The Generalized
Riemann-Hilbert Boundary Value Problem
for Non-Homogeneous Polyanalytic
Differential Equation of Order $n$
in the Sobolev Space $W_{n,p}(D)$

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Abstract. Given is a nonlinear non-homogeneous polyanalytic differential equation of order $n$
in a simply-connected domain $D$ in the complex plane. Initially we prove (under certain conditions) the existence of its general solution in $W_{n,p}(D)$ by first transforming it into a system of integro-differential equations. Next we prove the solvability of a generalized Riemann-Hilbert problem for the differential equation. This is effected by first reducing the boundary value problem posed to a corresponding one for a polyanalytic function. The latter is then transformed into $n$ classical Riemann-Hilbert problems for holomorphic functions, whose solutions are known in the literature.

Keywords: Polyanalytic functions, generalized Cauchy-Pompeiu integral operators of higher order, Riemann-Hilbert problem

AMS subject classification: 30 G 30, 35 J 40, 47 G 10

1. Introduction

We consider the following non-homogeneous polyanalytic differential equation of order $n$
in a given simply-connected bounded domain $D$ in the complex plane $\mathbb{C}$:

$$\frac{\partial^n w}{\partial z^n} = F\left(z, w, \left\{ \frac{\partial^{m+k} w}{\partial z^m \partial \bar{z}^k} \right\} \right)$$

$$n \geq m, k \in \mathbb{N}_0, m + k \leq n, (0, 0) \neq (m, k) \neq (0, n), n \in \mathbb{N}. \quad (1)$$

The right-hand side is a continuous function of its variables $z \in D$, $w$ and the partial derivatives of $w$ of order not exceeding $n$ and excluding $\frac{\partial^n w}{\partial z^n}$, which are denoted here by $\left\{ \frac{\partial^{m+k} w}{\partial z^m \partial \bar{z}^k} \right\}$. Following [5, 6] the general solution of equation (1) may be expressed in

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the form
\[
w(z) = \Phi(z) + T_{0,n,D}F\left(\zeta, w(\zeta), \left\{\frac{\partial^{m+k} w}{\partial \zeta^m \partial \zeta^k}\right\}\right)(z)
\]
\[
= \Phi(z) + \iint_D K_{0,n}(z - \zeta)F\left(\zeta, w(\zeta), \left\{\frac{\partial^{m+k} w}{\partial \zeta^m \partial \zeta^k}\right\}\right) d\xi d\eta
\]  
(2)

where \(\Phi\) is a polyanalytic function of order \(n\) in \(D\), and \(T_{0,n,D}\) is a generalized Cauchy-Pompeiu type singular integral operator:
\[
K_{m,n}(z) = \begin{cases}
\frac{(-m)!(-1)^m}{(n-1)!\pi} z^{m-1} \overline{z}^{n-1} & \text{if } m \leq 0 \\
\frac{(-n)!(-1)^n}{(m-1)!\pi} z^{m-1} \overline{z}^{n-1} & \text{if } n \leq 0 \\
\frac{z^{m-1} \overline{z}^{n-1}}{(m-1)!\pi (n-1)!} \left( \log |z|^2 - \sum_{r=1}^{m-1} \frac{1}{r} - \sum_{s=1}^{n-1} \frac{1}{s} \right) & \text{if } m, n \in \mathbb{N}.
\end{cases}
\]  
(3)

When \(m = 1\) or \(n = 1\), the corresponding summation in the formula is dropped. The kernel \(K_{m,n}\) of the integral operator \(T_{m,n,D}\) has no singularity on \(D\), except possibly at the origin. Moreover, it follows from the properties of the operators \(T_{m,k,D}\) \((m + k \leq n)\) that \(T_{0,n,D}f \in W_{n,p}(D)\), if \(f \in L_p(D)\) \((1 < p < \infty)\) (cf. [2, 5, 6]).

Suppose \(w \in W_{n,p}(D)\) is a solution of equation (1). Thus \(w\) may be expressed in the form (2), and hence we obtain
\[
\frac{\partial w}{\partial z} = \Phi_z + T_{-1,n,D}F, \quad \frac{\partial w}{\partial \zeta} = \Phi_\zeta + T_{0,n-1,D}F
\]
\[
\frac{\partial^k w}{\partial z^k} = \frac{\partial^k \Phi}{\partial z^k} + T_{-k,n,D}F, \quad \frac{\partial^k w}{\partial \zeta^k} = \frac{\partial^k \Phi}{\partial \zeta^k} + T_{0,n-k,D}F \quad (0 \leq k \leq n)
\]
and, in general,
\[
\frac{\partial^{m+k} w}{\partial z^m \partial \zeta^k} = \frac{\partial^{m+k} \Phi}{\partial z^m \partial \zeta^k} + T_{-m,n-k,D}F \quad (n \geq m, k; m + k \leq n).
\]

Consequently, we arrive at the following result (cf. [12, 15, 21]).

