Sobolev Inequalities
on
Sets with Irregular Boundaries

T. Kilpeläinen and J. Malý

Abstract. We derive (weighted) Sobolev-Poincaré inequalities for $s$-John domains and $s$-cusp domains, both with optimal exponents. These results are obtained as consequences of a more comprehensive criterion.

Keywords: Sobolev inequality, Poincaré inequality, embeddings, weighted Sobolev spaces, John domains, cusp domains

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1. Introduction

It is well known that the Sobolev space $W^{1,p} (\Omega)$ is continuously embedded into $L^q (\Omega)$ if $\Omega$ is a nice bounded domain in $\mathbb{R}^n$ and

$$1 \leq p < \infty \quad \text{and} \quad q(n-p) \leq np. \quad (1.1)$$

This fact, originally due to Sobolev, Gagliardo and Nirenberg, can nowadays be found in textbooks (cf. [12, 17]) and it is stated as the Sobolev-Poincaré inequality

$$\left( \int_\Omega |u - u_\Omega|^q \, dx \right)^\frac{1}{q} \leq C \left( \int_\Omega |\nabla u|^p \, dx \right)^\frac{1}{p}. \quad (1.2)$$

The weighted case of Sobolev’s imbedding has been developed by Nečas [14], Besov, Ilin and Nikol’skii [3, 4], Kufner [7], Maz’ya [12] and others. It is not very difficult to give examples of domains having cusps for which the Sobolev-Poincaré inequality (1.2) fails to hold or the range for its validity differs from (1.1). The question of this embedding in non-smooth domains $\Omega$ is addressed by many authors. To mention but a few, we would like to refer to the books [12, 13], and point out that Besov [1, 2] obtained embeddings in domains satisfying “flexible cone conditions”, Smith and Stegenga [15] proved Poincaré

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inequality with $q = p$ for $s$-John domains (that allow for twisted cusps of the type $t^s$ with certain $s \geq 1$). Maz'ya [10] (see also Labutin [8]) established the optimal embedding for $s$-cusps. Hajlasz and Koskela [5] proved the optimal Sobolev-Poincaré inequality in $s$-John domains with $p = 1$ and the next to the optimal one for $p > 1$. Their result also involves weights. We refer to [5] also for further historical notes and references.

In this note we complete the picture for $s$-John domains and give a proof for the optimal Sobolev-Poincaré inequality in $s$-John domains, thus improving the results in [5] (see Theorem 2.3). We study also the weighted case where the weight is a power of the distance to the boundary. The result is obtained as a consequence of a slightly more general criterion, which may be used to illustrate why the optimal exponent for $s$-John domains is worse than the optimal exponent for domains with a single $s$-cusp. We use Hedberg’s trick on the maximal operator [6], a truncation argument due to Maz’ya [11] and some ideas from Hajlasz and Koskela [5]. The main new ingredient of our proof is a careful grouping of chains around a curve which we call a worm.

The Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$ is denoted by $|E|$. If $u$ is an integrable function defined at least on $E$, we let $u_E$ stand for the average

$$u_E = \frac{\int_E u\,dx}{|E|} = \frac{1}{|E|} \int_E u\,dx.$$

The open $n$-dimensional ball with center at $x$ and radius $r$ is written as $B(x, r) = B_n(x, r)$. We use $\sharp F$ for the cardinality of a set $F$.

2. Main results

This section contains the results with proofs. We start with a general theorem and deduce the $s$-John domain result from it.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. We consider exponents $a, b, p, q$ satisfying

$$a \geq 0, \quad b \geq 1 - n \quad (2.1)$$

$$1 \leq p < q < \infty \quad (2.2)$$

$$\frac{1}{q} \geq \frac{1}{p} - \frac{1}{n}. \quad (2.3)$$

Let $\mu$ the measure on $\mathbb{R}^n$ with

$$\frac{d\mu}{dx} = \begin{cases} \rho^a & \text{in } \Omega \\ 0 & \text{outside } \Omega \end{cases}.$$

Here and in what follows $\rho(x) = \text{dist } (x, \mathbb{R}^n \setminus \Omega)$.

