Estimates for Quasiconformal Mappings onto Canonical Domains

Vo Dang Thao

Abstract. In this paper we establish estimates for $K$-quasiconformal mappings $z = g(w)$ of a domain bounded by two circles $|w| = 1, |w| = q$ and $n$ continua situated in $q < |w| < 1$ onto a circular ring $Q(g) < |z| < 1$ that has been slit along $n$ arcs on the circles $|z| = R_j(g)$ $(j = 1, \ldots, n)$ such that $|z| = 1$ and $|z| = Q$ correspond to $|w| = 1$ and $|w| = q$, respectively. The bounds in the estimates for $Q, R_j$ and $|g(w)|$ are explicitly given, most of them are optimal. They are deduced mainly from [17].

Keywords: $K$-quasiconformal mappings, Riemann moduli of a multiply-connected domain, monotony of the modulus of a doubly-connected domain

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1. Introduction and notations

The generalization of Carleman’s (see [1: p. 212], [2: p. 177], [12: p. 15]) area inequality for doubly-connected domains to multiply-connected domains in [15] improves many Grötzsch’s [4, 6, 8] and Rengel’s [13] significant circular slits theorems for conformal mappings. In [16] we establish further area inequalities for $K$-quasiconformal mappings (see the definition in [10: p. 16]). Combining this with Grötzsch’s [4, 5, 7] inequalities yields in [17] sharp or asymptotic sharp estimates for $K$-quasiconformal mappings of the circular ring $Q < |z| < 1$ with concentric circular slits onto domains lying in $q < |w| < 1$. In this paper we establish estimates for the inverse mappings of those studied in [17]. Here the consideration is partly similar to the case of conformal mappings ($K = 1$, see [14: pp. 121 - 124]) using two auxiliary functions introduced in Section 4.

Let now $B$ be any domain given in the $w$-plane, bounded by two circles $|w| = 1, |w| = q$ and $pn$ $(p, n \in \mathbb{N})$ boundary components $\sigma_1, \ldots, \sigma_{pn}$ lying in $(0 <) q < |w| < 1$, and transformed into itself by the rotation $t = e^{i\frac{2\pi}{pn}w}$. We shall write $B = B_q$ when all $\sigma_j$ are circular arcs concentric with the circular ring. Let $G$ be the family of all $K$-quasiconformal mappings $z = g(w)$ each of which maps $B$ onto a circular ring $Q(g) < |z| < 1$ that has been slit along $pn$ circular arcs $L_1(g), \ldots, L_{pn}(g)$ concentric with the

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circular ring such that $|z| = 1$, $|z| = Q$ and $L_j$ correspond to $|w| = 1$, $|w| = q$ and
\( \sigma_j \ (j = 1, \ldots, pn) \), respectively, $g(1) = 1$ and satisfies the $p$ times rotation symmetry
\[ g(e^{i \frac{2\pi}{p}} w) = e^{i \frac{2\pi}{p}} g(w) \quad (w \in B). \]  

It is clear that the symmetry condition is trivial for $p = 1$. Therefore we see that the
inverse mapping $f$ of $g$, $g \in G$, belongs to the family $F$ studied in [17], because from
(1.1) for all $z \in A$, $A = g(B)$, and $f = g^{-1}$, $g \in G$, the relation
\[ e^{i \frac{2\pi}{p}} f(z) = f(e^{i \frac{2\pi}{p}} z) \]
follows. Put
$$R_j(g) = |z| \text{ with } z \in L_j(g) \text{ and } g \in G$$
$$c_j = \min |w| \text{ and } d_j = \max |w| \text{ with } w \in \sigma_j \quad (j = 1, \ldots, pn)$$
$$c = \min \{c_1, \ldots, c_{pn} \} \text{ and } d = \max \{d_1, \ldots, d_{pn} \}$$
$$\mu = \sup \prod_{j=1}^{pn} \frac{r_{j}}{r''_{j}}, \text{ where } r_{j} < |w| < r''_{j} \text{ are pairwise disjoint sets in } B.$$  
Furthermore, denote

by $S$ the inner area of $B$
by $s_j$ the external area of the compact set bounded by $\sigma_j$.

Clearly,
\[ \frac{c}{qd} \leq \mu \leq \frac{1}{q} \]  
and
\[ S + s = \pi(1 - q^2) \quad \text{with} \quad s = \sum_{j=1}^{pn} s_j. \]  

Our task is to estimate the radii $Q(g)$ and $R_j(g)$ ($g \in G; j = 1, \ldots, pn$) that are nothing
but the Riemann moduli of the domain $B$ in the case $K = 1$ (see [11: p. 334]); as well
as $|g(w)|$ ($g \in G, w \in B$), by at most the quantities $K, p, q, c_j, d_j, s_j, \mu, S$ and $|w|$. The
bounds in the estimates will be explicitly calculated by simple functions or the auxiliary
function $R(p, t, s)$ introduced in Section 4. Most of them are the best among the bounds
that depend on the same quantities.

2. Estimates of $Q$

The estimate of $Q$ plays an important role in establishing estimates for other quantities.
Therefore we begin with this estimate.