**Theorem 1.** The function \(w \in W_{n,p}(D)\) \((2 < p < \infty)\) defined by equation (2) is a general solution of the non-homogeneous polyanalytic equation (1) if and only if for a given in the domain \(D\) polyanalytic function \(\Phi \in W_{n,p}(D)\) of order \(n\), \((w, \{h_{m,k}\})\) is a solution of the system
\[
w(z) = \Phi(z) + T_{0,n,D}F\left(\zeta, w(\zeta), \{h_{m,k}(\zeta)\}\right)(z)
\]
\[
h_{m,k}(z) = \frac{\partial^{m+k} \Phi}{\partial z^m \partial \zeta^k} + T_{-m,n-k,D}F\left(\zeta, w(\zeta), \{h_{m,k}(\zeta)\}\right)(z)
\]  
(4)
\[
n \geq m, k \in \mathbb{N}_0, m + k \leq n, (0, 0) \neq (m, k) \neq (0, n), n \in \mathbb{N}.
\]
We note in passing that the integral operators \( T_{m,k,D} \) \((m + k = 0 < m^2 + k^2)\) are of singular Calderon-Zygmund type, and may be viewed as analogues of Vekua-type integral operators \( \Pi_D \) and \( \overline{\Pi}_D \) defined by

\[
\Pi_D f(z) = \frac{-1}{\pi} \int_D \int \frac{f(\zeta)}{(\zeta - z)^2} d\xi d\eta
\]

\[
\overline{\Pi}_D f(z) = \frac{-1}{\pi} \int_D \int \frac{f(\zeta)}{(\zeta - \bar{z})^2} d\xi d\eta
\]

(cf. \([8, 11, 17, 18, 22]\)). They are singular and must be understood in the sense of Cauchy’s principal value. Moreover, they satisfy the Calderon-Zygmund inequality (cf. \([5, 6, 8, 17, 18, 22]\))

\[
\|T_{m,n,D} f\|_{p,D} \leq A_p \|f\|_{p,D}
\]

(5)

where

\[
A_p = \|T_{m,n,D}\|_p, \quad A_2 = 1 \quad (1 < p < \infty).
\]

On the other hand, if \( m + k > 0 \), then \( T_{m,k,D} \) are regular or weakly singular integral operators, and they may be viewed as generalizations of the Cauchy-Pompeiu integral operators \( T_D, \overline{T}_D, T_D^* \) and the potential operator \( P_D \) given by

\[
T_D f(z) = \frac{-1}{\pi} \int_D \int \frac{f(\zeta)}{(\zeta - z)^2} d\xi d\eta, \quad \overline{T}_D f(z) = \frac{-1}{\pi} \int_D \int \frac{f(\zeta)}{(\zeta - \bar{z})^2} d\xi d\eta
\]

\[
T_D^* f(z) = \frac{-1}{\pi} \int_D \int \frac{f(\zeta)}{|\zeta - z|} d\xi d\eta, \quad P_D f(z) = \frac{2}{\pi} \int_D f(\zeta) \log |\zeta - z| d\xi d\eta.
\]

Moreover, since \( \|K_{m,n}\|_{1,D} \leq C(m, n, D) = \text{const.} \), it follows from the convolution theorem of W. H. Young (see \([17]\), for instance) that \( T_{m,k,D} \) maps the Banach space \( L_p(D) \) \((1 \leq p \leq \infty)\) into itself, and the estimate

\[
\|T_{m,k,D} f\|_{p,D} \leq C(m, k, D) \|f\|_{p,D} \quad (1 \leq p \leq \infty, m + k > 0)
\]

(6)

holds.

2. Existence of the general solution

We make the following assumptions on the right-hand side of equation (1):

\( \textbf{(A1)} \) \( F(z, w, \{h_{m,k}\}) \) is a continuous function of its variables \( z \in D \), \( w \) and the partial derivatives of \( w \) of order not exceeding \( n \) and excluding \( \frac{\partial^n w}{\partial \bar{w}^n} \), which are denoted here by \( \{h_{m,k}\} \).

\( \textbf{(A2)} \) There exists a tuple \((w^*, \{h_{m,k}^*, k\}) \) \((w^*, \{h_{m,k}^*, k\} \in L_p(D), 2 < p < \infty)\) such that \( F(z, w^*, \{\bar{h}_{m,k}\}) \in L_p(D) \).

\( \textbf{(A3)} \) \( F(z, w, \{\bar{h}_{m,k}\}) \) satisfies a Lipschitz condition of the form

\[
|F(z, w(z), \{h_{m,k}(z)\}) - F(z, \bar{w}(z), \{\bar{h}_{m,k}(z)\})| \\
\leq L_1 \max \left\{ \max_{m+k<n} |h_{m,k}(z) - \bar{h}_{m,k}(z)|, |w(z) - \bar{w}(z)| \right\} \\
+ L_2 \max_{m+k=n} |h_{m,k}(z) - \bar{h}_{m,k}(z)|
\]

almost everywhere on \( D \). While \( 0 < L_2 < 1 \), \( L_1 \) may take any positive value.
Note. It follows from assumptions (A2) and (A3) that \( F(z, w, \{h_{m,k}\}) \in L_p(D) \) 
(2 \( < p < \infty \)) if \( w \) and all the elements of \( \{h_{m,k}\} \) belong to \( L_p(D) \).

We introduce the following Banach space \( L_p(D) \) (2 \( < p < \infty \)):

\[
L_p(D) = \{ (w, \{h_{m,k}\}) \mid w, h_{m,k} \in L_p(D) \}
\]

\[
\gamma \max_{m+k<n} \|h_{m,k}\|_{p,D}, \max_{m+k=n} \|h_{m,k}\|_{p,D} \quad (\gamma > 0).
\]