We shall define a worm. This is a pair $(\gamma, \Delta)$, where $\gamma : [0, \ell] \to \Omega$ is a curve joining $y = \gamma(0)$ to $x_0 = \gamma(\ell)$, parametrized by its arc-length, and $\Delta = \{\xi_k\}, 0 = \xi_0 < \xi_1 < \xi_2 < \ldots < \xi_m = \ell$, is a finite partition of $[0, \ell]$. With each worm we associate its
parameters: the number $m$ of the partition intervals $[\xi_{k-1}, \xi_k]$, and for $k = 1, \ldots, m$ the quantities
\[
\ell_k = \xi_k - \xi_{k-1} \\
R_k = \sup \{ |\gamma(t) - y| : t \in [\xi_{k-1}, \xi_k] \} \\
r_k = \inf \{ \rho(\gamma(t)) : t \in [\xi_{k-1}, \xi_k] \}.
\]

**Theorem 2.1.** Let $a, b, p, q$ satisfy (2.1) - (2.3). Let
\[
\frac{1}{q} \geq \frac{n - p + b}{p(n + a)}. 
\]
Suppose that there is a constant $A > 0$ and a point $x_0 \in \Omega$ such that for each $y \in \Omega \setminus B(x_0, \frac{\rho(x_0) \ell}{2})$ there is a worm $(\gamma, \Delta)$ joining $y$ to $x_0$, with parameters $m$, $\{\ell_k\}$, $\{R_k\}$, $\{r_k\}$ and constants $\tau_1, \ldots, \tau_m \in (0, 1)$ (both parameters and $\tau_k$'s may depend on $y$), such that
\[
\rho(y) \leq 3R_k \quad (k = 1, \ldots, m) \\
(1 + A^{-1})\tau_{k-1} \leq \tau_k \leq A\tau_{k-1} \quad (k = 2, \ldots, m) \\
A^{-1} (\mu(B(y, 3R_k)))^{\frac{1}{\ell}} \leq \tau_k \leq A\frac{\ell}{(\mu(B(y, 3R_k)))^{\frac{1}{\ell}}}.
\]
Then there is a constant $C = C(n, p, a, b, A, \Omega) > 0$ such that
\[
\left( \int_{\Omega} |u - \bar{u}_a|^q \rho^q \, dx \right)^{\frac{1}{q}} \leq C \left( \int_{\Omega} |\nabla u|^p \rho^b \, dx \right)^{\frac{1}{p}}
\]
for each $u \in C^1(\Omega)$ where $\bar{u}_a = \frac{1}{\mu(\Omega)} \int_{\Omega} u \, d\mu$.

We start the proof with the following lemma.

**Lemma 2.2.** Suppose that $\Omega$ is a bounded open set. Let $z, z' \in \Omega$ and let $\gamma : [\xi, \xi'] \to \mathbb{R}^n$ be a path of the length $\ell$ that joins $z$ and $z'$. Suppose that $b \geq 1 - n$ and that $\rho \geq r$ on $\gamma$. Let $u \in C^1(\Omega)$. Then
\[
|u_{B(z, 1, \rho(\gamma))} - u_{B(z', 1, \rho(\gamma'))}| \leq Cr^{\frac{b-n}{p}} \ell^{\frac{1-n}{p}} \left( \int_{D_{\gamma}} |\nabla u|^p \rho^b \, dx \right)^{\frac{1}{p}}
\]
where $D_{\gamma} = \cup_{t \in [\xi, \xi']} B\left(\gamma(t), \frac{1}{\rho(\gamma(t))}\right)$.

