De Rham’s Singular Function
and
Related Functions
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Abstract. For de Rham’s singular function we derive new properties, in particular some formulas which express its self-similarity. Inversions and compositions of de Rham’s function are considered as well as generalizations of de Rham’s functional equations which have a connection to the \((3n + 1)\)-iteration of Collatz.

Keywords: De Rham’s singular function, inverse singular functions, compositions of such functions, functional equations, Collatz problem

AMS subject classification: 39 B 22, 39 B 62, 26 A 30, 26 A 48

1. Introduction

It is well known that for a fixed \(a \in (0, 1)\) the system of functional equations

\[
\begin{cases}
\varphi\left(\frac{t}{2}\right) = a \varphi(t) \\
\varphi\left(\frac{t + 1}{2}\right) = a + (1 - a) \varphi(t)
\end{cases} \quad \text{\(t \in [0, 1]\)}
\] (1.1)

has a unique bounded solution. This solution \(\varphi\) is continuous, strictly increasing with \(\varphi(0) = 0\) and it has the representation

\[
\varphi\left(\sum_{j=0}^{\infty} 2^{-\gamma_j}\right) = \sum_{j=0}^{\infty} a^{\gamma_j} q^j
\] (1.2)

where \(q = \frac{1-a}{a}\), \(\gamma_j \in \mathbb{N}\) and \(\gamma_0 < \gamma_1 < \gamma_2 < \ldots\), in particular \(\varphi\left(\frac{1}{2}\right) = a\) and \(\varphi(1) = 1\). In the case of need it will be denoted more precisely by \(\varphi_a\). The case \(a = \frac{1}{2}\) is elementary, namely \(\varphi(t) = t\). However, in the case of \(a \neq \frac{1}{2}\) the solution \(\varphi\) has the interesting property that it is a strictly singular function, i.e., a continuous and strictly increasing function with derivative zero almost everywhere. This solution \(\varphi\) was first constructed by de Rham [9], so that it is called de Rham’s function (cf. [3], where a detailed history of the whole context can be found). Formula (1.2) defines a continuous solution of system (1.1) also in the case of complex \(a\) with \(|a| < 1\) and \(|1-a| < 1\).

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In the following we derive some new properties for de Rham’s singular function \( \varphi \) and for some similar functions which are solutions of generalized de Rham’s functional equations. We consider the self-similarity of de Rham’s function, show that the inverse of a singular function is also singular, and deal with compositions of the studied functions. Finally, a connection to the \((3n + 1)\)-iteration of Collatz is pointed out.

2. De Rham’s singular function

If for \( t \in [0, 1] \) we introduce the dyadic representation \( t = 0.d_1d_2 \cdots \) with \( d_j \in \{0, 1\} \), then according to de Rham [8], the formula \( \varphi(0) = 0 \) and representation (1.2) can be gathered up as

\[
\varphi(t) = \sum_{j=0}^{\infty} a^{j+1} d_{j+1} q^{d_1 + \ldots + d_j} \tag{2.1}
\]

since we have \( d_{j+1} = 1 \) for \( j + 1 = \gamma \) and \( d_{j+1} = 0 \) else, so that \( d_1 + \ldots + d_j = i \). The series in (2.1) appears also in [6], however, in another context. For \( t = \frac{k}{2^n} \) sum (2.1) terminates:

\[
\varphi\left(\frac{k}{2^n}\right) = \sum_{j=0}^{t-1} a^{j+1} d_{j+1} q^{d_1 + \ldots + d_j}. \tag{2.2}
\]

For non-negative integers \( k \) with the dyadic representation \( k = \delta_1 \delta_2 \cdots \delta_n \) (\( \delta_j \in \{0, 1\} \)) we need the binary sum-of-digits function (cf. [1])

\[
\nu(k) = \delta_1 + \ldots + \delta_n \tag{2.3}
\]

which satisfies the equations

\[
\begin{align*}
\nu(2k) &= \nu(k) \\
\nu(2k + 1) &= \nu(k) + 1
\end{align*} \tag{2.4}
\]

Next, we shall show that the terms

\[
x_n(t) = d_{n+1} q^{d_1 + \ldots + d_n} \tag{2.5}
\]

appearing in (2.2) are step functions with special properties, so that (2.1) is a representation of \( \varphi \) by a series of step functions.