Next we define a mapping \( \mathbb{P} \) in \( L_p(D) \) through the right-hand side of (4). For any tuple \( (w, \{h_{m,k}\}) \in L_p(D) \) we set

\[
(W, \{H_{m,k}\}) = \mathbb{P}(w, \{h_{m,k}\})
\]

\[
W(z) = \Phi(z) + T_{0,n,D}F(\zeta, w(\zeta), \{h_{m,k}(\zeta)\})(z)
\]

\[
H_{m,k}(z) = \frac{\partial^{m+k}}{\partial z^m \partial \bar{z}^k} \Phi + T_{-m,-n,k,D}F(\zeta, w(\zeta), \{h_{m,k}(\zeta)\})(z)
\]

\[
\gamma \|W - \bar{W}\|_p \leq \gamma \|T_{0,n,D}\|_p \|F(z, w, \{h_{m,k}\}) - F(z, \bar{w}, \{\bar{h}_{m,k}\})\|_{p,D}
\]

\[
\leq \gamma \|T_{0,n,D}\|_p \left( L_1 \max_{0<k<n} \|h_{m,k} - \bar{h}_{m,k}\|_{p,D}, \|w - \bar{w}\|_{p,D} \right)
\]

\[
+ L_2 \max_{m+k=n} \|h_{m,k} - \bar{h}_{m,k}\|_{p,D}
\]

\[
\leq \|T_{0,n,D}\|_p (L_1 + \gamma L_2) \|w, \{h_{m,k}\} - (\bar{w}, \{\bar{h}_{m,k}\})\|.
\]

Similarly we obtain

\[
\gamma \|H_{m,k} - \bar{H}_{m,k}\|_{p,D} \leq \|T_{-m,-n,k,D}\|_p (L_1 + \gamma L_2) \|w, \{h_{m,k}\} - (\bar{w}, \{\bar{h}_{m,k}\})\|
\]

\[
\|H_{a,b} - \bar{H}_{a,b}\|_p \leq \|T_{-a,-n,b,D}\|_p \left( \frac{1}{\gamma} L_1 + L_2 \right) \|w, \{h_{m,k}\} - (\bar{w}, \{\bar{h}_{m,k}\})\|
\]

for \( 0 < m + k < n \) and \( \alpha + \beta = n \) with \( (\alpha, \beta) \neq (0, n) \). On account of the relations

\[
\|T_{-m,-n,k,D}\| = \begin{cases} C(m, k, D) & \text{for } 0 < m + k < n \\ \|I_D\|_p & \text{for } m + k = n \end{cases} \quad (1 < p < \infty),
\]
where \( \Pi_D \) is the strongly singular Vekua-type integral operator, we arrive at the estimate

\[
\| (W, \{H_{m,k}\}) - (\tilde{W}, \{H_{m,k}\}) \| \\
\leq \left( \frac{1}{\gamma}L_1 + L_2 \right) \max \left\{ \gamma \| T_{0,n,D} \|_p, \gamma \max_{m+k<n} \| T_{-m,n-k,D} \|_p, \| \Pi_D \|_p \right\} \\
\times \| (w, \{h_{m,k}\}) - (\tilde{w}, \{\tilde{h}_{m,k}\}) \|
\]

and \( \mathbb{P} \) is contractive in \( \mathcal{L}_p(D) \) \( (2 < p < \infty) \) if

\[
\left( \frac{1}{\gamma}L_1 + L_2 \right) \max \left\{ \gamma \| T_{0,n,D} \|_p, \gamma \max_{m+k<n} \| T_{-m,n-k,D} \|_p, \| \Pi_D \|_p \right\} < 1. \tag{9}
\]

This condition may be satisfied if the constants \( L_1, L_2 \) and \( \gamma \) can be chosen properly and the domain \( D \) made sufficiently small. It is known (cf. [5, 6, 8, 12, 17, 18], for instance) that

\[
\| \Pi_D \|_p \geq 1 \quad (1 < p < \infty) \quad \text{and} \quad \| \Pi_D \|_2 = 1.
\]

Thus for a chosen \( \mathcal{L}_p(D) \) \( (2 < p < \infty) \) we need \( L_2, 0 < L_2 < 1 \), such that \( L_2 \| \Pi_D \|_p < 1 \). Next we choose the constant \( \gamma > 0 \) large enough so that, for the given \( L_1 > 0, (\frac{1}{\gamma}L_1 + L_2)\| \Pi_D \|_p < 1 \) also holds. Finally, since \( \| T_{0,n,D} \|_p \) and \( \| T_{-m,n-k,D} \|_p \), \( 0 < m + k < n \), vary directly with the area of the domain \( D \), we may satisfy estimate (9) eventually be reducing the size of \( D \).

If estimate (9) is realized, then \( \mathbb{P} \) has a unique fixed element \( (\hat{w}, \{\hat{h}_{m,k}\}) \in \mathcal{L}_p(D) \) \( (2 < p < \infty) \) and \( \hat{w} \) is the general solution of equation (1) corresponding to the given in \( D \) polyanalytic function \( \Phi \in W_{n,p}(D) \) of order \( n \). Moreover, \( \hat{w} \in W_{n,p}(D) \) \( (2 < p < \infty) \):

\[
\hat{w}(z) = \Phi(z) + T_{0,n,D}F(\gamma, \hat{w}(\gamma), \{\hat{h}_{m,k}(\gamma)\})(z)
\]

\[
\hat{h}_{m,k}(z) = \frac{\partial^{m+k} \Phi}{\partial z^m \partial \bar{z}^k} + T_{-m,n-k,D}F(\gamma, \hat{w}(\gamma), \{\hat{h}_{m,k}(\gamma)\})(z)
\]

\[
n \geq m, k \in \mathbb{N}_0, m + k \leq n, (0, 0) \neq (m, k) \neq (0, n), n \in \mathbb{N}.
\]

**Theorem 2.** Under assumptions (A1) - (A3) and (9) the non-homogeneous polyanalytic differential equation (1) admits a uniquely defined solution \( w \in W_{n,p}(D) \) \( (2 < p < \infty) \) given by equation (2) for every prescribed in the domain \( D \) polyanalytic function \( \Phi \in W_{n,p}(D) \). This solution defines a mapping from \( \Phi \mapsto w = R(\Phi) \).
3. The generalized Riemann-Hilbert problem for polyanalytic functions

We consider the following boundary value problem for a polyanalytic function \( \Phi \) of order \( n \):

\[
\frac{\partial^n \Phi}{\partial z^n} = 0 \quad \text{on } D = \{ z : |z| < 1 \}
\]

\[
\text{Re} \left[ (a_k + ib_k) \frac{\partial^{n-1} \Phi}{\partial x^{n-k} \partial y^{k-1}} \right](t) = c_k(t) \quad \text{on } \partial D \quad (k = 1, \ldots, n)
\]

(10)

where \( a_k, b_k, c_k \in W_{1-\frac{1}{p}}(\partial D) \) \( (2 < p < \infty) \) are prescribed real-valued functions on \( \partial D \). Moreover, \( (a_k + ib_k)(t) \neq 0 \) for all \( t \in \partial D \).