**Theorem 1.** Under the above hypothesis and notations we have, for every $g \in G$, the estimate
\[ \left( 1 + \frac{S}{\pi q^2} \right)^{-\frac{K}{p}} \leq Q(g) \leq \mu^{-\frac{K}{p}}, \]  
where the equality on the left-hand side holds if and only if $B = B_0$ and $g(w) = w |w|^{K-1}$ ($w \in B$), and the equality on the right-hand side holds if and only if $B = B_0$ and $g(w) = w |w|^{K-1}$ ($w \in B$).
**Proof.** Applying [17: Theorem 3.1] to the mapping \( f = g^{-1}, \ g \in G \), we have

\[
S \geq \pi q^2 (Q - \frac{1}{\bar{K}}) - 1,
\]

hence the lower bound of \( Q \) in (2.1) follows. Here the equality holds if and only if \( f(z) = z, z \in A \), i.e. \( B = B_0 \) and \( g(w) = w |w|^{-1} \) (\( w \in B \)). On the other hand, applying [16: Theorem 1] to the mapping \( g \in G \), we obtain

\[
\pi \geq \pi Q^2 \mu \bar{K},
\]

hence the upper bound of \( Q \) in (2.1) follows. Here with the help of [17: Formula (2.5)] we have the assertion on the occurrence of the equality \( \Box \)

**Corollary 1.** By (1.3), the lower bound of \( Q \) in (2.1) may be written in the form

\[
Q(g) \geq q^K \left( 1 - \frac{s}{\pi} \right)^{-\frac{K}{\pi}} \quad (g \in G)
\]

hence

\[
Q(g) \geq q^K \quad (g \in G).
\]

Equality in (2.2) and (2.3) can only occur if \( B = B_0 \) and \( g(w) = w |w|^{-1} \) (\( w \in B \)).

**Corollary 2.** From (2.1) and (1.2) we obtain the estimate

\[
Q(g) \leq \left( \frac{dq}{c} \right)^{\frac{K}{\pi}} \quad (g \in G)
\]

where the equality can only occur if \( c = d \) and \( g(w) = w |w|^{-1} \) (\( w \in B \)).

### 3. Lower bounds of \( R_j \)

Since \( R_j(g) > Q(g) \) (\( g \in G; \ j = 1, \ldots, pm \)), with the help of (2.2) or (2.3) we can get lower bounds of \( R_j \). However, we want to establish other relations that, in certain situations, may give sharper estimates.

**Theorem 2.** Under the hypothesis and notations given in Section 1, for every \( g \in G \) with \( Q(g) = Q > q^K \) we have the estimates

\[
R_j(g) > Q \left[ \frac{ps_j}{\pi (Q \bar{K} - q^2)} \right]^\frac{K}{\pi} \quad (j = 1, \ldots, pn)
\]

and

\[
\max_{1 \leq j \leq pm} R_j(g) > Q \left[ \frac{s}{\pi (Q \bar{K} - q^2)} \right]^\frac{K}{2}.
\]

**Proof.** First, we notice that by (1.1), each circle \( |z| = R_j \) contains at least \( p \) slits belonging to the boundary of \( A = g(B), g \in G \). By [17: Formula (3.1)], we therefore have

\[
ps_j \leq \pi R_j^2 \left( 1 - q^2 Q^{-\frac{K}{\pi}} \right),
\]

hence for every \( g \in G \) with \( q < Q \frac{1}{\bar{K}}, \) i.e., \( Q(g) > q^K \), estimate (3.1) follows. Similarly we obtain, with the help of [17: formula (3.2)], estimate (3.2) \( \Box \)
4. The auxiliary functions $R(p, t, s)$ and $T(p, r, s)$

In order to establish other estimates of $R_j, Q$ and $|g(w)|$ we will introduce the following two real functions.

**Definition.** The real functions

$$
\begin{align*}
  r &= R(p, t, s) \ (0 \leq s < t < 1) \\
  t &= T(p, r, s) \ (0 \leq s < r < 1)
\end{align*}
\quad (p \in \mathbb{N})
$$

are defined in such a way that the circular ring $s < |w| < 1$ with $p$ radial slits

$$
P_j = \left\{ w \big| s \leq |w| \leq t \text{ and } \arg w = j \frac{2\pi}{p} \right\} \quad (j = 0, \ldots, p - 1)
$$

and the circular ring $r < |z| < 1$ can be schlicht conformally mapped onto each other.