**Proposition 2.1.** For \( t \in [0, 1) \) functions (2.5) satisfy the recursions

\[
\begin{align*}
x_0(t) &= 0, & x_{n+1}(t) &= x_n(2t) & \text{for } 0 \leq t < \frac{1}{2} \\
x_0(t) &= 1, & x_{n+1}(t) &= q x_n(2t - 1) & \text{for } \frac{1}{2} \leq t < 1
\end{align*} \tag{2.6}
\]

and if we extend \( x_0(t) \) for \( t \geq 1 \) by

\[
x_0(t) = q^{\nu(k)} x_0(t - k) \tag{2.7}
\]
where \( k \in \mathbb{N} \) and \( k = [t] \), then
\[
x_n(t) = x_0(2^n t)
\]  \hfill (2.8)
for \( 0 \leq t < 1 \).

**Proof.** For the numbers \( d_j \) from the dyadic representation \( t = 0.d_1d_2 \cdots \) let us write \( d_j = d_j(t) \). In the case of \( 0 \leq t < 1 \) we have \( d_1 = 0 \) and \( d_j(2t) = d_{j+1}(t) \) for \( j \in \mathbb{N} \). In the case of \( \frac{1}{2} \leq t < 1 \) we have \( d_1 = 1 \) and \( d_j(2t - 1) = d_{j+1}(t) \). Hence, (2.5) immediately implies (2.6). Solving (2.6) recursively, we find \( x_n(t) = q^{\nu(k)}x_0(2^n t - k) \) for \( k \leq 2^n t \leq k + 1 \leq 2^n \) and according to (2.7) finally (2.8) \( \blacksquare \)

**Proposition 2.2.** The solution \( \varphi \) of system (1.1) satisfies the functional equations
\[
\varphi \left( \frac{k + \tau}{2^\ell} \right) = \varphi \left( \frac{k}{2^\ell} \right) + a^\ell q^{\nu(k)}\varphi(\tau)
\]  \hfill (2.9)
where \( \ell \in \mathbb{N}, k = 0, 1, \ldots, 2^\ell - 1, \tau \in [0,1] \), and for \( t = \frac{k}{2^\ell} \) with \( k = 0, 1, \ldots, 2^\ell \) it has the finite representation
\[
\varphi \left( \frac{k}{2^\ell} \right) = a^\ell \sum_{j=0}^{k-1} q^{\nu(j)}.
\]  \hfill (2.10)

**Proof.** In view of (2.3) representation (2.1) can be written as
\[
\varphi(t) = \sum_{j=0}^{t-1} a^{j+1}d_{j+1}q^{d_1+\ldots+d_j} + a^\ell q^{\nu(k)} \sum_{j=0}^{\infty} a^{j+1}d_{t+j+1}q^{d_{t+1}+\ldots+d_{t+j}}
\]
with \( k = [2^\ell t] \). Substituting \( t = \frac{k+\tau}{2^\ell} \) with \( \tau \in [0,1] \), the first sum on the right-hand side is equal to \( \varphi(\frac{k}{2^\ell}) \) in view of (2.2). Since \( \tau = 2^\ell t - k \) has the dyadic representation \( \tau = 0.d_{t+1}d_{t+2} \cdots \), the last series is equal to \( \varphi(\tau) \), so that (2.9) is proved. Now, in view of \( \varphi(1) = 1 \), representation (2.10) follows from (2.9) with \( \tau = 1 \) by summation \( \blacksquare \)

Note that equations (2.2) and (2.10) are quite different in their external shape. Equation (2.9) has the following counterpart with respect to the left of \( \frac{k}{2^\ell} \):
\[
\varphi_a \left( \frac{k - \tau}{2^\ell} \right) = \varphi_a \left( \frac{k}{2^\ell} \right) - a^\ell q^{\nu(k-1)}\varphi_{1-a}(\tau)
\]  \hfill (2.11)
where \( k = 1, 2, \ldots, 2^\ell \) and \( \tau \in [0,1] \), which can easily be derived from (2.9) by means of the later formula (2.12). Equations (2.9) and (2.11) express very distinctively the self-similarity of de Rham’s function (with respect to the dyadic points), which is well known in the theory of fractals (cf. [5]).