A polyanalytic function \( \Phi \) of order \( n \) may be expressed as

\[
\Phi = \Phi(z, \bar{z}) = \sum_{\rho=0}^{n-1} \bar{z}^\rho \varphi_\rho(z) \quad (\varphi_\rho \text{ holomorphic}).
\]

Thus

\[
\frac{\partial^{n-1} \Phi}{\partial x^{n-k} \partial y^{k-1}} = i^{k-1} \left( \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \right) \left( \frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} \right)^{k-1} \sum_{\rho=0}^{n-1} \bar{z}^\rho \varphi_\rho(z)
\]

\[
= i^{k-1} \sum_{\alpha=0}^{n-k} \sum_{\beta=0}^{k-1} (-1)^\beta \binom{n-k}{\alpha} \binom{k-1}{\beta} \times \sum_{\rho=\alpha+\beta}^{n-1} \frac{\rho!}{(\rho - \alpha - \beta)!} \bar{z}^{\alpha+\beta} \frac{d^{n-\alpha-\beta-1}}{dz^{n-\alpha-\beta-1}} \varphi_\rho(z).
\]

Since on \( \partial D \bar{t} = \frac{1}{t} \) we shall replace \( \bar{z} \) by \( \frac{1}{z} \) in the expression above and then reduce the given boundary conditions for the polyanalytic function \( \Phi \) to \( n \) equivalent Riemann-Hilbert boundary value problems for some holomorphic functions \( G_k \) \( (k = 1, \ldots, n) \) which are defined in terms of the holomorphic functions \( \varphi_\rho \) \( (\rho = 0, \ldots, n-1) \). Thus

\[
\frac{\partial^{n-1} \Phi}{\partial x^{n-k} \partial y^{k-1}} = i^{k-1} \sum_{\alpha=0}^{n-k} \sum_{\beta=0}^{k-1} (-1)^\beta \binom{n-k}{\alpha} \binom{k-1}{\beta} \times \sum_{\rho=\alpha+\beta}^{n-1} \frac{\rho!}{(\rho - \alpha - \beta)!} \bar{z}^{\alpha+\beta} \frac{d^{n-\alpha-\beta-1}}{dz^{n-\alpha-\beta-1}} \varphi_\rho(z).
\]

Hence

\[
\text{Re} \left[ (a_k + ib_k) \frac{\partial^{n-1} \Phi}{\partial x^{n-k} \partial y^{k-1}} \right](t)
\]

\[
= \text{Re} \left[ (a_k + b_k)(t) t^{1-n} i^{k-1} \sum_{\alpha=0}^{n-k} \sum_{\beta=0}^{k-1} (-1)^\beta \binom{n-k}{\alpha} \binom{k-1}{\beta} \times \sum_{\rho=\alpha+\beta}^{n-1} \frac{\rho!}{(\rho - \alpha - \beta)!} t^{n+\alpha+\beta-\rho-1} \frac{d^{n-\alpha-\beta-1}}{dz^{n-\alpha-\beta-1}} \varphi_\rho(t) \right],
\]
i.e.
\[
\text{Re} \left[ (a_k + ib_k)(t) i^{k-1} t^{1-n} G_k(t) \right] = c_k(t) \quad (k = 1, \ldots, n) \tag{11}
\]
where
\[
G_k(z) = \sum_{\alpha=0}^{n-k-1} \sum_{\beta=0}^{k-1} (-1)^\beta \binom{n-k}{\alpha} \binom{k-1}{\beta} \times \sum_{\rho=\alpha+\beta}^{n-1} \frac{\rho!}{(\rho - \alpha - \beta)!} z^{n+\alpha+\beta-\rho-1} \frac{d^{n-\alpha-\beta-1}}{dz^{n-\alpha-\beta-1}} \varphi_\rho(z). \tag{12}
\]

The solution of the Riemann-Hilbert problem (11) is known (see [9, 11, 14, 16]). If \( \kappa_k := \text{index } [(a_k - ib_k), \partial D] \geq 0 \), then the general solution \( G_k \) is given with the aid of the Schwarz integral as
\[
z^{1-n} G_k(z) = \frac{X_k(z)}{2\pi i} \left[ \int_{\partial D} \frac{i^{1-k} c_k(t)}{(a_k + ib_k)(t) X_k^+(t)} \frac{t + z}{(t - z)^j} dt + P_{\kappa_k}(z) \right] \tag{13}
\]
where \( P_{\kappa_k} \) is a polynomial of degree not exceeding \( \kappa_k \) and \( X_k \) is the canonical solution of the corresponding homogeneous problem
\[
X_k(z) = z^{\kappa_k} \exp \Gamma_k(z),
\]
\[
\Gamma_k(z) = \frac{1}{4\pi i} \int_{\partial D} \log \left( (-1)^{k-2} t^{-2\kappa_k} \frac{(a_k - ib_k)(t)}{(a_k + ib_k)(t)} \right) \frac{t + z}{(t - z)^j} dt.
\]

If any of the \( \kappa_k \) is negative, then the corresponding Riemann-Hilbert problem has a unique solution, bounded at infinity for instance, if and only if the conditions
\[
\int_{\partial D} \frac{c_k(t)}{(a_k + ib_k)(t) X_k^+(t)} t^j dt = 0 \quad (j = 0, \ldots, -2\kappa_k - 2) \tag{14}
\]
are fulfilled, and in that case the solution is given by (13) as well, with the obvious modification that we set \( P_{\kappa_k}(z) \equiv 0 \) (cf. [9, 11, 16]).