**Proof.** Write $B = B(z, \frac{1}{\rho(\gamma)})$ and $B' = B(z', \frac{1}{\rho(\gamma')})$. We construct a chain $\{B_i\}$, $B_i = B(z_i, \frac{1}{\rho(\gamma)})$ of balls and denote $\hat{B}_i = B(z_i, \frac{1}{\rho(\gamma)})$. For the construction, it is enough to determine points $t_i$ such that $z_i = \gamma(t_i)$. If $t_1, \ldots, t_{j-1}$ are selected, we find the next as
\[
t_j = \sup \left\{ t \in [t_{j-1}, \xi'] : B\left(\gamma(t), \frac{1}{\rho(\gamma(t))}\right) \cap \hat{B}_{j-1} \neq \emptyset \right\}.
\]
If $t_j = \xi'$, we set $j_{\text{max}} = j$, $t_j = \xi'$ and terminate the construction.
We observe that the balls \( B(z_i, \frac{1}{4} \rho(z_i)) \) \( (i < j_{\text{max}}) \) are disjoint, and since their radii are bounded away from zero and \( \Omega \) is bounded, the sequence really terminates after a finite number of steps. Fix \( x \in \Omega \) and denote \( I(x) = \{ i < j_{\text{max}} : x \in B_i \} \). Let \( i \in I(x) \). Then
\[
\rho(z_i) \leq \rho(x) + |x - z_i| \leq \rho(x) + \frac{1}{2} \rho(z_i) \\
\rho(x) \leq \rho(z_i) + |x - z_i| \leq \rho(z_i) + \frac{1}{2} \rho(z_i)
\]
and thus
\[
\rho(z_i) \leq 2 \rho(x) \quad \text{and} \quad \rho(x) \leq 2 \rho(z_i). \quad (2.9)
\]
For any \( y \in \hat{B}_i \) we have \( |y - x| \leq \rho(z_i) \leq 2 \rho(x) \) which means that \( \bigcup_{i \in I(x)} \hat{B}_i \subset B(x, 2\rho(x)) \). Since \( \hat{B}_i \) \( (i \in I(x)) \) are disjoint, we have
\[
|B(x, \frac{1}{8} \rho(x))| \|I(x)\| \leq \sum_{i \in I(x)} |\hat{B}_i| \leq |B(x, 2\rho(x))|
\]
which implies \( \|I(x)\| \leq 16^n \). Thus we have proven that
\[
\sum_{i=1}^{j_{\text{max}}} \chi_{B_i} \leq 16^n + 1. \quad (2.10)
\]
Next, consider \( i \in \{1, \ldots, j_{\text{max}}\} \) and notice that there is a point \( x \in \overline{\hat{B}_{i-1} \cap \hat{B}_i} \). Then, as above, we infer that (2.9) holds and
\[
B(x, \frac{1}{8} \rho(x)) \subset B(x, \frac{1}{4} \rho(z_{i-1})) \cap B(x, \frac{1}{4} \rho(z_i)) \subset B_{i-1} \cap B_i
\]
\[
B_{i-1} \cup B_i \subset B(x, \rho(z_{i-1})) \cup B(x, \rho(z_i)) \subset B(x, 2\rho(x))
\]
so that
\[
|B_{i-1} \cup B_i| \leq 16^n |B_{i-1} \cap B_i|. \quad (2.11)
\]
Also, it is clear that
\[
\sum_{i=1}^{j_{\text{max}}} \rho(z_i) \leq Ct. \quad (2.12)
\]
Using (2.11) and the Poincaré inequality we have
\[
|u_{B_i} - u_{B_{i-1}}| \leq |u_{B_i} - u_{B_i \cap B_{i-1}}| + |u_{B_i \cap B_{i-1}} - u_{B_{i-1}}|
\]
\[
\leq \int_{B_i \cap B_{i-1}} |u - u_{B_i}| \, dx + \int_{B_i \cap B_{i-1}} |u - u_{B_{i-1}}| \, dx
\]
\[
\leq \frac{|B_{i-1}|}{|B_i \cap B_{i-1}|} \int_{B_i} |u - u_{B_i}| \, dx + \frac{|B_i|}{|B_i \cap B_{i-1}|} \int_{B_{i-1}} |u - u_{B_{i-1}}| \, dx
\]
\[
\leq C \rho(z_i)^{\frac{1}{p}} \left( \int_{B_i} |\nabla u|^p \, dx \right)^{\frac{1}{p}} + C \rho(z_{i-1})^{\frac{1}{p}} \left( \int_{B_{i-1}} |\nabla u|^p \, dx \right)^{\frac{1}{p}}.
\]
Hence we can estimate by using (2.10) and (2.12) that

\[
|u_{B'} - u_B| \leq \sum_{i=2}^{j_{\text{max}}} |u_{B_i} - u_{B_{i-1}}|
\]

\[
\leq C \sum_{i=1}^{j_{\text{max}}} \rho(z_i)^{1 - \frac{n}{p}} \left( \int_{B_i} |\nabla u|^p dx \right)^{\frac{1}{p}}
\]