**Proposition 2.3.** The solution \( \varphi_a \) from system (1.1) with \( 0 < t < 1 \) and \( 0 < a < 1 \) is also strictly increasing with respect to \( a \). It has the property
\[
\varphi_{1-a}(t) = 1 - \varphi_a(1-t).
\]  \hfill (2.12)

The family of all curves \( y = \varphi_a(t) \) with \( 0 < a < 1 \) fills out the whole open square \( 0 < t, y < 1 \).
Proof. If $h$ is a differentiable strictly increasing function of $a$ with $0 < h < 1$ for $0 < a < 1$, then the function $a \mapsto a + (1 - a)h(a)$ is strictly increasing. Since $\varphi_a\left(\frac{k}{2^t}\right) = a$ is a strictly increasing polynomial, the specialization of system (1.1)

$$\varphi_a\left(\frac{k}{2^t+1}\right) = a \varphi_a\left(\frac{k}{2^t}\right)$$

$$\varphi_a\left(\frac{2^t + k}{2^{t+1}}\right) = a + (1 - a) \varphi_a\left(\frac{k}{2^t}\right)$$

with $0 < k < 2^t$ shows by induction that all functions $\varphi_a\left(\frac{k}{2^t}\right)$ are also strictly increasing polynomials in $a$. Hence, at arbitrarily fixed $t \in (0,1)$, the function $a \mapsto \varphi_a$ is at least (improper) increasing, and we have to exclude intervals of constancy. In order to do this we show that, for $|a| < 1$ and $|1 - a| < 1$, the function $\varphi_a$ is holomorphic. Namely, choosing $|a| < 1$ and $|1 - a| \leq 1 - \varepsilon < 1$ in representation (1.2) with $t = \sum_{j=0}^{\infty} 2^{-\gamma_j}$ we obtain the estimate

$$|\varphi_a(t)| \leq \sum_{j=0}^{\infty} |a|^{\gamma_j-1} - 1 - a| \leq \sum_{j=0}^{\infty} (1 - \varepsilon)^j = \frac{1}{\varepsilon}$$

in view of $j < \gamma_j$. This implies that series (1.2) of polynomials is uniformly convergent in every compact subset of the domain $\{a : (|a| < 1) \cap (|1 - a| < 1)\}$. Consequently, in this domain $\varphi_a$ is holomorphic. If it would by constant in a certain real interval, then it would be constant everywhere. But this is impossible since in view of $j < \gamma_j$ representation (1.2) implies $\lim_{a \to 0} \varphi_a(t) = 0$ and $\lim_{a \to 1} \varphi_a(t) = 1$ for $0 < t < 1$. Moreover, the both last relations imply in connection with the continuity that the curves fill out the whole open unit square.

If in system (1.1) we replace the constant $a$ by $1 - a$ and $t$ by $1 - t$, we obtain

$$\varphi_{1-a}\left(\frac{1 - t}{2}\right) = (1 - a) \varphi_{1-a}(1 - t)$$

$$\varphi_{1-a}\left(\frac{1 - t}{2}\right) = 1 - a + a \varphi_{1-a}(1 - t),$$

and if we further replace $\varphi_{1-a}(1 - t) = 1 - \varphi(t)$, we again obtain system (1.1), only with interchanged equations. Since in the space of continuous functions system (1.1) is uniquely solvable, the proposition is proved.

Figure 1: The graphs of de Rham’s function for $a = 0.1(0.1)0.9$
Proposition 2.3 is illustrated by means of Figure 1, which shows de Rham’s function for different parameters $a$ (cf. also [6]).

Let us mention a connection to a functional equation, which was studied by Klemmt [4], and which gives us a new possibility to prove that $\varphi$ is a singular function in the case of $a \neq \frac{1}{2}$. The equations in system (1.1) easily imply for $0 < t < 1$

$$\varphi'(\frac{t}{2}) + \varphi'(\frac{t+1}{2}) = 2\varphi'(t)$$

almost everywhere. According to [4], $\varphi'$ must be constant almost everywhere: $\varphi'(t) = c$ with $c \geq 0$. Hence $ct \leq \varphi(t)$, and in view of $\varphi(\frac{1}{2^2}) = a^n$ for $n \in \mathbb{N}$ we obtain $0 \leq c \leq (2a)^n$ and therefore $c = 0$ in the case of $0 < a < \frac{1}{2}$. The case $\frac{1}{2} < a < 1$ can be transferred to the foregoing one by means of (2.12).

3. Related functions

Since de Rham’s function $\varphi$ is continuous and strictly increasing in $t$, its inverse $\varphi^{-1}$ exist and we can deal with it.

