10] on a generalization of Browder's degree for strongly nonlinear elliptic inclusions of $(S)_+$ type there is a gap in their construction of approximate single-valued scalar functions $g_\varepsilon(\cdot, \cdot)$ (see [10: p. 24418], where in fact it is impossible to use "line segments to make continuous connections" for defining their auxiliary function $\eta_8^*(r)$). This gap can be closed by using our "applied" $\varepsilon$-approximate $CM$-selection Theorems 2.2 and 3.1 together with Remark 2.1/(2) of Section 2.

2. $CM$-Selectors

For the convenience of the reader, we give the basic definitions and notations following [2, 5]. Let $(X, \rho)$ be a metric space. For $x \in X$, $M \subset X$ and $\varepsilon > 0$ we denote by $d(x,M) = \inf \{ \rho(x,y) : y \in M \}$ the distance from $x$ to $M$, by $U_\varepsilon(M) = \{ y \in X : d(y,M) < \varepsilon \}$ the $\varepsilon$-neighbourhood of $M$ and by $B(x,r) = B_X(x,r)$ the open ball with center $x$ and radius $r$. The distance in the product $X \times Y$ of metric spaces is defined by $d((x,y),(x_1,y_1)) = \rho(x,x_1) + \rho(y,y_1)$.
max\{ρ_X(x, x_1), ρ_Y(y, y_1)\}. We assume that each multifunction considered has non-empty values, unless stated to the contrary. The graph of a multifunction \( F : X \to 2^Y \) is the set \( \text{Gr } F = \{(x, y) \in X \times Y : y \in F(x)\} \). If \( A \subset X \), then \( F(A) \) denotes the set \( \cup_{x \in A} F(x) \).

Let \( X, Y \) be metric spaces. A multifunction \( F : X \to 2^Y \) is called

- **upper or lower semicontinuous at** \( x_0 \) if for any open set \( V \subset Y \) with \( F(x_0) \subset V \) or \( F(x_0) \cap V \neq \emptyset \) one can find an open neighbourhood \( U \subset X \) of \( x_0 \) such that \( F(x) \subset V \) or \( F(x) \cap V \neq \emptyset \), respectively, for all \( x \in U \).

- **upper or lower semicontinuous**, if it is upper or lower semicontinuous, respectively, at every \( x \in X \);

- **H-upper or H-lower semicontinuous at** \( x_0 \) if for any \( \varepsilon > 0 \) one can find \( \delta > 0 \) such that \( F(B(x_0, \delta)) \subset U_\varepsilon(F(x_0)) \) or \( F(x_0) \subset U_\varepsilon(F(x)) \), respectively, for all \( x \in B(x_0, \delta) \).

- **H-upper or H-lower semicontinuous**, if it is H-upper or H-lower semicontinuous at every \( x \in X \), respectively;

If \( F \) is upper semicontinuous, then it is H-upper semicontinuous; the converse is true if \( F \) takes compact values. If \( F \) is H-lower semicontinuous, then \( F \) is lower semicontinuous; the converse is true if \( F \) takes compact values.

If \( Y \) is a normed space, we denote by \( \text{conv } D \) and \( \overline{\text{conv } D} \) the convex hull and the closed convex hull of a subset \( D \) of \( Y \), respectively.

The main purpose of this section is to prove a theorem, which is intermediate between two famous continuous selection theorems: the Michael theorem [13] and the Cellina theorem [7]. To formulate it, there is a need for a new notion which we immediately introduce.

**Definition 2.1.** Let \( F, G : X \to 2^Y \) be two multifunctions, where \( X \) and \( Y \) are metric spaces, and let \( \varepsilon > 0 \) be an arbitrary positive number. By an \( \varepsilon \)-**approximate CM-selector** for the pair \((F, G)\) we mean a continuous function \( f : X \to Y \) which is both a selector for \( G \) (i.e., \( f(x) \in G(x) \) for all \( x \in X \)) and an \( \varepsilon \)-approximate selector (\( \varepsilon \)-selector in short) for \( F \) (i.e. \( \text{Gr } f \subset U_\varepsilon(\text{Gr } F) \)).

**Remark.** If \( Y \) is a normed space, then \( f : X \to Y \) is an \( \varepsilon \)-selector for \( F \) if and only if \( f(x) \in F(B_X(x, \varepsilon)) + B_Y(0, \varepsilon) \) for all \( x \in X \).

**Theorem 2.1.** Let \( X \) be a metric space and \( Y \) a Banach space. Assume that \( F, G : X \to 2^Y \) are multifunctions, \( F \) H-upper semicontinuous with convex values and \( G \) lower semicontinuous with closed convex values, and such that \( F(x) \cap G(x) \neq \emptyset \) for all \( x \in X \). Then for every \( \varepsilon > 0 \) there exists an \( \varepsilon \)-approximate CM-selector for the pair \((F, G)\).

**Proof.** The proof of Theorem 2.1 will be carried out in two steps.

**Step 1:** Suppose first that \( Y \) is a normed space and \( G : X \to 2^Y \) has convex values only. We claim that then for every \( \varepsilon_1, \varepsilon_2 > 0 \) there exists a continuous map \( f : X \to Y \) such that \( \text{Gr } f \subset U_{\varepsilon_1}(\text{Gr } F), f(x) \in U_{\varepsilon_2}(G(x)) \) for every \( x \in X \), and \( f(F) \subset \text{conv } F(X) \).