\[
\leq C \sum_{i=1}^{j_{\text{max}}} \rho(z_i)^{1 - \frac{n}{p} + \frac{1 - n - b}{p}} \left( \int_{B_i} (\rho(z_i)^2 |\nabla u|^p dx \right)^{\frac{1}{p}}
\]

\[
\leq C \sum_{i=1}^{j_{\text{max}}} r^{1 - \frac{n - b}{p}} \rho(z_i)^{1 - \frac{1}{p}} \left( \int_{B_i} \rho^b |\nabla u|^p dx \right)^{\frac{1}{p}}
\]

\[
\leq C r^{1 - \frac{n - b}{p}} \left( \sum_{i=1}^{j_{\text{max}}} \rho(z_i) \right)^{1 - \frac{1}{p}} \left( \int_{\Omega} \rho^b |\nabla u|^p dx \right)^{\frac{1}{p}}
\]

\[
\leq C r^{1 - \frac{n - b}{p}} \left( \int_{D_n} \rho^b |\nabla u|^p dx \right)^{\frac{1}{p}}
\]

(2.13)

since \( b + n \geq 1 \). The lemma is proven.

**Proof of Theorem 2.1.** Denote \( B_0 = B(x_0, \frac{1}{2} \rho(x_0)) \) and let \( u \in C^1(\Omega) \). We may assume that

\[
|\{u \geq 0\} \cap B_0| \geq \frac{1}{2} |B_0| \quad \text{and} \quad |\{u \leq 0\} \cap B_0| \geq \frac{1}{2} |B_0|.
\]

(2.14)

We will also assume as we may that

\[
\int_{\Omega} |\nabla u|^p \rho^b dx = 1.
\]

(2.15)

We shall first establish a weak type estimate

\[
\mu(A_\lambda) \leq C \lambda^{-q},
\]

(2.16)

where \( A_\lambda = \{x \in \Omega : |u(x)| > \lambda\} \) and \( \lambda > 0 \). First observe that since the median of \( u \) is zero in \( B_0 \) by (2.14), we have

\[
\int_{B_0} |u|^p dx \leq c \int_{B_0} |\nabla u|^p dx
\]

(2.17)

(see [17: Theorem 4.4.4]). Hence

\[
|u_{B_0}| \leq \left( \int_{B_0} |u|^p dx \right)^{\frac{1}{p}} \leq c \left( \int_{B_0} |\nabla u|^p dx \right)^{\frac{1}{p}} \leq c_0,
\]

(2.18)
where \( \epsilon_0 \) is independent of \( u \). Since \( \mu(\Omega) < \infty \) it suffices to establish (2.16) for \( \lambda > 3\epsilon_0 \).

To do so, we fix \( \lambda > 3\epsilon_0 \) and divide \( A_\lambda \) into three parts: write \( B_y = B(y, \frac{1}{2}\rho(y)) \) and let

\[
E_\lambda = \{ y \in A_\lambda \setminus B_0 : |u_{B_y}| > \frac{1}{2}\lambda \}
\]

\[
F_\lambda = A_\lambda \setminus (B_0 \cup E_\lambda).
\]

The third part is

\[
A_\lambda \cap B_0.
\]

We treat \( E_\lambda \) first. Fix \( y \in E_\lambda \) and let \( (\gamma, \{\xi_k\}) \) be a worm in \( \Omega \) that connects \( y \) to \( x_0 \), with parameters \( m, \{\ell_k\}, \{R_k\}, \{r_k\} \), and obeys the bounds of the theorem. We apply Lemma 2.2 to paths \( \gamma_k = \gamma|_{[\xi_k-1,\xi_k]} \) and points \( z = z_k = \gamma(\xi_k-1) \) and \( z' = z'_k = \gamma(\xi_k) \). Let \( x = \gamma(t) \) with \( t \in [\xi_k-1,\xi_k] \). Then by (2.5)

\[
\rho(x) \leq \rho(y) + |y - x| \leq 4R_k
\]

and thus

\[
B(x, \frac{1}{2}\rho(x)) \subset B(y, R_k + 2R_k)
\]

\[
D_{\gamma_k} \subset B(y, 3R_k).
\]