**Proposition 3.1.** If $f$ is a strictly singular function, then the inverse $g = f^{-1}$ is also strictly singular.

**Proof.** Since $g$ is strictly increasing, $g$ is differentiable almost everywhere with $g'(\tau) \geq 0$. For arbitrary $0 < \alpha < \beta$ let $E_{\alpha, \beta}$ be the set of all $\tau$ such that $g'(\tau)$ exists and $\alpha \leq g'(\tau) \leq \beta$. Denote by $|E_{\alpha, \beta}|$ the Lebesgue measure of the measurable set $E_{\alpha, \beta}$. According to $f'(g(\tau)) = \frac{1}{g'(\tau)}$ we have $f'(t) \geq \frac{1}{\beta}$ for all $t \in g(E_{\alpha, \beta})$, which implies that $|g(E_{\alpha, \beta})| = 0$ since $f$ is singular. In view of $f'(\tau) \geq \alpha$ for $\tau \in E_{\alpha, \beta}$ we have the estimate $\alpha |E_{\alpha, \beta}| \leq |g(E_{\alpha, \beta})|$ (cf. [7: p. 234]). Consequently, $|E_{\alpha, \beta}| = 0$ for $0 < \alpha < \beta$. Since the set $E$ of all $\tau$ with $g'(\tau) > 0$ is representable as countable union of such sets, we obtain $|E| = 0$. Hence $g' = 0$ almost everywhere.

There is another possibility to prove Proposition 3.1 by means of measure theory. Namely, if $f$ is an increasing singular function, then it generates a Stieltjes measure which is singular with respect to the Lebesgue measure. If, moreover, $x = f(t)$ is continuous and strictly increasing, then the inverse function $t = f^{-1}(x)$ generates automatically also a measure singular to the Lebesgue measure, i.e. $f^{-1}$ is also a singular function.

In particular, for fixed $a \neq \frac{1}{2}$ the inverse $\varphi^{-1}$ of de Rham’s function is also strictly singular with respect to $t$. System (1.1) implies that

$$\begin{align*}
\varphi^{-1}(at) &= \frac{1}{2}\varphi^{-1}(t) \\
\varphi^{-1}(a + (1-a)t) &= \frac{1}{2} + \frac{1}{2}\varphi^{-1}(t)
\end{align*}$$

(3.1)

(cf. [2]). Moreover, from (2.12) we obtain that

$$\varphi^{-1}_{1-a}(t) = 1 - \varphi^{-1}_{a}(1-t)$$

(3.2)
for $0 \leq t \leq 1$.

Systems (1.1) and (3.1) can be generalized by

$$
\begin{align*}
\varphi (ct) &= a \varphi (t) \\
\varphi (c + (1 - c)t) &= a + (1 - a) \varphi (t)
\end{align*}
$$

(t \in [0, 1]) \tag{3.3}

with fixed $0 < a, c < 1$.

**Proposition 3.2.** The following assertions are valid:

(i) The composition $\varphi (t) = \varphi_a (\varphi_c^{-1}(t))$ is the unique bounded solution of the functional equations (3.3).

(ii) This solution is continuous, strictly increasing and maps $[0, 1]$ onto $[0, 1]$.

**Proof.** It can easily be checked that the composition $\varphi_a \varphi_c^{-1}$ satisfies equations (3.3):

$$
\varphi_a (\varphi_c^{-1}(ct)) = \varphi_a \left( \frac{1}{c} \varphi_c^{-1}(t) \right) = a \varphi_a (\varphi_c^{-1}(t))
$$

and

$$
\varphi_a (\varphi_c^{-1}(c + (1 - c)t)) = \varphi_a \left( \frac{1}{2} + \frac{1}{c} \varphi_c^{-1}(t) \right) = a + (1 - a) \varphi_a (\varphi_c^{-1}(t)).
$$

Moreover, $\varphi = \varphi_a \varphi_c^{-1}$ has in fact properties (ii).

Now, let $\varphi$ be a further solution of equations (3.3). For $0 \leq t \leq 1$ we put $d(t) = |\varphi(t) - \varphi_a (\varphi_c^{-1}(t))|$. Assume that there exists a point $t_0 \in [0, 1]$ with $d(t_0) > 0$. If $t_0 < c$, then for $t_1 = \frac{1}{c} t_0$ we have $t_1 \in [0, 1]$ and the first equation of (3.3) implies that $d(t_1) = \frac{1}{c} d(t_0)$. In the case of $c < t_0 < 1$ the point $t_1 = \frac{t_0 - c}{1 - c}$ lies in $[0, 1]$ and from the second equation of (3.3) we obtain that $d(t_1) = \frac{1}{c} d(t_0)$. Putting $m = \min\{\frac{1}{c}, \frac{1}{1 - a}\}$ and

$$
t_{n+1} = \begin{cases} 
\frac{1}{c} t_n & \text{for } 0 \leq t_n \leq c \\
\frac{t_n - c}{1 - c} & \text{for } c < t_n \leq 1
\end{cases}
$$

where $n \in \mathbb{N}$, we get $d(t_n) \geq m^n d(t_0)$. However, in view of $m > 1$ this is a contradiction to the boundedness of $\varphi$. 