For the proof fix \( \varepsilon_1, \varepsilon_2 > 0 \). Let \( y_x \) be an arbitrary element of \( F(x) \cap G(x) \) with \( x \in X \). \( F \) is H-upper semicontinuous, so for \( x \in X \) there is \( \delta_1(x) > 0 \) such that
\[ \delta_1(x) < \varepsilon_1 \text{ and } F(B(x, \delta_1(x))) \subset U_{\varepsilon_1}(F(x)). \] The multifunction \( G \) is lower semicontinuous, therefore for \( x \in X \) there exists \( \delta_2(x) > 0 \) such that \( B(y_x, \varepsilon_2) \cap G(x') \neq \emptyset \), i.e. \( y_x \in U_{\varepsilon_2}(G(x')) \) for \( x' \in B(x, \delta_2(x)) \). Denote \( \delta(x) = \min\{\delta_1(x), \delta_2(x)\} \) and \( U_x = B(x, \frac{1}{2}\delta(x)) \) for \( x \in X \). Since \( (U_x)_{x \in X} \) is an open covering of the metric space \( X \) and \( X \) is paracompact by the Stone theorem \([12]\), we can find a locally finite refinement \((W_i)_{i \in I}\) of \((U_x)_{x \in X}\) and a continuous partition of unity \((\phi_i)_{i \in I}\) subordinate to \((W_i)_{i \in I}\). For each \( i \in I \) choose \( x_i \in X \) such that \( \phi_i \equiv 0 \) on \( X \setminus U_{x_i} \). Denote \( \delta(x_i) = \delta_i, U_{x_i} = U_i \) and \( y_{x_i} = y_i \) for \( i \in I \). Define the function \( f : X \to Y \) by \( f(x) = \sum_{i \in I} \phi_i(x) y_i \). Evidently, \( f \) is continuous, and as \( f(x) \) is a convex combination of elements of \( F(X) \), we have \( f(x) \in conv F(X) \) for every \( x \in X \).

Observe that \( f \) is an \( \varepsilon_1 \)-selector of \( F \). Indeed, let \( x \in X \) and denote \( I(x) = \{i \in I : \phi_i(x) \neq 0\} \). The set \( I(x) \) is finite and we have \( f(x) = \sum_{i \in I(x)} \phi_i(x) y_i \). Define \( j \in I(x) \) so that \( \delta_j = \max_{i \in I(x)} \delta_i \). If \( i \in I(x) \), then \( \phi_i(x) > 0 \) and hence \( x \in U_i \). Now

\[ \rho(x_i, x_j) \leq \rho(x_i, x) + \rho(x, x_j) < 2(\frac{1}{2}\delta_j) = \delta_j, \]

so \( x_i \in B(x_j, \delta_i) \), and therefore \( y_i \in U_{\varepsilon_i}(F(x_j)) \) for \( i \in I(x) \). Consequently, \( f(x) \in U_{\varepsilon_1}(F(x_j)) \) as the \( \varepsilon \)-neighbourhood \( U_{\varepsilon_1}(F(x_j)) \) of the convex set \( F(x_j) \) in the normed space \( Y \) is convex. On the other hand, \( x \in B(x_j, \varepsilon_1) \) because \( \delta_i < \varepsilon_1 \). Finally, \( (x, f(x)) \in U_{\varepsilon}(Gr F(x)) \) for every \( x \in X \), i.e. \( f \) is an \( \varepsilon_1 \)-selector for \( F \).

For the proof of the remaining part of our statement, let again \( x \in X \). If \( i \in I(x) \), then \( x \in U_i \) and hence \( y_i \in U_{\varepsilon_2}(G(x)) \). Therefore \( f(x) = \sum_{i \in I(x)} \phi_i(x) y_i \in U_{\varepsilon_2}(G(x)) \) as the set \( G(x) \) and hence also \( U_{\varepsilon_2}(G(x)) \) is convex.

**Step 2:** Assume now that \( Y \) is even a Banach space and that \( G \) takes closed convex values. We claim that for every \( \varepsilon > 0 \) there exists a \( CM \)-selector \( f \) for the pair \((F, G)\). Indeed, fix \( \varepsilon > 0 \). By Step 1 there exists a continuous map \( f_1 : X \to Y \) such that \( Gr f_1 \subset U_{\varepsilon}(Gr F) \) and \( f_1(x) \in U_{\varepsilon}(G(x)) \) for \( x \in X \). Consider the multifunction \( G_1 : X \to 2^Y \) defined by \( G_1(x) = \overline{G(x) \cap B(f_1(x), \frac{\varepsilon}{2})} \). Of course, \( G_1 \) has non-empty closed convex values. Moreover, \( G_1 \) is lower semicontinuous (see, e.g., [5; Proposition 1.1.5]). Thus, by the famous Michael theorem, \( G_1 \) has a continuous selector \( f : X \to Y \). Note that \( f \) is also a selector for \( G \) as \( G_1(x) \subset \overline{G(x)} = G(x) \) for \( x \in X \).