Since \( \lambda > 3\epsilon_0 \), we have

\[
\lambda \leq 6 |u_{B_y} - u_{B_0}|
\]

\[
\leq 6 \sum_{k=1}^{m} |u_{B_{\gamma_k}} - u_{B_{\gamma_k}}|
\]

\[
\leq C \sum_k r_k^{\frac{1-k-n}{p}} \ell_k^{\frac{p-1}{p}} \left( \int_{B(y,3R_k)} \rho^{h-a} |\nabla u|^p \right)^{\frac{1}{p}}.
\]

We split the last sum into two parts by \( K = K(y) \) that is to be determined. First we notice that by (2.6) and (2.2)

\[
\sum_{k > K} \tau_k^{-1} \leq C \tau_{K+1}^{-1} \quad \text{and} \quad \sum_{k \leq K} \tau_k^{\frac{p}{p-1}} \leq C \tau_K^{\frac{p}{p-1}}.	ag{2.19}
\]

If \( K < m \), due to our normalization of \( u \), (2.7) and (2.19) we have

\[
\sum_{k > K} r_k^{\frac{1-k-n}{p}} \ell_k^{\frac{p-1}{p}} \left( \int_{B(y,3R_k)} \rho^{h-a} |\nabla u|^p \right)^{\frac{1}{p}}
\]

\[
\leq \left( \int_{\Omega} \rho^{h} |\nabla u|^p dx \right)^{\frac{1}{p}} \sum_{k > K} r_k^{\frac{1-k-n}{p}} \ell_k^{\frac{p-1}{p}}
\]

\[
= \sum_{k > K} r_k^{\frac{1-k-n}{p}} \ell_k^{\frac{p-1}{p}}
\]

\[
\leq C \sum_{k > K} \tau_k^{-1}
\]

\[
\leq C \tau_{K+1}^{-1}.	ag{2.20}
\]
Before treating the second part of the sum, we set

\[ f = |\nabla u|^p \rho^{\beta - a} \quad \text{and} \quad g(x) = \left( \sup_{r > 0} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} f \, d\mu \right)^{\frac{1}{p'}}.
\]

Since the maximal operator with respect to \( \mu \) is of weak type \( (1, 1) \) (see, e.g., \[9: \text{Theorem 2.19}\] or \[16: \text{p. 44/1.8.17}\]) and \( \|f\|_{L^1(\mu)} = 1 \), we have

\[ \mu(\{g^p > t\}) \leq C \frac{1}{t} \quad \text{for} \quad 0 < t < \infty. \quad (2.21) \]

We estimate

\[
\begin{align*}
\sum_{k \leq K} r_k^{1 - \frac{\beta - a}{p}} \ell_k^{p - 1} \left( \int_{B(y, 3R_k)} \rho^{\beta - a} |\nabla u|^p d\mu \right)^{\frac{1}{p'}} & \\
& \leq \sum_{k \leq K} r_k^{1 - \frac{\beta - a}{p}} \ell_k^{p - 1} \left( \mu(B(y, 3R_k)) \right)^{\frac{1}{p'}} g(y) \\
& \leq C \sum_{k \leq K} \tau_k^{-1} \tau_k^{\frac{\beta}{p}} g(y) \\
& \leq C \tau_K^{-1 + \frac{\beta}{p}} g(y).
\end{align*}
\]

Now we specify the choice of \( K \), distinguishing three cases. If \( \tau_1^{-\frac{\beta}{p}} \leq g(y) \), we choose \( K = 0 \). Then the sum over all \( k = 1, \ldots, m \) reduces to (2.20) and we have \( \lambda \leq C\tau_1^{-1} \leq C g(y)^{\frac{\beta}{p}} \). If \( \tau_m^{-\frac{\beta}{p}} \geq g(y) \), we choose \( K = m \). Then the sum over \( k = 1, \ldots, m \) is treated in (2.22), and we have

\[ \lambda \leq C\tau_m^{-1 + \frac{\beta}{p}} g(y) \leq C g(y)^{\frac{\beta}{p} - 1} g(y) = C g(y)^{\frac{\beta}{p}}. \]

The remaining case is that \( \tau_m^{-\frac{\beta}{p}} < g(y) < \tau_1^{-\frac{\beta}{p}} \). Then we choose the integer \( K < m \) so that \( \tau_{K+1}^{-\frac{\beta}{p}} \leq g(y) < \tau_K^{-\frac{\beta}{p}} \). Using (2.20) and (2.22) we obtain

\[ \lambda \leq C\tau_{K+1}^{-1} + C\tau_K^{-1 + \frac{\beta}{p}} g(y) \leq C g(y)^{\frac{\beta}{p}}. \]