We investigate the possibility for the satisfaction of the solvability conditions (14). For this purpose we consider the modified Riemann-Hilbert problem (cf. [4, 25])
\[
\text{Re} \left[ i^{k-1} t^{1-n} (a_k + ib_k)(t) G_k(t) \right] = c_k(t) - \sum_{s=\kappa_k+1}^{-\kappa_k-1} \lambda_s t^s \quad \text{on } \partial D \tag{11}'
\]
where \( \lambda_{-s} = \overline{\lambda_s} \) are constants yet to be determined appropriately. The modified problem is uniquely solvable for \( \kappa_k < 0 \), and the solution \( G_k \) to the original Riemann-Hilbert problem (11) has the representation
\[
z^{1-n} G_k(z) = \frac{X_k(z)}{2\pi i} \int_{\partial D} \frac{i^{1-k} c_k(t) t^j}{(a_k + ib_k)(t) X_k^+(t)} \frac{t + z}{(t - z)^j} dt \tag{13}'
\]
(cf. [4, 25]).
In order that a Riemann-Hilbert problem with non-negative index to be uniquely solvable $2k_k + 1$ point conditions need to be imposed on the solution $G_k$. These conditions can be expressed in terms of the solution $\Phi$ of the given polyanalytic equation (10). Suppose $r$ among the $n$ Riemann-Hilbert problems (11) have non-negative indices, whose sum is $N$. Then, we demand that

$$\text{Im} \left[ i^{k-1} (a_k + ib_k)(\tau_j) \tau_j^{1-n} G_k(\tau_j) \right] = dj \quad (j = 1, 2, \ldots, N + r)$$

$$\tau_j \in \partial D, \tau_m \neq \tau_n \text{ for } m \neq n, d_j \in \mathbb{R} \quad (15)$$

It can be shown that $z^{1-n} G_k(z) \in W_{1,p}(0)$ ($2 < p < \infty, k = 1, \ldots, n$) and estimates of the form

$$\|z^{1-n} G_k\|_{p,D} \leq C_k(a_k, b_k, p, D) \|c_k\|_{p, \partial D}$$

$$\|z^{1-n} G_k\|_{1,p,D} \leq K_k(a_k, b_k, p, D) \|c_k\|_{1-p, \partial D}$$

hold (cf. [12, 14, 15]).

Suppose we have determined all $n$ holomorphic functions $G_k$ ($k = 1, \ldots, n$) uniquely. We proceed to compute the required polyanalytic function $\Phi$ by expressing the holomorphic functions $\varphi_p$ ($p = 0, \ldots, n - 1$) in terms of $G_k$. We shall make use of the following three facts:

1. Derivatives of $\Phi$ with respect to $x, y$ can easily be expressed by the holomorphic functions $G_k$. It follows from (12) that

$$\frac{\partial^{n-j} \Phi}{\partial x^{n-q-v-j} \partial y^{q+v}} = i^{q+v} \left( \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \right)^{n-q-v-j} \left( \frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} \right)^{q+v} \sum_{\rho=0}^{n-1} \varphi_\rho(z)$$

$$= i^{q+v} \sum_{\alpha=0}^{n-q-v-j} \sum_{\beta=0}^{q+v} (-1)^\beta \binom{n-q-v-j}{\alpha} \binom{q+v}{\beta}$$

$$\times \sum_{\rho=0}^{n-1} \frac{\rho!}{(\rho - \alpha - \beta)!} \frac{d^{\rho-a-\beta-1}}{d z^{\rho-a-\beta-1}} \varphi_\rho(z)$$

$$= i^{q+v} z^{1-n} G_{q+v+j}(z)$$

on $\partial D$ (i.e. $z = \frac{1}{z}$).

2. Derivatives of $\Phi$ with respect to $z, \bar{z}$ can be expressed by the derivatives with respect to $x, y$, and hence in terms of $G_k$. Indeed, on $\partial D$ we have

$$\frac{\partial^{n-j} \Phi}{\partial z^{n-k} \partial \bar{z}^{k-j}} = 2i^{n-k} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)^{n-k} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)^{k-j} \Phi$$

$$= 2i^{n-k} \sum_{q=0}^{n-k} \sum_{\nu=0}^{k-j} (-1)^q q^{a+v} \binom{n-k}{q} \binom{k-j}{\nu} \frac{\partial^{n-j} \Phi}{\partial x^{n-q-v-j} \partial y^{q+v}}$$

$$= 2i^{n-k} \sum_{q=0}^{n-k} \sum_{\nu=0}^{k-j} (-1)^q q^{a+v} \binom{n-k}{q} \binom{k-j}{\nu} \frac{\partial^{n-j} \Phi}{\partial x^{n-q-v-j} \partial y^{q+v}}$$

$$= 2i^{n-k} \sum_{q=0}^{n-k} (-1)^q q^{a+v} \binom{n-k}{q} \sum_{\nu=0}^{k-j} (-1)^\nu \binom{k-j}{\nu} G_{q+v+j}(z)$$

$$= 2i^{n-k} \sum_{q=0}^{n-k} (-1)^q q^{a+v} \binom{n-k}{q} G_{q+v+j}(z)$$

(17)
for \( n \geq k \geq j \).