It remains to show that \( Gr f \subset U_{\varepsilon}(Gr F) \). Indeed, let \( x \in X \). Since \((x, f_1(x)) \in U_{\varepsilon}(Gr F) \), we have \( \rho(x', x) < \frac{\varepsilon}{2} \) and \( \rho(y, f_1(x)) < \frac{\varepsilon}{2} \) for some \( x' \in X \) and \( y \in F(x') \). Hence \( \rho(y, f(x)) \leq \rho(y, f_1(x)) + \rho(f_1(x), f(x)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \), because \( f(x) \in G_1(x) \subset \overline{B(f_1(x), \frac{\varepsilon}{2})} \). Thus we have \( d((x, f(x)), (x', y)) < \varepsilon \), and consequently \( f \) is an \( \varepsilon \)-selector for \( F \).}

**Remarks 2.1.** 1) Theorem 2.1 is true also in the more general setting of Cellina’s Theorem 1 from [8], i.e. when \( X \) is a paracompact, uniform space with countable base (in particular, metric space) and \( Y \) is a complete metric, locally convex space (i.e. Fréchet space). The \( L_\phi \)-decomposable nonconvex-valued version of the theorem is valid too (cf. [6]; the results were announced in H. T. Nguyểin [14] and are accepted for publication in [16]).

2) From Theorem 2.1 follows in addition the possibility to construct a \( CM \)-selector \( f \) such that \( f(a) = f_0(a) \) (\( a \in A \)), where \( A \subset X \) is a fixed closed set (in particular, \( A \) is a fixed finite or closed countable set) and \( f_0 : A \to Y \) is a fixed continuous
function such that \( f_0(a) \in F(a) \cap G(a) \) (\( a \in A \)). For a proof put \( G_0(x) = \{ f_0(x) \} \) for \( x \in A \) and \( G_0(x) = G(x) \) for \( x \not\in A \). By [13] (see also [5]), \( G_0 \) is lower semicontinuous just as \( G \). Applying the statement of Theorem 2.1 for the pair \((F,G_0)\), we get its CM-selector \( f \), which clearly is a CM-selector for the pair \((F,G)\) with the additional property \( f(x) = f_0(x) \) (\( x \in A \)). The existence of \( \varepsilon \)-approximate selectors with this property, for an \( H \)-upper semicontinuous multifunction (as in Cellina’s selection theorem [7]) seems to be unnoticed before (see recent references in the books [5, 11], and recent papers, for example [10]), although it is well-known that a lower semicontinuous multifunction of Michael’s theorem has a continuous selector satisfying the additional property.

The following "applied" \( \varepsilon \)-approximate CM-selection theorem (and it together with the above Remark 2.1/(2)) is an example of how Theorem 2.1 can be applied to constructing \( \varepsilon \)-approximate continuous selectors satisfying some additional conditions.

**Theorem 2.2.** Let \( X \) be a Banach space and \( X^* \) be its dual. Assume that \( F : X \to 2^{X^*} \) is a \( H \)-upper semicontinuous multifunction with convex values and that \( g : X \to \mathbb{R} \) is a continuous non-negative function. Define \( G : X \to 2^{X^*} \) by

\[
G(x) = \begin{cases} 
  \{ y \in X^* : \langle x, y \rangle \leq g(x) \} & \text{if } x \neq 0 \\
  X^* & \text{if } x = 0 \text{ and } g(0) > 0 \\
  \{ 0 \} & \text{if } x = 0 \text{ and } g(0) = 0.
\end{cases}
\]

Assume that \( F(x) \cap G(x) \neq \emptyset \) for all \( x \in X \). Then \( G \) is lower semicontinuous, and the pair \((F,G)\) has an \( \varepsilon \)-approximate CM-selector for every \( \varepsilon > 0 \).

**Proof.** It suffices to show that \( G \) satisfies the assumptions of Theorem 2.1. It is clear that \( G \) has non-empty closed convex values. It remains to show that it is lower semicontinuous. Indeed, assume that \( G \) is not lower semicontinuous at some \( x_0 \in X \). Then there exist an open set \( V \subset X^* \) such that \( G(x_0) \cap V \neq \emptyset \) and a sequence \((x_n) \subset X \setminus \{ 0 \} \), which converges to \( x_0 \) and such that \( G(x_n) \cap V = \emptyset \) for \( n \in \mathbb{N} \). Therefore, for every \( y \in V \) we have \( \langle x_n, y \rangle > g(x_n) \) for \( n \in \mathbb{N} \) and hence \( \langle x_0, y \rangle \geq g(x_0) \) by the continuity of \( \langle \cdot, \cdot \rangle \) and \( g \). This is a contradiction if \( g(0) > 0 \) and \( x_0 = 0 \).

Assume now that \( x_0 \neq 0 \) and take \( y_0 \in G(x_0) \cap V \). Then \( B(y_0, r) \subset V \) for some \( r > 0 \). From the above it follows that \( \langle x_0, y_0 \rangle = g(x_0) \). On the other hand, by the classical Hahn-Banach theorem [9], there exists \( z \in X^* \) with \( \| z \| = 1 \) and \( \langle x_0, z \rangle = \| x_0 \| \neq 0 \). Then for \( y = y_0 - \frac{r}{\| z \|} z \) we have \( y \in B(y_0, r) \subset V \) and \( \langle x_0, y \rangle = g(x_0) - \frac{r}{\| z \|} \| x_0 \| < g(x_0) \) which is a contradiction.