**Proposition 3.3.** The solution $\varphi = \varphi_a \varphi_c^{-1}$ of system (3.3) has the representation

$$
\varphi \left( \sum_{j=0}^{\infty} c^{\gamma_j} q_c^j \right) = \sum_{j=0}^{\infty} a^{\gamma_j} q_a^j \tag{3.4}
$$

where $q_a = \frac{1 - a}{a}$, $q_c = \frac{1 - c}{c}$, $\gamma_j \in \mathbb{N}$ and $\gamma_0 < \gamma_1 < \gamma_2 < \ldots$. Moreover, for $\ell \in \mathbb{N}$ and $k = 0, 1, \ldots, 2^\ell - 1$ we have

$$
\varphi \left( \varphi_c \left( \frac{k}{2^\ell} \right) + c^\ell q_c^{\nu(k)} \varphi_c (\tau) \right) = \varphi_a \left( \frac{k}{2^\ell} \right) + a^\ell q_a^{\nu(k)} \varphi_a (\tau) \tag{3.5}
$$

for $0 \leq \tau \leq 1$. In the case of $a \neq c$ the solution $\varphi$ of system (3.3) is strictly singular and its derivative is 0 whenever it exists.
Proof. Representation (3.4) follows from (1.2) in view of \( \varphi(\varphi_c(t)) = \varphi_a(t) \) with \( t = \sum_{j=0}^\infty 2^{-\gamma_j} \). From \( \varphi(\varphi_c(t)) = \varphi_a(t) \) with \( t = \frac{k+\tau}{2^k} \) we also get (3.5) by twofold use of (2.9), but once with \( c \) instead of \( a \).

Let be \( x \in [0, 1] \) such that \( \varphi'(x) \) exists. For \( n \in \mathbb{N} \) choose integers \( k_n \) with \( 0 \leq k_n \leq 2^n - 1 \) and \( x_n = \varphi_c \left( \frac{k_n}{2^n} \right) \), \( y_n = \varphi_c \left( \frac{k_n + 1}{2^n} \right) \) so that \( x_n \leq x \leq y_n \). From (2.9) and (3.5) with \( \tau = 0 \) respectively \( \tau = 1 \) we obtain

\[
D_n = \frac{\varphi(y_n) - \varphi(x_n)}{y_n - x_n} = \frac{a^n q_a^{\varphi(k_n)}}{c^n q_c^{\varphi(k_n)}} \to \varphi'(x) \quad (n \to \infty)
\]

owing to \( \varphi(0) = 0 \) and \( \varphi(1) = 1 \). Now, putting \( z_n = x_n + c(y_n - x_n) \), there are two possibilities, either \( x_n \leq x \leq z_n \) or \( z_n < x \leq y_n \). From (2.9) and (3.5) with \( \tau = 0 \), \( \tau = \frac{1}{2} \) respectively \( \tau = 1 \) we get in view of \( \varphi(c) = a \) that

\[
\frac{\varphi(z_n) - \varphi(x_n)}{z_n - x_n} = \frac{a}{c} D_n \quad \text{and} \quad \frac{\varphi(y_n) - \varphi(z_n)}{y_n - z_n} = \frac{1 - a}{1 - c} D_n .
\]

At least one of both possibilities mentioned before occurs infinitely many times. Consequently, \( \varphi'(x) = \frac{a}{c} \varphi'(x) \) or \( \varphi'(x) = \frac{1 - a}{1 - c} \varphi'(x) \). Hence \( \varphi'(x) = 0 \) in view of \( a \neq c \).}

Denoting the solution of system (3.3) by \( \varphi_{a,c} \), we easily see the validity of the relations

1. \( \varphi_{a,b}(\varphi_{b,c}(t)) = \varphi_{a,c}(t) \)
2. \( \varphi_{a,c}^{-1}(t) = \varphi_{c,a}(t) \)
3. \( \varphi_{a,c}(1 - t) = 1 - \varphi_{1-a,1-c}(t) \)

for arbitrary \( 0 < a, b, c < 1 \).