Hence we always have \( \lambda \leq C g(y)^{\frac{\beta}{p}} \) for every \( y \in E_\lambda \). Therefore by (2.21)

\[ \mu(E_\lambda) \leq \mu(\{g^p > (\lambda/C)^q\}) \leq C\lambda^{-q}. \quad (2.23) \]

Next, we estimate the measure of \( F_\lambda \). Using the Besicovitch covering theorem (cf. \[9: \text{Theorem 2.7}\]) we can cover \( F_\lambda \) with balls \( B_{x_i} = B(x_i, \frac{1}{2} \rho(x_i)) \) so that \( x_i \in F_\lambda \) and
\[
\sum_i \chi_{B_{r_{x_i}}} \leq N. \text{ Then } |u - u_{B_{x_i}}| \geq \frac{1}{2} \lambda \text{ on } F_{\lambda} \text{ whence we have by using the Sobolev-Poincaré inequality that }
\]

\[
\mu(F_{\lambda}) \leq \sum_i \mu(B_{x_i} \cap F_{\lambda}) \leq \sum_i \int_{B_{x_i} \cap F_{\lambda}} \rho^a dx \leq C \sum_i \rho(x_i)^a \int_{B_{x_i} \cap F_{\lambda}} |u - u_{B_{x_i}}|^q dx \leq \lambda^{-q} \sum_i \rho(x_i)^{a+q+n(1-\frac{a}{p})} \left( \int_{B_{x_i}} |\nabla u|^p dx \right)^{\frac{q}{p}} \leq C \lambda^{-q} \left( \int_{\Omega} |\nabla u|^p \rho^b dx \right)^{\frac{q}{p}} \leq C \lambda^{-q}.
\]

since \( p(\frac{a+n}{q} + 1 - \frac{n}{p}) \geq b \) by (2.4).

Finally, combining (2.17) and the usual Sobolev inequality in the ball \( B_0 \), we obtain the weak type estimate \( \mu(A_{x} \cap B_0) \leq C \lambda^{-q} \). Hence by estimates (2.23) and (2.24)

\[
\mu(A_{x}) \leq \mu(E_{\lambda}) + \mu(F_{\lambda}) + \mu(A_{x} \cap B_0) \leq C \lambda^{-q}.
\]

In conclusion, (2.16) holds for all \( \lambda > 0 \) or, without normalization (2.15),

\[
\sup_{\lambda > 0} \lambda \mu(\{|u| > \lambda\})^{\frac{1}{q}} \leq C \left( \int_{\Omega} |\nabla u|^p \rho^b dx \right)^{\frac{q}{p}}. \tag{2.25}
\]

A truncation argument shows that the weak type estimate (2.25) implies the desired embedding. Indeed, for each \( t > 0 \) the truncated functions

\[
u_t(x) = \begin{cases} 
\frac{1}{2}t & \text{if } |u(x)| > t \\
|u(x)| - \frac{1}{2}t & \text{if } \frac{1}{2}t < |u(x)| < t \\
0 & \text{if } |u(x)| < \frac{1}{2}t 
\end{cases}
\]

satisfy (2.14). Thus we may use (2.25) to conclude

\[
\left( \int_{\{|u| > 2t\}} |u|^q d\mu \right)^{\frac{1}{q}} \leq C t \mu(\{|u| > t\})^{\frac{1}{q}} \leq C t \mu(\{|u_t| \geq \frac{1}{2}t\})^{\frac{1}{q}}
\]
\[
\leq C \left( \int_{\Omega} |\nabla u|^p \rho^b \, dx \right)^{\frac{1}{p}}
\]

\[
= C \left( \int_{\{\frac{1}{2}t < |u| \leq t\}} |\nabla u|^p \rho^b \, dx \right)^{\frac{1}{p}}.
\]