The lower semicontinuity of \( G \) at \( x_0 = 0 \) when \( g(0) = 0 \) is obvious, since in this case by definition \( G(0) = \{ 0 \} \). So if \( G(0) \cap V \neq \emptyset \) where \( V \) is open, then \( 0 \in V \). But of course \( 0 \in G(x) \) for every \( x \in X \), hence \( G(x) \cap V \neq \emptyset \) for every \( x \in X \).

Remark that the inequality \( \langle x, y \rangle \leq g(x) \) is called in the literature one-sided estimate.
3. CM-selectors for multifunctions of two variables

Let \((\Omega, \mathcal{A}, \mu)\) be a measure space with a complete, \(\sigma\)-finite measure \(\mu\) on a \(\sigma\)-algebra \(\mathcal{A}\). Let \(X\) and \(Y\) be separable complete metric spaces. A multifunction \(F : \Omega \to 2^X\) with closed values is called measurable if the set \(\{x \in X : F(x) \cap U \neq \emptyset\}\) is measurable for every open subset \(U\) of \(X\). For other equivalent notions of measurability of multifunctions see, e.g., [2, 5].

We recall that \(f : \Omega \times X \to Y\) is called a Carathéodory function if \(f(s, \cdot)\) is continuous for almost all \(s \in \Omega\) and \(f(\cdot, x)\) is measurable for all \(x \in X\). Following e.g. [2], a multifunction \(F : \Omega \times X \to 2^Y\) is called \(H\)-upper Carathéodory if \(F(s, \cdot)\) is \(H\)-upper semicontinuous for almost all \(s \in \Omega\) and \(F(\cdot, x)\) is measurable for all \(x \in X\).

Further, a multifunction \(F : \Omega \times X \to 2^Y\) is called \((\text{mod } 0)\)-measurable if \(F(\cdot, \cdot)\) is measurable on \((\Omega \setminus D_0) \times X\) with respect to the algebra \(\mathcal{A} \times \mathcal{B}(X)\) where \(D_0\) is some measurable set with \(\mu(D_0) = 0\) and \(\mathcal{B}(X)\) is the algebra of all Borel subsets of \(X\). More information concerning multifunctions of two variables can be found e.g. in [2].

The following "applied" \(\varepsilon\)-approximate CM-selection theorem is a parametric version of Theorem 2.2. Note that Remark 2.1/(2) is valid also for Theorem 3.1, and is useful in applications.

**Theorem 3.1.** Let \(F : \Omega \times \mathbb{R}^m \to 2^{\mathbb{R}^m}\) be an \(H\)-upper Carathéodory and \((\text{mod } 0)\)-measurable multifunction, taking convex compact values. Further, let \(g : \Omega \times \mathbb{R}^m \to \mathbb{R}\) be a non-negative Carathéodory function, define \(G : \Omega \times \mathbb{R}^m \to 2^{\mathbb{R}^m}\) by

\[
G(s, x) = \begin{cases} 
\{y \in \mathbb{R}^m : \langle x, y \rangle \leq g(s, x)\} & \text{if } x \neq 0 \\
\{0\} & \text{if } x = 0 \text{ and } g(s, 0) > 0 \\
& \text{if } x = 0 \text{ and } g(s, 0) = 0.
\end{cases}
\]

for any \(s \in \Omega\), and assume that \(F(s, x) \cap G(s, x) \neq \emptyset\) for almost all \(s \in \Omega\) and all \(x \in \mathbb{R}^m\). Then for every positive measurable function \(\varepsilon : \Omega \to \mathbb{R}_+\) there exists a Carathéodory function \(f : \Omega \times \mathbb{R}^m \to \mathbb{R}^m\) such that \(f(s, \cdot)\) is a CM-selector for the pair \((F(s, \cdot), G(s, \cdot))\) with respect to \(\varepsilon(s) > 0\) for almost all \(s \in \Omega\).

**Proof.** Let \(\varepsilon : \Omega \to \mathbb{R}_+\) be an arbitrary measurable positive function. Define \(\hat{F}(s, x) = F(s, \overline{B}(x, 1_2 \varepsilon(s)))\) for \(s \in \Omega\) and \(x \in \mathbb{R}^m\). It is easy to show that \(\hat{F}(s, \cdot)\) is \(H\)-upper semicontinuous and has closed values for almost all \(s \in \Omega\). We contend also that \(\hat{F}(\cdot, x)\) is measurable for all \(x \in X\). In fact, it is known that the properties of \(F\) ensure that the multifunction \(s \mapsto \hat{F}(s, Z(s))\) is measurable for every measurable multifunction \(Z : \Omega \to 2^{\mathbb{R}^m}\) (see, e.g., [2]). Taking \(Z(s) = \overline{B}(x, 1_2 \varepsilon(s))\) for \(s \in \Omega\) and fixed \(x \in \mathbb{R}^m\) we obtain that the multifunction \(F(\cdot, \overline{B}(x, 1_2 \varepsilon(\cdot)))\), and hence also \(\hat{F}(\cdot, x)\) is measurable.