3. The holomorphic functions \( \varphi_\rho \) can be expressed by the derivatives of the polyanalytic function \( \Phi \) with respect to \( z \) and \( \overline{z} \).

Thus

\[
\frac{\partial^{n-1}\Phi}{\partial \overline{z}^{n-1}} = \frac{\partial^{n-1}}{\partial \overline{z}^{n-1}} \sum_{\rho=0}^{n-1} \overline{z}^{\rho} \varphi_\rho(z) = (n-1)! \varphi_{n-1}(z).
\]

On the other hand, it follows from (17), with \( k = n \) and \( j = 1 \), that

\[
\frac{\partial^{n-1}\Phi}{\partial \overline{z}^{n-1}} = (2z)^{1-n} \sum_{\nu=0}^{n-1} (-1)^{\nu} \binom{n-1}{\nu} G_{\nu+1}(z) \quad \text{on } \partial D.
\]

Hence we may conclude that

\[
\varphi_{n-1}(z) = \frac{1}{(n-1)!} (2z)^{1-n} \sum_{\nu=0}^{n-1} (-1)^{\nu} \binom{n-1}{\nu} G_{\nu+1}(z) \quad \text{on } \partial D.
\]

Next we have, on the one hand,

\[
\frac{\partial^{n-2}\Phi}{\partial \overline{z}^{n-2}} = \frac{\partial^{n-2}}{\partial \overline{z}^{n-2}} \left( \overline{z}^{n-2} \varphi_{n-2}(z) + \overline{z}^{n-1} \varphi_{n-1}(z) \right) = (n-2)! \varphi_{n-2}(z) + (n-1)! \overline{z} \varphi_{n-1}(z).
\]

On the other hand, we deduce from (17), with \( k = n \) and \( j = 2 \), that

\[
\frac{\partial^{n-2}\Phi}{\partial \overline{z}^{n-2}} = 2^{2-n} z^{1-n} \sum_{\nu=0}^{n-2} (-1)^{\nu} \binom{n-2}{\nu} G_{\nu+2}(z) \quad \text{on } \partial D.
\]

So we can obtain for \( \varphi_{n-2} \) the representation

\[
\varphi_{n-2}(z) = \frac{1}{(n-2)!} \left[ 2^{2-n} z^{1-n} \sum_{\nu=0}^{n-2} (-1)^{\nu} \binom{n-2}{\nu} G_{\nu+2}(z) - (n-1)! \overline{z} \varphi_{n-1}(z) \right]
\]

on \( \partial D \).

Similarly we compute \( \varphi_{n-3}, \ldots, \varphi_1, \varphi_0 \). Suppose we have computed \( \varphi_{n-1}, \varphi_{n-2}, \ldots, \varphi_{n-j+1} \). Then we compute \( \varphi_{n-j} \) as

\[
\frac{\partial^{n-j}\Phi}{\partial \overline{z}^{n-j}} = \frac{\partial^{n-j}}{\partial \overline{z}^{n-j}} \sum_{\rho=0}^{n-1} \overline{z}^{\rho} \varphi_\rho(z) = (n-j)! \varphi_{n-j}(z) + \sum_{\rho=n-j+1}^{n-1} \varphi_\rho(z) \frac{\partial^{n-j}\overline{z}^\rho}{\partial \overline{z}^{n-j}}.
\]

On the other hand, for \( k = n \) formula (17) yields

\[
\frac{\partial^{n-j}\Phi}{\partial \overline{z}^{n-j}} = 2^{n-j} z^{1-n} \sum_{\nu=0}^{n-j} (-1)^{\nu} \binom{n-j}{\nu} G_{\nu+j}(z).
\]
We thus arrive at the general representation

\[ \varphi_{n-j}(z) = \frac{1}{(n-j)!} \left[ 2^{j-n} z^{1-n} \sum_{\nu=0}^{n-j} (-1)^\nu \binom{n-j}{\nu} G_{\nu+j}(z) - \sum_{\rho=n-j+1}^{n-1} \varphi_{\rho}(z) \frac{\partial^{n-j-\rho}}{\partial z^{n-j}} \right] \]

for \( \varphi_{n-j} \) \( (j = 1, \ldots, n) \). Hence all \( n \) holomorphic functions \( \varphi_j \) \( (j = 0, \ldots, n-1) \) are uniquely determinable, and with them the polyanalytic function \( \Phi \) as well. Furthermore, since \( a_k, b_k, c_k \in W_{1-\frac{1}{p},p}(\partial D) \) \( (2 < p < \infty) \), we conclude that \( z^{1-n} G_k(z) \in W_{1,p}(D) \) (cf. [1, 10, 12, 14, 15, 19, 20]). It thus follows from (12) that

\[ t^{1-n} G_1(t) = \sum_{\alpha=0}^{n-1} \binom{n-1}{\alpha} \sum_{j=\alpha}^{n-1} \frac{j!}{(j-\alpha)!} t^{\alpha-j} \frac{d^{n-\alpha-1}}{dt^{n-\alpha-1}} \varphi_j(t) \in W_{1-\frac{1}{p},p}(\partial D) \]

and, in particular,

\[ \frac{d^{n-\alpha-1}}{dt^{n-\alpha-1}} \varphi_j(t) \in W_{1-\frac{1}{p},p}(\partial D) \quad (j, \alpha = 0, \ldots, n-1). \]

Hence

\[ \frac{d^{n-1}}{dt^{n-1}} \varphi_j \in W_{1-\frac{1}{p},p}(\partial D) \quad (j = 0, 1, \ldots, n-1). \]

It now follows from the properties of traces of functions that

\[ \varphi_j \in W_{n-\frac{1}{p},p}(\partial D) \quad \text{and} \quad \varphi_j, \Phi \in W_{n,p}(D) \quad (j = 1, \ldots, n) \]

and the estimates

\[
\begin{aligned}
&\|\Phi\|_{p,D} \leq C_1(p,D) \max_k \|c_k\|_{p,\partial D} \\
&\|\Phi\|_{j,p,D} \leq C_2(p,D) \max_k \|c_k\|_{1-\frac{1}{p},p,\partial D}
\end{aligned}
\]

(18)

hold (cf. [1, 10, 12, 14, 15, 19, 20]).