Hence

\[
\int_{\Omega} |u|^q \rho^a \, dx \leq \sum_{j=-\infty}^{\infty} \int_{\{2^j < |u| \leq 2^{j+1}\}} |u|^q \rho^a \, dx
\]

\[
\leq C \sum_{j=-\infty}^{\infty} \left( \int_{\{2^{j-1} < |u| \leq 2^j\}} |\nabla u|^p \rho^b \, dx \right)^{\frac{q}{p}}
\]

\[
\leq C \left( \int_{\Omega} |\nabla u|^p \rho^b \, dx \right)^{\frac{q}{p}},
\]

and the theorem is proved, since \(\int_{\Omega} |u - \bar{u}_a|^q \rho^a \, dx \leq C \int_{\Omega} |u|^q \rho^a \, dx\)

Following Smith and Stegenga [15] we call a bounded domain \(\Omega\) an \textit{s-John domain} \((s \geq 1)\), if there is a point \(x_0 \in \Omega\) and a constant \(c_0 \geq 1\) such that each point \(x \in \Omega\) can be joined to \(x_0\) in \(\Omega\) by a rectifiable curve (called an \textit{s-John core}) \(\gamma : [0, \ell] \to \Omega\) such that \(\gamma\) is parametrized by the arc length, \(\gamma(0) = x, \gamma(\ell) = x_0\), and dist \((\gamma(t), \partial \Omega) \geq c_0^{-1}t^s\) for all \(t \in [0, \ell]\).

The next theorem improves the main result of [5].

**Theorem 2.3.** Suppose that \(\Omega\) is an \textit{s-John domain}. Let \(a, b, p, q\) satisfy (2.1) - (2.3) and

\[
\frac{1}{q} \geq \frac{s(n + b - 1) - p + 1}{p(n + a)}.
\]

Then there is a constant \(C = C(n, p, q, a, b, \Omega) > 0\) such that

\[
\left( \int_{\Omega} |u - \bar{u}_a|^q \rho^a \, dx \right)^{\frac{1}{q}} \leq C \left( \int_{\Omega} |\nabla u|^p \rho^b \, dx \right)^{\frac{1}{p}}
\]

for each \(u \in C^1(\Omega)\).

**Proof.** We will verify the assumptions of Theorem 2.1. First we notice that \(s \geq 1\) implies

\[
\frac{1}{q} \geq \frac{s(n + b - 1) - p + 1}{p(n + a)} \geq \frac{n + b - p}{p(n + a)}
\]

so that (2.4) is true. For fixed \(y \in \Omega \setminus B(x_0, \frac{1}{2} \rho(x_0))\), the \textit{s-John core} \(\gamma\) on \([0, \ell]\) gives us the desired worm: Let \(d = \sup \{|\gamma(t) - y| : t \in [0, \ell]\}\). Find the integer \(m\) with \(3d < 2^m \rho(y) \leq 6d\). Since

\[
\rho(y) \leq \rho(x_0) + |y - x_0| \leq 3|y - x_0| \leq 3d,
\]

we have \(m \geq 1\). Set

\[
\xi_k = \sup \left\{ t \in [0, \ell] : |\gamma(s) - y| \leq 2^{k-m}d \text{ for all } s \in [0, t] \right\}.
\]
Then \((\gamma, \{\xi_k\})\) is a worm with parameters \(m, \{\ell_k\}, \{R_k\}, \{r_k\}\), and
\[
\begin{align*}
\ell_k & \leq \xi_k \\
\xi_k & \geq R_k = 2^{k-m}d \\
r_k & \geq c_0 \xi_k^a.
\end{align*}
\]
The inequality \(\rho(y) \leq 6 \cdot 2^{-m}d \leq 3R_k\) verifies (2.5). Since
\[
\frac{n + a}{q} \geq \frac{s(n + b - 1) + 1 - p}{p}
\]
we have by choosing \(\tau_k = 2^{(k-m)\frac{n+a}{q}}\) that
\[
\mu(B(y, R_k))^{\frac{1}{q}} \leq R_k \cdot \frac{n+a}{q} \leq C\tau_k
\]
and
\[
\frac{r_k^{\frac{n+a-1}{p}} \ell_k^{\frac{p-1}{p}}}{\tau_k^{\frac{p}{q}}} \leq (c_0 \xi_k^{-s} \frac{n+a-1}{p})^{\frac{p-1}{p}} \leq C\tau_k^{-\frac{n+a}{q}} \leq C\tau_k^{-1}.
\]
Hence the claim follows from Theorem 2.1.