Denote \(H(s, x) = \hat{F}(s, x) \cap G(s, x)\) \((s \in \Omega, x \in \mathbb{R}^m)\) and let \(C(\mathbb{R}^m, \mathbb{R}^m)\) be the separable metric space of all continuous functions from \(\mathbb{R}^m\) into \(\mathbb{R}^m\), with the topology of uniform convergence on compact sets. From Theorem 2.2 it follows that \(H(s, \cdot)\) has a continuous selector for almost all \(s \in \Omega\), so the multifunction \(\Phi : \Omega \to 2^{C(\mathbb{R}^m, \mathbb{R}^m)}\) defined by \(\Phi(s) = \{f \in C(\mathbb{R}^m, \mathbb{R}^m) : f(x) \in H(s, x) \text{ for all } x \in \mathbb{R}^m\}\) has a.e. non-empty values. We noted earlier that \(\hat{F}(s, \cdot)\) is \(H\)-upper semicontinuous for almost all
\( s \in \Omega, \) and therefore for almost all \( s \in \Omega \) and for any continuous function \( f : \mathbb{R}^m \rightarrow \mathbb{R}^m \) the implication
\[
f(r) \in \tilde{F}(s, r) \ (\forall \ r \in \mathbb{Q}^m) \quad \implies \quad f(x) \in \tilde{F}(s, x) \ (\forall \ x \in \mathbb{R}^m)
\]
is valid. In view of this fact, and because \( H \) takes closed values, we can write
\[
\text{Gr} \ \Phi = \left\{(s, f) \in \Omega \times C(\mathbb{R}^m, \mathbb{R}^m) : d(f(x), H(s, x)) = 0 \ (\forall \ x \in \mathbb{R}^m)\right\}
\]
\[= \left\{(s, f) \in \Omega \times C(\mathbb{R}^m, \mathbb{R}^m) : d(f(r), H(s, r)) = 0 \ (\forall \ r \in \mathbb{Q}^m)\right\}
\]
\[= \bigcap_{r \in \mathbb{Q}^m} \left\{(s, f) \in \Omega \times C(\mathbb{R}^m, \mathbb{R}^m) : d(f(r), H(s, r)) = 0\right\}.
\]

Clearly, \( H(\cdot, x) \) is measurable for all \( x \in \mathbb{R}^m \), so for each \( r \) of the countable set \( \mathbb{Q}^m \) the function \( (s, f) \mapsto d(f(r), H(s, r)) \) is a Carathéodory function from \( \Omega \times C(\mathbb{R}^m, \mathbb{R}^m) \) into \( \mathbb{R} \). Hence as is well-known (see, e.g., [2, 5]) we have
\[
\left\{(s, f) \in \Omega \times C(\mathbb{R}^m, \mathbb{R}^m) : d(f(r), H(s, r)) = 0\right\} \in A \otimes B(C(\mathbb{R}^m, \mathbb{R}^m))
\]
where \( B(M) \) is the algebra of all Borel subsets of a metric space \( M \). It follows that \( \text{Gr} \ \Phi \in A \otimes B(C(\mathbb{R}^m, \mathbb{R}^m)) \). By the von Neumann-Aumann selection theorem (see [5]) \( \Phi \) has a measurable selector \( h : \Omega \rightarrow C(\mathbb{R}^m, \mathbb{R}^m) \). Set \( f(s, x) = (h(s))(x) \). Then \( f \) is a Carathéodory function with the desired properties.

Theorem 3.1 (and a more general parametric version of Theorem 2.1) can also be proved in the framework of Fréchet or Banach spaces (by a different but rather complicated technique). Its various modifications cover multi-valued versions of many generalized Hammerstein one-sided estimates and all generalized sign conditions.

4. The Dirichlet problem for multi-valued elliptic differential systems with strong non-linearities

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \ (n \geq 2) \) and \( F : \Omega \times \mathbb{R}^m \rightarrow 2^\mathbb{R}^m \) some multifunction of two variables \( (s, u) \in \Omega \times \mathbb{R}^m \). We shall consider the problem
\[
\begin{align*}
-\Delta_m u(s) &\in F(s, u(s)) \quad \text{for a.a. } s \in \Omega \\
u|_{\partial \Omega} &= 0
\end{align*}
\]
where \( \Delta_m = (\Delta, \ldots, \Delta) \) is an \( m \)-vector Laplacian.