Next, we consider the generalization of system (1.1)

\[
\begin{align*}
g \left( \frac{t}{2} \right) &= a g(t) & \quad (0 \leq t \leq 1) \\
g \left( \frac{t+1}{2} \right) &= a + c g(t) & \quad (0 < t \leq 1)
\end{align*}
\]

(3.6)

with \( |a| < 1 \) and \( |c| < 1 \). A bounded solution of system (3.6) must have the particular values \( g(0) = 0 \), \( g(1) = \frac{a}{1-c} \) and \( g(\frac{1}{2}) = \frac{a^2}{1-c} \), where also \( g(+0) = 0 \). However, in the case of \( a + c \neq 1 \) it cannot be right-continuous in all points, since \( g(\frac{1}{2} + 0) = a \neq g(\frac{1}{2}) \). However, system (3.6) possesses the left-continuous solution

\[
g \left( \sum_{j=0}^\infty 2^{-\gamma_j} \right) = \sum_{j=0}^\infty a^{\gamma_j} q^j \quad (3.7)
\]

with \( 1 \leq \gamma_j < \gamma_{j+1} \) and \( q = \frac{2^\gamma}{a} \) (cf. (1.2)). On the other side, for

\[
t_n = \sum_{j=0}^n 2^{-\gamma_j} = \sum_{j=0}^{n-1} 2^{-\gamma_j} + \sum_{j=0}^\infty 2^{-\gamma_n - j - 1}
\]

(3.8)
with $n \geq 0$ we have
\[
g(t_n) = \sum_{j=0}^{n-1} a^{\gamma_j} q^j + \sum_{j=0}^{\infty} a^{\gamma_n+j} q^{n+j} \quad \text{and} \quad g(t_n + 0) = \sum_{j=0}^{n} a^{\gamma_j} q^j,
\]
and in view of
\[
\sum_{j=0}^{\infty} a^{\gamma_n+j} q^{n+j} = \frac{a^{\gamma_n+1} q^n}{1 - aq} = \frac{a^{\gamma_n+1} q^n}{1 - c}
\]
consequently
\[
g(t_n + 0) - g(t_n) = \frac{a^{\gamma_n} q^n}{1 - c} (1 - a - c).
\]
Hence, the solution $g$ is discontinuous at all dyadic points, so far as $a \neq 1 - c$. However, it is bounded and Lebesgue integrable as limit of uniformly converging step functions. According to (3.10) it is not increasing for $a + c > 1$.

**Proposition 3.4.** In the case of $0 < a, c$ and $a + c < 1$ the solution $g$ of system (3.6) is strictly increasing and continuous except in the dyadic points $t = t_n$ from (3.8) where
\[
g(t_n - 0) = g(t_n) < g(t_n + 0)
\]
with jumps (3.10). Moreover, $g' = 0$ almost everywhere.

**Proof.** Assuming that $t, t' \in (0, 1]$ have the representations
\[
t = \sum_{j=0}^{\infty} 2^{-\gamma_j} \quad \text{and} \quad t' = \sum_{j=0}^{\infty} 2^{-\gamma_j'}
\]
with $\gamma_j$ as before respectively $\gamma_j'$, then $t > t'$ if and only if there exists an integer $m$ such that $\gamma_j = \gamma_j'$ for $j = 0, ..., m - 1$ and $\gamma_m \geq \gamma_m + 1$. Owing to (3.7) we have
\[
g(t) = \sum_{j=0}^{\infty} a^{\gamma_j} q^j \geq \sum_{j=0}^{m-1} a^{\gamma_j} q^j + a^{\gamma_m} q^m
\]
since $q = \frac{c}{a} > 0$. Moreover, $\gamma_m \geq \gamma_m + 1$ implies that $\gamma_{m+j} \geq \gamma_{m+1} + j$ for all $j \geq 0$ so that in view of $0 < aq = c < 1$ we get
\[
g(t') = \sum_{j=0}^{m-1} a^{\gamma_j} q^j + \sum_{j=m}^{\infty} a^{\gamma_j} q^j \leq \sum_{j=0}^{m-1} a^{\gamma_j} q^j + \sum_{j=0}^{\infty} a^{\gamma_{m+1}+j} q^{m+j}.
\]
Hence, according to (3.9) and $a < 1 - c$ we obtain $g(t) > g(t')$, i.e. $g$ is strictly increasing.