Because of the monotony of the modulus of a doubly-connected domain (see [2: p. 176]) we have the following monotones of the auxiliary function $R(p, t, s)$ with $p \in \mathbb{N}$:

$$
\begin{align*}
  s &< R(p, t, s) < t \ (0 \leq s < t < 1) \\
  R(p, t_1, s) &< R(p, t_2, s) \ (0 \leq s < t_1 < t_2 < 1) \\
  R(p, t, s_1) &< R(p, t, s_2) \ (0 \leq s_1 < s_2 < t < 1) \\
  R(p, t, s) &> R(1, t, s) \ (0 \leq s < t < 1; p \geq 2)
\end{align*}
$$

With the help of Hirsch’s [9: p. 316] and Nehari’s [11: p. 295] formulae, I myself found in [14: pp. 101 - 104] the following expression for $R(p, t, s)$:

$$
R(p, t, 0) = \exp \left\{ \frac{-\pi K'(w)}{2pK(w)} \right\} \quad (0 < t < 1; \ p \in \mathbb{N})
\quad (4.4)
$$

with

$$
K(k) = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}}
$$

and

$$
K'(k) = K(\sqrt{1 - k^2})
$$

and for $0 < s < t < 1$

$$
R(p, t, s) = \exp \left\{ \frac{-\pi K'(u)}{2pK(u)} \right\}
\quad (4.4)
$$

with

$$
u = 1 + h - \sqrt{h(2 + h)}
$$

where

$$
h = \frac{(1 - k)(1 - ak)}{k(1 + a)}, \quad k = 4s^p \prod_{n=1}^{\infty} \left[ \frac{1 + s^{4pn}}{1 + s^{2p(2n-1)}} \right]^4
$$

$$
a = sn \left( b + \frac{i2pb}{\pi} \log \frac{t}{s} k \right), \ b = K(k).$$
Here \( sn(z, k) \) means the Jacobian elliptic sinus with the parameter \( k \). Another expression for \( R(p, t, s) \) was shown by Graeser in [3: pp. 77 - 78].

In view of [14: Formula (1.21)] we have the estimate

\[
4^{-\frac{1}{s}} t < R(p, t, s) < t \quad (0 \leq s < t < 1; p \in \mathbb{N}),
\]

hence \( R(p, t, s) \approx t \) when \( p \to \infty \).

The evaluation of \( K(t^p) \) and \( K'(t^p) \) (see [18: p. 177]) yields the asymptotic behaviour of \( R(p, t, 0) \):

\[
R(p, t, 0) \approx 4^{-\frac{1}{s}} t \quad \text{when } t \to 0
\]

and

\[
1 - R(p, t, 0) \approx \frac{\pi^2}{2p \log \frac{q}{p(1-t)}} \quad \text{when } t \to 1.
\]

Successive approximations for \( R(1, t, 0) \) are given by Lehto [10: p. 64]. The expression for \( T(p, r, s) \), that is not needed here, was shown by Thao ([14: pp. 102 - 105] or [17: p. 61]).

5. Other estimates of \( R_j, Q \) and \( |g(w)| \)

Using the auxiliary functions studied in Section 4, other estimates for \( R_j, Q \) and \( |g(w)| \) will be given. In particular, when \( s_j = 0 \) or \( s = 0 \), they may be sharper than ones of (3.1) and (2.2).

**Theorem 3.** Under the hypothesis and notations given in Sections 1 and 4, for every \( g \in G \), \( w \in B \) and \( j = 1, \ldots, p_n \), we have the estimates

\[
R^K(p, d_j, q) < R_j(g) < Q(g)R^{-K}(p, \frac{q}{c_j}, q) \quad (5.1)
\]

\[
Q(g) > R^K(p, d_j, q)R^K\left(p, \frac{q}{c_j}, q\right) \quad (5.2)
\]

\[
R^K(p, |w|, q) < |g(w)| < Q(g)R^{-K}\left(p, \frac{q}{|w|}, q\right). \quad (5.3)
\]

**Proof.** Considering the mapping \( f = g^{-1} \), \( g \in G \), with the help of [17: Theorem 6.1] we obtain

\[
d_j < T(p, R_j^{\frac{1}{s}}, q) = t \quad \text{and} \quad \frac{q}{c_j} < T\left[p, \left(\frac{Q}{R_j}\right)^{\frac{1}{s}}, q\right] = t'.
\]

Hence the definition of the auxiliary functions and the monotony (4.2) yield the relations

\[
R_j^{\frac{1}{s}} = R(p, t, q) > R(p, d_j, q) \quad \text{and} \quad \left(\frac{Q}{R_j}\right)^{\frac{1}{s}} = R(p, t', q) > R\left(p, \frac{q}{c_j}, q\right).
\]

Thus we have estimate (5.1). Estimate (5.2) is just a consequence of (5.1). Using [17: Formula (6.10)], we obtain similiary estimate (5.3). \( \blacksquare \)
Corollary 3. From Theorem 3 and (4.5), for every \( g \in G, w \in B \) and \( j = 1, \ldots, pn \) we obtain the simple estimates

\[
4^{-\frac{k}{p}} d_j^k < R_j(g) < 4^{\frac{k}{p}} Q(g) \left( \frac{c_j}{q} \right)^K 
\]  
\[
Q(g) > 4^{-2^{\frac{k}{p}}} \left( \frac{qd_j}{c_j} \right)^K 
\]  
\[
4^{-\frac{k}{p}} |w|^K < |g(w)| < 4^{\frac{k}{p}} Q(g) \left( \frac{|w|}{q} \right)^K . 
\]  

In view of (4.3) and (4.6) we see that the coefficients in (5.4) - (5.6) are the best possible.

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References


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