In what follows we shall denote the scalar product and norm in the Euclidean space \( \mathbb{R}^m \) by \((\cdot, \cdot)\) and \(|\cdot|\), respectively, and the scalar product and norm in the Lebesgue space \( L_2 = L_2(\Omega, \mathbb{R}^m) \) by \((\cdot, \cdot)\) and \(|\cdot|\), respectively. As usual, \( H^1 = H^1(\Omega, \mathbb{R}^m) \) is the Sobolev space defined by the norm \(|u|_1 = |u| + |\nabla u|\), while \( H^0 = H^0(\Omega, \mathbb{R}^m) \) is the closure of \( C_0^\infty(\Omega, \mathbb{R}^m) \) with respect to this norm. Denote by \( H^{-1} \) the dual space to \( H^0 \) with respect to the \( L_2 \)-pairing \((\cdot, \cdot)\). Given a Young function \( M : \Omega \times [0, +\infty) \rightarrow [0, +\infty) \),
the term Orlicz space (see, e.g., [17]) will refer to the space \(L_M = L_M(\Omega, \mathbb{R}^m)\) (of all equivalence classes) of all measurable functions \(u\) on \(\Omega\) taking values in \(\mathbb{R}^m\), which is equipped with the Luxemburg norm \(\|u\|_M = \inf\{k > 0 : \int_{\Omega} M(s, \|u(s)\|/k) \, ds \leq 1\}\). In particular, we shall be interested in Orlicz spaces \(X\) with the property that \(X \subset L_2 \subset X'\) where \(X'\) denotes the Köthe associate space of \(X\) (see, e.g., [4]). Remember that if \(M(s, \alpha) = |\alpha|^p\) \((1 \leq p < +\infty)\), we get \(L_M = L_p\).

Throughout this section, we denote by \(Z\) the special Lebesgue or Orlicz space

\[
Z = \begin{cases} 
L_{\frac{2n}{n-2}}^2 & \text{if } n > 2 \\
L_{2n} & \text{if } n = 2 
\end{cases}
\]  

(4.2)

where \(N(s, \alpha) = \exp(|\alpha|^2) - 1\). By the Sobolev exact embedding theorem (the case \(n > 2\)) and the Pokhozaev-Trudinger exact embedding theorem (the case \(n = 2\)), the Sobolev space \(H_0^1\) is always continuously non-compactly embedded into \(Z\) (see, e.g., [18]). By e.g. [4: Lemma 1], \(H_0^1\) is compactly embedded into \(X\), if \(X\) is an Orlicz space such that the space \(Z\) is absolutely continuously embedded into the space \(X\), i.e. the elements of the unit ball of \(Z\) have uniformly absolutely continuous norms in \(X\):

\[
\lim_{\text{mes}(D) \to 0} \sup_{\|u\|_Z \leq 1} \|P_D u\|_X = 0.
\]

Here \(P_D\) denotes the multiplication operator by the characteristic function of a measurable set \(D\). Following e.g. [2], we define the multi-valued superposition operator \(N_F\) by

\[
N_F(u) = \{v : v \text{ is measurable and } v(s) \in F(s, u(s)) \text{ a.e.}\}.
\]

We shall use one of the following acting conditions:

\(\text{(AC}1\) \(m = 1\) (i.e. the case of scalar equations), \(X = Z\), and the multi-valued superposition operator \(N_F\) acts from \(Z\) into \(2^{Z'}\) where \(Z' = L_{\frac{2n}{n+2}}\) if \(n > 2\) and \(Z' = L_{N^*}\) with \(N^*\) the dual to the Young function \(N\) if \(n = 2\).

\(\text{(AC}2\) \(m > 1\) (i.e. the case of a system of equations), and either

(a) \(n > 2\), \(Z \subset X\) strictly, the multi-valued superposition operator \(N_F\) acts from \(X\) into \(2^{Z'}\), \(Z\) is absolutely continuously embedded into \(X\)

or

(b) \(n = 2\), \(Z \subset X\), the multi-valued superposition operator \(N_F\) acts from \(X\) into \(2^{Z'}\), and the equality \(\lim_{\text{mes}(D) \to 0} \sup_{y \in N(x), \|y\|_X \leq r} \langle y, P_D z \rangle = 0\)

holds for each \(z \in Z\) and \(r > 0\).

Later on, we denote by \(\mu_\Delta\) the first Dirichlet eigenvalue of the Laplacian \(-\Delta\) on \(\Omega\).

The main result for this section is the following

**Theorem 4.1.** Let \(Z\) be the space in (4.2) and \(X\) be an Orlicz space such that

\[
H_0^1 \subset Z \subset X \subset L_2 \subset X' \subset Z' \subset H^{-1}
\]

(4.4)

continuously. Suppose condition (AC1) for the case \(m = 1\) and condition (AC2) for the case \(m > 1\). Suppose in addition that the following conditions are satisfied:
1) $F(\cdot, \cdot)$ has non-empty compact convex values and is an $H$-upper Carathéodory as well as an $(\text{mod } 0)$-measurable multifunction.

2) For almost all $s \in \Omega$ there exists $w \in F(s, u)$ such that the one-sided inequality

$$
(u, w) \leq \gamma(u, u) + \delta(s)
$$

holds where $0 < \gamma < \mu_\Delta$ and $\delta \in L_1(\Omega, \mathbb{R})$ is positive.

Then problem (4.1) has at least one solution $u_* \in H_0^1$.

The proof of Theorem 4.1 will be given in Section 5.