It follows that the intervals $(g(t_n), g(t_n + 0))$ are disjoint. Since the set of all dyadic points is dense in $[0,1]$ the union
\[
G = \bigcup_{\ell=0}^{\infty} \bigcup_{k=0}^{2^{\ell-1}} \left( g\left( \frac{2k+1}{2^{\ell+1}} \right), g\left( \frac{2k+1}{2^{\ell+1}} + 0 \right) \right)
\]
is an open Cantor set with Lebesgue measure

\[ |G| = \sum_{t_n} a^{\gamma_n} q^n \frac{1}{1-c} (1 - a - c) \]

(cf. (3.10) where we have to sum over all dyadic \( t_n \) of \( (0,1) \)). Since there are \( \binom{k}{n} \) possibilities for \( \gamma_n \) to be equal to \( k + 1 \) we find that

\[ \sum_{n=0}^{\infty} a^{\gamma_n} q^n = \sum_{n=0}^{\infty} q^n \sum_{k=n}^{\infty} \binom{k}{n} \frac{a^{k+1}}{k!} = \sum_{k=0}^{\infty} a^{k+1}(1 + q)^k = \frac{a}{1 - a - c} \]

in view of \( aq = c \). Therefore we obtain that \( |G| = \frac{a}{1-c} = g(1) \). Consequently, the increasing function \( g \) cannot have further jumps.

For the set \( M = [0,1] \setminus \{ t_n \} \) we have \( |M| = 1 \) and \( |g(M)| = 0 \) which implies that \( g' = 0 \) almost everywhere (cf. [7: p. 234]). Hence, the proposition is proved

**Remarks.**

1. \( P = [0, \frac{a}{1-c}] \setminus G \) is a perfect Cantor set with measure zero.

2. Note that the boundary points \( t = 0 \) and \( t = 1 \) do not belong to the points (3.8).

3. The results can easily be transferred to the case that the first equation in system (3.6) is valid for \( 0 \leq t < 1 \) and the second equation for \( 0 \leq t \leq 1 \), where the solution is determined by \( g(1) = \frac{a}{1-c} \), \( g\left( \sum_{j=0}^{n} 2^{-\gamma_j} \right) = \sum_{j=0}^{n} a^{\gamma_j} q^j \) and right continuity with \( q \) and \( \gamma_j \) as before.

**Supplement.** Finally, we consider the generalization of systems (1.1) and (3.6)

\[
\begin{align*}
    f\left( \frac{t}{2} \right) &= a f(t) \\
    f\left( \frac{t+1}{2} \right) &= b + c f(t)
\end{align*}
\]

(3.12)

where we admit that the solution is not defined for all \( t \in (0,1) \).

**Proposition 3.5.** For \( a \neq 0 \), \( |a| < 1 \), \( |c| < 1 \) and \( 0 < t \leq 1 \), system (3.12) has the left-continuous solution

\[
    f\left( \sum_{j=0}^{\infty} 2^{-\gamma_j} \right) = \frac{b}{a} \sum_{j=0}^{\infty} a^{\gamma_j} q^j
\]

with \( \gamma_j \in \mathbb{N} \), \( \gamma_j < \gamma_{j+1} \) and \( q = \frac{a}{a} \). If \( 1 < a \), \( 0 < b \) and \( 0 < c < 1 \), then every \( y > f(1) = \frac{b}{1-c} \) has infinitely many inverse images under \( f \).

**Proof.** If \( g \) is the solution (3.7) of system (3.6), then \( f = \frac{b}{a} g \) is the solution of system (3.12). If \( |q| < 1 \), but \( |a| > 1 \), then the right-hand side of (3.13) can diverge, and \( f \) remains undefined at the corresponding points of \( (0,1] \). However, for \( c \neq 1 \) the solution of system (3.12) always possesses the value \( f(1) = \frac{b}{1-c} \). Now, let \( 1 < a \), \( 0 < b \), \( 0 < c < 1 \) and \( y > f(1) \). We look for a sequence \( \gamma_j \) such that

\[
    \frac{b}{a} \left( \sum_{j=0}^{k-1} a^{\gamma_j} q^j + a^{\gamma_k} q^k \right) < y \leq \frac{b}{a} \left( \sum_{j=0}^{k-1} a^{\gamma_j} q^j + a^{\gamma_k+1} q^k \right)
\]