4. The generalized Riemann-Hilbert problem for equation (1)

We now take up the following boundary value problem for the function \( w \):

\[ \frac{\partial^n w}{\partial z^n} = F \left( z, w, \left\{ \frac{\partial^{n+k} w}{\partial z^m \partial \bar{z}^k} \right\} \right) \text{ on } D \]

\[ \Re \left[ (a_k + ib_k) \frac{\partial^{n-1} w}{\partial z^{n-k} \partial \bar{y}^{k-1}} \right](t) = c_k(t) \text{ on } \partial D \quad (k = 1, \ldots, n) \]

(19)

\[ n \geq m, k \in \mathbb{N}_0, m + k \leq n, (0,0) \neq (m,k) \neq (0,n), n \in \mathbb{N} \]
where $a_k, b_k, c_k \in W_{1-\frac{1}{p},p}(\partial D) \ (2 < p < \infty)$ are prescribed real-valued functions on $\partial D$ with $(a_k + ib_k)(t) \neq 0$ for all $t \in \partial D$.

It was shown earlier that for every polyanalytic function $\Phi \in W_{n,p}(D) \ (2 < p < \infty)$ there exists a unique solution $w \in W_{n,p}(D)$ to the partial differential equation (1). This solution is represented by (2). We shall now exploit the arbitrariness of the polyanalytic function $\Phi$ to construct the solution of the boundary value problem (1), (19). For this purpose we shall write $\Phi$ as

$$
\Phi = \Phi_c + \Phi_{(w,h)}
$$

where $\Phi_c, \Phi_{(w,h)}$ are solutions of the boundary value problems

$$
\text{Re} \left[ (a_j + ib_j) \frac{\partial^{n-1} \Phi_c}{\partial x^{n-j} \partial y^{j-1}} \right](t) = c_j(t) \quad \text{on} \quad \partial D \\
\text{Re} \left[ (a_j + ib_j) \frac{\partial^{n-1} \Phi_{(w,h)}}{\partial x^{n-j} \partial y^{j-1}} \right](t) = -\text{Re} \left[ (a_j + ib_j) \frac{\partial^{n-1}}{\partial x^{n-j} \partial y^{j-1}} T_{0,n,D}F(\cdot, w, \{h_{m,k}\}) \right](t)
$$

$$
= g_{(w,h),j}(t) \quad \text{on} \quad \partial D
$$

for $j = 1, \ldots, n$. Since $F(z, w, \{h_{m,k}\}) \in L_p(D) \ (2 < p < \infty)$, then $T_{0,n,D}F \in W_{n,p}(D)$ (cf. [5, 6]). Moreover,

$$
\frac{\partial^{n-1}}{\partial x^{n-k} \partial y^{k-1}} T_{0,n,D}F(z) = i^{k-1} \sum_{\alpha=0}^{n-k} \binom{n-k}{\alpha} \sum_{\beta=0}^{k-1} (-1)^\beta \binom{k-1}{\beta} \frac{\partial^{n-1}}{\partial x^{n-\alpha-\beta} \partial y^{\alpha+\beta}} T_{0,n,D}F(z)
$$

$$
= i^{k-1} \sum_{\alpha=0}^{n-k} \binom{n-k}{\alpha} \sum_{\beta=0}^{k-1} (-1)^\beta \binom{k-1}{\beta} T_{a+\beta+1-n, n-\alpha-\beta, D}F(z)
$$

$$
i.e. g_{(w,h),j} \in W_{1-p,p}(\partial D) \ (2 < p < \infty; j = 1, \ldots, n) \ (cf. \ [1, 10, 11, 12, 19, 20]).
$$

Polyanalytic functions which satisfy boundary conditions of the form (20) have been constructed earlier, and we deduce from there that $\Phi_c, \Phi_{(w,h)} \in W_{n,p}(D) \ (2 < p < \infty)$ and, in particular, the estimates

$$
\|\Phi_{(w,h)}\|_{p,D} \leq C(p, D) \|g_{(w,h),j}\|_{p,\partial D} \leq C_1(p, D) \|T_{0,n,D}F\|_{1,p,D}
$$

$$
\|\Phi_{(w,h)}\|_{k,p,D} \leq C_2(p, D) \|T_{0,n,D}F\|_{n+1-k, p,D}
$$

hold for $k, j = 1, \ldots, n$ and $2 < p < \infty$.

We now define a mapping $Q$ in the Banach space $L_p(D) \ (2 < p < \infty)$. For any tuple $(w, \{h_{m,k}\}) \in L_p(D)$ we set

$$
(W, \{H_{m,k}\}) = Q(w, \{h_{m,k}\})
$$
where

\[ W(z) = \Phi_c(z) + \Phi_{(w, h)}(z) + T_{0,n,D}F(\zeta, w(\zeta), \{h_{m,k}(\zeta)\})(z) \]

\[ H_{m,k}(z) = \frac{\partial^{m+k}}{\partial z^m \partial \overline{z}^k} \left( \Phi_c(z) + \Phi_{(w, h)}(z) \right) + T_{-m,n-k,D}F(\zeta, w(\zeta), \{h_{m,k}(\zeta)\})(z) \]

\[ n \geq m, k \in \mathbb{N}_0, m + k \leq n, (0, 0) \neq (m, k) \neq (0, n), n \in \mathbb{N}. \]

The operator \( Q \) is uniquely defined, and it maps the Banach space \( L_p(D) \ (2 < p < \infty) \) into itself. Moreover, the following result holds.