**Remarks 4.1.**

1) Sufficient (and necessary) conditions guaranteeing that the multi-valued superposition (Nemytskij) operator $N_F$ acts as desired in conditions (AC1) - (AC2) of Theorem 4.1 are completely analogous to those for the single-valued superposition operator (for the latter case see, e.g., [4]). For example, when $m = 1$ and $n > 2$ we may assume the polynomial growth condition

$$
\sup_{w \in F(s, u)} |w| \leq a(s) + b \left| u \right|^{\frac{n+2}{n-2}}
$$

for some $a \in L_{\frac{n}{n+2}}(\Omega, \mathbb{R})$ and $b \in [0, +\infty)$; when $m = 1$ and $n = 2$ we may assume the analogous non-polynomial exponential growth condition, using the Young function $N$ and its dual Young function $N^\ast$. It is well-known that all the exponents in (4.6) as well as in the above non-polynomial exponential growth condition are critical (and the inclusion under consideration is non-compact-type strongly nonlinear) since they all correspond to the exact continuous non-compact embeddings in the above-mentioned Sobolev/Pokhozaev-Trudinger theorems (see the discussions for the single-valued case, e.g., in [4, 15]).

2) The compact-type nonlinear inclusions were treated, e.g., in [3] and the references cited therein.

3) Analogous existence results are valid for more complicated strongly nonlinear inclusions such as multi-valued versions of strongly nonlinear problems, which were studied, e.g., in [4, 15] and the references cited therein.

**Example 4.1.** Let $n > 2$ and $D \subset \mathbb{R}$ be a fixed closed non-empty set (finite or countable, or uncountable such as a Cantor "middle thirds" set). Put $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1(u) = u^{\frac{n+2}{n-2}}$ for $u \leq 0$ and $\varphi_1(u) = 0$ for $u > 0$, and $\varphi_2(u) = 0$ for $u \in D$ and $\varphi_2(u) = 1$ for $u \not\in D$ (then the set of discontinuity points of $\varphi$ coincides with $D$). Define [2] the so-called (Krasovskij) convexification of the discontinuous $\varphi$ by $\varphi^*(u) = \cap_{\eta > 0} \overline{\varphi}(\varphi([u - \eta, u + \eta]))$. Given any $h \in L_2(\Omega)$ with $h \not\in L_p(\Omega)$ for all $p > n$, Theorem 4.1 allows us to state the solvability result for (4.1) with $F(s, u) = \varphi^*(u) + h(s)$, while the example cannot be treated by [3], and papers cited therein.
5. Proof of Theorem 4.1

We shall divide the proof of Theorem 4.1 into 7 steps.

**Step 1:** By Theorem 3.1 (if $F(s, \cdot)$ is independent of $s$, it is sufficient to apply Theorem 2.2) there exists for each $\varepsilon > 0$ a Carathéodory function $g_\varepsilon(\cdot, \cdot) : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ such that

$$ \text{Gr} \, g_\varepsilon(s, \cdot) \subseteq \left\{ (u, v) \in \mathbb{R}^m \times \mathbb{R}^m : d((u, v), \text{Gr} \, F(s, \cdot)) < \varepsilon \right\} \text{ a.e.} \quad (5.1) $$

and, moreover, $g_\varepsilon$ satisfies the one-sided estimate

$$ (u, g_\varepsilon(s, u)) \leq \gamma(u, u) + \delta(s) \quad (5.2) $$

where $\gamma$ and $\delta(\cdot)$ are the same as in (4.5) and do not depend on $\varepsilon$. Choosing $\varepsilon_n = \frac{1}{n}$ we define so-called **CM-relaxations** $f_n$ by

$$ f_n(s, u) = \begin{cases} \frac{g_{\frac{1}{n}}(s, u)}{n} & \text{if } |g_{\frac{1}{n}}(s, u)| \leq n \\ g_{\frac{1}{n}}(s, u) & \text{if } |g_{\frac{1}{n}}(s, u)| > n. \end{cases} \quad (5.3) $$

By (5.2) and (5.3), $f_n$ are Carathéodory functions and

$$ f_n(s, u) = \theta_n(s, u) g_{\frac{1}{n}}(s, u) \quad (5.4) $$

where $0 < \theta_n(s, u) < 1$ and

$$ (u, f_n(s, u)) \leq \gamma(u, u) + \delta(s). \quad (5.5) $$

**Step 2:** As is well-known (see, e.g., [18]), the operator $L$ generated by the Laplacian $-\Delta_m$ is continuous and invertible from $H^1_0$ into $H^{-1}$, and

$$ \langle Lu, u \rangle \geq \alpha \| u \|_1^2 \quad (u \in H^1_0) \quad (5.6) $$

for some $\alpha > 0$ (see, e.g., [18]). Remember that the solvability of (4.1) in $H^1_0$ means the existence of $u \in H^1_0$ and $v \in N_F(u)$ such that $v \in H^{-1}$ and $Lu = v$. Now we consider the approximate single-valued problem in $H^1_0$

$$ -\Delta_m u(s) = f_n(s, u(s)) \quad \text{a.e.} \quad (5.7) $$

where $f_n$ is defined in (5.4) and satisfies (5.5). Clearly, the single-valued superposition operator $F_n$, where $F_n(x) = f_n(\cdot, x(\cdot))$, maps the space $L_2$ into itself. In the presence of (5.5) via CM-selections, by [4: Lemma 5], the continuous compact operator $L^{-1}F_n$ has a fixed point $u_n \in L_2$ (i.e. $u_n = L^{-1}F_n u_n$) such that

$$ \|u_n\|_{L_2}^2 \leq \|u_n\|_1^2 \leq \frac{d}{c} \quad (5.8) $$
where \( d = \| \delta(\cdot) \|_{L_1} \) and \( c = \alpha (\mu_\Delta - \gamma) \mu_\Delta^{-1} \) do not depend on \( n \in \mathbb{N} \).