(3.14)
for infinitely many $k$. For $k = 0$ this inequality means

$$\frac{ba^{\gamma_0-1}}{1-c} < y \leq \frac{ba^{\gamma_0}}{1-c}$$

and determines $\gamma_0$ uniquely in view of $a > 1$. If $\gamma_0, \ldots, \gamma_n$ are already determined, we choose an arbitrary integer $k > n$ depending on $n$, define $\gamma_j = \gamma_n + j - n$ for $j = n + 1, \ldots, k - 1$, and determine $\gamma_k$ out of (3.14). The last step is uniquely possible, since in view of

$$a^{\gamma_n}q^n = \sum_{j=n}^{k-1} a^{\gamma_j}q^j + \frac{a^{\gamma_{k-1}+1}q^k}{1-c}$$

there always exits such a $\gamma_k \geq \gamma_{k-1} + 1$. In this way we find infinitely many sequences $\gamma_j$ such that the right-hand side of (3.13) is equal to $y$.

Let $|a| > 1$. If we define $\ell = \lim_{j \to \infty} \frac{d_j}{q}$, the series at the right-hand side of (3.13) is convergent for $\ell < 1 - \frac{\ln |c|}{\ln |a|}$ and divergent for $\ell > 1 - \frac{\ln |c|}{\ln |a|}$ in view of the root test. In the "periodic" case $\gamma_j + k = r_j + q_k$ for sufficiently great $j$ and $k = 0, 1, \ldots, p - 1$, where $t = \sum_{j=0}^{\infty} 2^{-\gamma_j}$ is rational, we have $\ell = \frac{r}{p}$.

From system (3.12) we can derive further functional equations. Namely, for $k = 0, 1, \ldots, 2^\ell - 1$ with $\ell \in \mathbb{N}$, the dyadic representation $k = d_1d_2 \cdots d_n$, $d_j \in \{0, 1\}$, where $d_1 = 0$ is allowed, and $0 < t < 1$ we find

$$f\left(\frac{t+k}{2^\ell}\right) = b \sum_{j=0}^{\ell-1} a^j d_{j+1} t^{d_1 + \cdots + d_j} + a^\ell q^\mu(k) f(t)$$

(cf. (2.9) and (2.2)). For $|a| > 1$ this formula shows that $f$ is unbounded in every subinterval of $(0, 1]$, since $f(\frac{t}{2^n}) = a^n f(t)$.

Let us mention a curious connection to the $(3n + 1)$-problem of L. Collatz, which for negative $n$ is equivalent to the $(3n - 1)$-problem, i.e. to the iteration of the function

$$t(n) = \begin{cases} \frac{1}{3}n & \text{for } n \text{ even} \\ \frac{1}{2}(3n-1) & \text{for } n \text{ odd.} \end{cases} \quad (3.15)$$

The iterates of $n \in \mathbb{N}$ under $t$ have the fixed point 1 and the two cycles $(5, 7, 10)$ as well as $(17, 25, 37, 55, 82, 41, 61, 91, 136, 68, 34)$, and one conjectures that all $t$-trajectories eventually end in one of these three sets (cf. [10: p. 13]). It suffices to restrict ourselves to odd $n$ and to replace $t(n)$ for such $n$ by $T(n) = 2^{-p_n}(3n-1)$ if $2^{p_n}|(3n-1)$ but $2^{p_n+1} \not| (3n-1)$, $p_n \in \mathbb{N}$. The equation for $T$ can be inverted by

$$n = \frac{1}{3} \left(1 + 2^{p_n}T(n)\right). \quad (3.16)$$

We denote the iterates of $T$ by $T_k(n) = T(T_{k-1}(n))$ with $T_0(n) = n$, and for a fixed $n$ we introduce the notations $\gamma_0 = 1$ and $\gamma_k = 1 + p_{T_0(n)} + \cdots + p_{T_{k-1}(n)}$ for $k \geq 1$. Then (3.16) implies the representation

$$n = \frac{1}{6} \left(2^{\gamma_0} + \frac{1}{3} 2^{\gamma_1} + \cdots + \frac{1}{3^{k-1}} 2^{\gamma_{k-1}} + \frac{1}{3^{k-1}} 2^{\gamma_k} T_k(n)\right)$$

for every odd $n$, and for $k \to \infty$ the right-hand side converges to the right-hand side of (3.13) with $a = 2$, $b = \frac{1}{3}$, $c = \frac{2}{3}$ and therefore $q = \frac{1}{3}$.

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References


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