**Theorem 3.** If \( (w, \{h_{m,k}\}) \) is a fixed point of the operator \( Q \), then \( w \) is the solution of the given differential equation (1) which also satisfies the boundary conditions (19).

We next derive the conditions to be imposed in order that \( Q \) has a fixed point. Suppose \( (W, \{H_{m,k}\}), (\overline{W}, \{\overline{H}_{m,k}\}) \) are the respective images of \( (w, \{h_{m,k}\}), (\overline{w}, \{\overline{h}_{m,k}\}) \in L_p(D) \ (2 < p < \infty) \). If we set

\[ \varphi = \Phi_{(w, h)} - \Phi_{(\overline{w}, \overline{h})} \quad \text{and} \quad f = F(z, w, \{h_{m,k}\}) - F(z, \overline{w}, \{\overline{h}_{m,k}\}), \]

then

\[ W - \overline{W} = \varphi + T_{0,n,D}f, \quad H_{m,k} - \overline{H}_{m,k} = \frac{\partial^{m+k} \varphi}{\partial z^m \partial \overline{z}^k} + T_{-m,n-k,D}F \]

and

\[ \gamma \|W - \overline{W}\|_{p,D} \]

\[ \leq \gamma \left( C_1(p, D) \|T_{0,n,D}\|_{1,p} + \|T_{0,n,D}\|_p \right) \|f\|_{p,D} \]

\[ \leq \left( L_1 \max \left\{ \max_{m+k<n} \|h_{m,k} - \overline{h}_{m,k}\|_{p,D}, \|w - \overline{w}\|_{p,D} \right\} \right) \gamma \left( C_1(p, D) \|T_{0,n,D}\|_{1,p} + \|T_{0,n,D}\|_p \right) \]

\[ + L_2 \max_{m+k=n} \|h_{m,k} - \overline{h}_{m,k}\|_{p,D} \left( C_1(p, D) \|T_{0,n,D}\|_{1,p} + \|T_{0,n,D}\|_p \right) \gamma (L_1 + \gamma L_2) \|(w, \{h_{m,k}\}) - (\overline{w}, \{\overline{h}_{m,k}\})\| \]

Similary we arrive at

\[ \gamma \|H_{m,k} - \overline{H}_{m,k}\|_{p,D} \leq \left( C_2(p, D) \|T_{0,n,D}\|_{n-m-k,p} + \|T_{-m,n-k,D}\|_p \right) \]

\[ \times (L_1 + \gamma L_2) \|(w, \{h_{m,k}\}) - (\overline{w}, \{\overline{h}_{m,k}\})\| \]

and

\[ \|H_{\alpha,\beta} - \overline{H}_{\alpha,\beta}\|_{p,D} \leq \left( C_3(p, D) \|T_{0,n,D}\|_{1,p} + \|T_{-\alpha,n-\beta,D}\|_p \right) \]

\[ \times \left( \frac{1}{\gamma} L_1 + L_2 \right) \|(w, \{h_{m,k}\}) - (\overline{w}, \{\overline{h}_{m,k}\})\| \]

for \( 0 < m + k < n, \alpha + \beta = n \) and \((\alpha, \beta) \neq (0, n)\). Consequently, on account of relations (8), we arrive at the estimate

\[ \|(W, \{H_{m,k}\}) - (\overline{W}, \{\overline{H}_{m,k}\})\|_p \leq \kappa \|(w, \{h_{m,k}\}) - (\overline{w}, \{\overline{h}_{m,k}\})\| \]  
(22)
where \( \left( \frac{1}{\gamma} L_1 + L_2 \right)^{-1} \kappa \) is the maximum of the three quantities

\[
\gamma \left( C_1(p, D) \| T_{0,n,D} \|_{1,p} + \| T_{0,n,D} \|_p \right)
\]

\[
\gamma \max_{m+k \leq n} \left\{ C_2(p, D) \| T_{0,n,D} \|_{n-m-k+1,p} + \| T_{-m,n-k,D} \|_p \right\}
\]

\[
C_3(p, D) \| T_{0,n,D} \|_{1,p} + \| \Pi_D \|_p
\]

If \( \kappa < 1 \), then the mapping \( \mathcal{Q} \) is contractive in \( \mathcal{L}_p(D) \) \( (2 < p < \infty) \) and it has therefore exactly one fixed element \( (w, \{ h_{m,k} \}) \in \mathcal{L}_p(D) \), by the Banach fixed point theorem.

The contractiveness of \( \mathcal{Q} \) imposes certain restrictions on the constants \( L_1, L_2, \gamma \) and the size of the domain \( D \). Going through an argument similar to the one presented earlier for the case of the existence of a general solution, we can secure the contractiveness of \( \mathcal{Q} \), and hence the existence of a solution \( w \in W_{n,p}(D) \) \( (2 < p < \infty) \) of the boundary value problem posed. It is easy to establish that the solution is unique.

**Theorem 4.** Under the assumptions (A1) - (A3), (15) and \( \kappa < 1 \) the generalized Riemann-Hilbert boundary value problem (1), (19) admits a unique solution \( w \in W_{n,p}(D) \) \( (2 < p < \infty) \).

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**References**


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