**Step 3:** In the presence of (5.8) and (5.5) via CM-selections, by the same argument [4], we deduce that the inequality

\[
\int_{\Omega} |(F_n u_n(s), u_n(s))| \, ds \leq 2\sigma
\]

(5.9)
holds for \( u_n \) from Step 2, where \( \sigma = d(1 + \frac{2}{c}) \) does not depend on \( n \in \mathbb{N} \).

**Step 4:** We claim additionally that for \( u_n \) from Step 2 we have the equality

\[
\lim_{\operatorname{mes}(D) \to 0} \sup_{n} \langle F_n u_n, P_D z \rangle = 0
\]

(5.10)
for each \( z \in Z \). In the presence of (5.9) and (5.5) via CM-selections, this can be verified directly as in the single-valued case [4].

**Step 5:** From (5.10) via the Dunford-Pettis type \( \sigma(Z', Z) \)-weak precompactness and \( \sigma(Z', Z) \)-weak completeness theorems in \( Z' \) (see, e.g., [9, 16]) it follows that there exist some subsequence \( n_k \) and some \( v_s \in Z' \) such that \( \langle F_{n_k} u_{n_k}, z \rangle \to \langle v_s, z \rangle \) for each \( z \in Z \). Further, since \( L^{-1} \) acts continuously from \( H^{-1} \) into \( H_0^1 \) and \( H_0^1 \subset Z \subset Z' \subset H^{-1} \) continuously (see (4.2)), \( L^{-1} \) acts continuously from \( Z' \) into \( Z \). So the dual operator \( (L^{-1})^* \) acts continuously from \( Z' \) into \( Z \). Remark that \( Z' \subset Z \) continuously and \( (Z')^* = (Z')' = Z \), since \( Z \) is a perfect space and \( Z' = (Z')^0 \) (see [17]) by our choice of \( Z \) in (4.2). Therefore, \( (L^{-1})^* \) acts continuously from \( Z' \) into \( Z \), and so \( \langle L^{-1} v, z \rangle = \langle v, (L^{-1})^* z \rangle \) for all \( v, z \in Z' \). Consequently,

\[
\langle L^{-1}(F_{n_k} u_{n_k}), z \rangle = \langle F_{n_k} u_{n_k}, (L^{-1})^* z \rangle \to \langle v_s, (L^{-1})^* z \rangle = \langle L^{-1} v_s, z \rangle
\]

for each \( z \in Z' \), as \( k \to +\infty \). So \( L^{-1}(F_{n_k} u_{n_k}) \) converges in the weak topology \( \sigma(Z, Z') \) to \( L^{-1} v_s \).

**Step 6:** From (5.8) and the Rellich-Kondrashov theorem (see [18]) we get that \( \{u_{n_k}\}_k \) is precompact in \( L_2 \) and in measure. Therefore we may choose a subsequence of \( \{n_k\}_k \), which we shall for simplicity denote again by \( \{n_k\}_k \), such that \( u_{n_k} \) converges in \( L_2 \) to \( u_s \) and \( u_{n_k}(s) \to u_s(s) \) for almost all \( s \in \Omega \), for some measurable function \( u_s \in L_2 \). From (5.8) we get that \( n_k, u_{n_k} \in Z \). From our choice of \( n_k \), we get also that (see Step 5) \( \langle F_{n_k} u_{n_k}, z \rangle \to \langle v_s, z \rangle \) \( (k \to +\infty) \) for each \( z \in Z = (Z')^* \), i.e. \( F_{n_k} u_{n_k} \to v_s \) in the weak topology \( \sigma(Z', (Z')^*) \). Invoking e.g., [1: Theorem 8] about sequential strong-weak continuous dependence, we get \( v_s \in N_F(u_s) \). We draw attention of the reader to the fact that for to apply the theorem it is crucial that in definition (5.3) of the functions \( f_n \) there is involved property (5.1) of CM-selections. Let us remark here that in the single-valued case of \( F(\cdot, \cdot) \) one can get by the Nemytskij theorem (see [4, 15, 17]) also the convergence in measure of \( F_{n_k} u_{n_k} \) to \( N_F(u_s) \) (and so one can get immediately \( v_s = N_F(u_s) \)); in the multi-valued case this is not true.

**Step 7:** From Step 2 we get \( u_{n_k} = L^{-1}(F_{n_k} u_{n_k}) \). From the results of Steps 5 and 6 we then obtain \( u_s = L^{-1} v_s \) (since by the well-known Hahn-Saks-Vitali theorem[9] the limit in measure and the \( \sigma(Z, Z') \)-weak limit coincide in \( Z \)) and \( v_s \in N_F(u_s) \). Therefore \( u_s \in L^{-1} N_F(u_s) \), and so \( Lu_s \in N_F(u_s) \).
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References


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