**Remark.** The exponent \(q\) of Theorem 2.3 is the best possible in the class of \(s\)-John domains (see [5]).

**Example 2.4.** An example of an \(s\)-John domain is an \(s\)-cusp domain. Surprisingly, the optimal embedding exponent for the \(s\)-cusp obtained in [8, 10, 13] is better than that for general \(s\)-John domains. The reason is that complicated \(s\)-John domains may contain "rooms and corridors" so that the upper estimate for \(\mu(B(y, R) \cap \Omega)\) must be more carefully examined. We show that the optimal embedding for \(s\)-cusp domains can be deduced from Theorem 2.1. Let us write \(x \in \mathbb{R}^n\) as \(x = (\hat{x}, x^*)\), where \(\hat{x} \in \mathbb{R}^{n-1}\) and \(x^*\) is the last coordinate of \(x\). We will consider the \(s\)-cusp domain
\[
\Omega = \left\{x \in \mathbb{R}^n : |\hat{x}| \leq (x^*)^a \text{ and } 0 < x^* < 2\right\}
\]
and show that if (2.1) - (2.3) are verified, Theorem 2.1 yields embedding of \(W^{1,p}(\Omega, \rho^b)\) into \(L^q(\Omega, \rho^a)\), where
\[
\frac{1}{q} \geq \frac{s(n + b - 1) + p + 1}{p(s(n + a - 1) + 1)}.
\]
We choose \(x_0 = e_n = (0,1)\). If \(y \in \Omega \setminus B(x_0, \frac{1}{2}\rho(x_0))\), we set \(\ell = \ell(y) = |\hat{y}| + |y^* - 1|\) and define the worm curve \(\gamma : [0, \ell] \to \Omega\) as
\[
\gamma(t) = \begin{cases} 
(1 - \frac{t}{|\hat{y}|})\hat{y}, y^* & \text{if } 0 \leq t \leq |\hat{y}| \\
(1 + \frac{\ell - t}{|\hat{y}|}(y^* - 1)) e_n & \text{if } |\hat{y}| \leq t \leq \ell.
\end{cases}
\]
In other words, worm curve starts at \(y\), goes first on line segment connecting \(y\) with \(y^* e_n\) and then turns to the line segment connecting \(y^* e_n\) with \(e_n\). We find a partition \(\{\xi_0, \ldots, \xi_m\}\) of \([0, \ell]\) in such a way that \(\xi_0 = 0\),
\[
\xi_0 = 0, \xi_k = 2^{k-m} \ell \quad (k = 1, \ldots, m)
\]
\[
\rho(y) < \xi_1 < 2\rho(y),
\]
where the last is what determines \( m \) and guarantees (2.5).

From now we treat only the interesting case that \( y^* < 1 \). Then

\[
\ell_1 = \xi_1, \ell_k = \frac{1}{2} \xi_k \quad (k = 2, \ldots, m)
\]

\[
\ell_k^p \leq r_k
\]

\[
\xi_k \leq R_k \leq 2 \xi_k
\]

\[
B(y, R_k) \cap \Omega \subset B_{n-1}(\hat{y}, C \ell_k) \times (y^*- R_k, y^* + R_k)
\]

\[
\rho \leq C \ell_k \quad \text{on } B(y, R_k).
\]

Set \( \tau_k = (\xi_k^{n+a-1} \ell_k)^{\frac{1}{n}} \). It is easy to observe that \( \tau_k \) satisfy (2.6). From (2.26) we obtain

\[
\frac{n+a-1}{p} \ell_k^{\frac{p}{n}} \geq \frac{n+a-1}{q} \ell_k^{\frac{1}{q}} \geq C \tau_k.
\]

The additional information provided by (2.26)\( _4 \)–\( _5 \) has no counterpart in the case of a general \( s \)-John domain. We use it to estimate \( \mu(B(y, 3R_k)) \):

\[
C \mu(B(y, R_k))^{\frac{1}{n}} \leq C(R_k r_k^{n-1+a})^{\frac{1}{n}} \leq C(\xi_k r_k^{n-1+a})^{\frac{1}{n}} \leq C \tau_k.
\]

Hence (2.7) is verified and Theorem 2.1 yields the result.

References


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