Seifert Klein bottles for knots with common boundary slopes

Luis G Valdez-Sánchez

Abstract We consider the question of how many essential Seifert Klein bottles with common boundary slope a knot in $S^3$ can bound, up to ambient isotopy. We prove that any hyperbolic knot in $S^3$ bounds at most six Seifert Klein bottles with a given boundary slope. The Seifert Klein bottles in a minimal projection of hyperbolic pretzel knots of length 3 are shown to be unique and $\pi_1$–injective, with surgery along their boundary slope producing irreducible toroidal manifolds. The cable knots which bound essential Seifert Klein bottles are classified; their Seifert Klein bottles are shown to be non-$\pi_1$–injective, and unique in the case of torus knots. For satellite knots we show that, in general, there is no upper bound for the number of distinct Seifert Klein bottles a knot can bound.

AMS Classification 57M25; 57N10

Keywords Seifert Klein bottles, knot complements, boundary slope

Dedicated to Andrew J Casson on the occasion of his 60th birthday

1 Introduction

For any knot in $S^3$ all orientable Seifert surfaces spanned by the knot have the same boundary slope. The smallest genus of such a surface is called the genus of the knot, and such a minimal surface is always essential in the knot exterior. Moreover, by a result of Schubert–Soltsien [21], any simple knot admits finitely many distinct minimal genus Seifert surfaces, up to ambient isotopy, while for satellite knots infinitely many isotopy classes may exist (cf [7, 14]).

The smallest genus of the nonorientable Seifert surfaces spanned by a knot is the crosscap number of the knot (cf [4]). Unlike their orientable counterparts, nonorientable minimal Seifert surfaces need not have a unique boundary slope. In fact, by [13], any knot $K \subset S^3$ has at most two boundary slopes $r_1, r_2$ corresponding to essential (incompressible and boundary incompressible, in the
geometric sense) Seifert Klein bottles, and if so then \( \Delta(r_1, r_2) = 4 \) or \( 8 \), the latter distance occurring only when \( K \) is the figure-8 knot. The knots for which two such slopes exist were classified in [19] and, with the exception of the \((2, 1, 1)\) and \((-2, 3, 7)\) pretzel knots (the figure-8 knot and the Fintushel-Stern knot, respectively), all are certain satellites of 2-cable knots. Also, a minimal crosscap number Seifert surface for a knot need not even be essential in the knot exterior (cf [13]).

In this paper we study the uniqueness or non-uniqueness, up to ambient isotopy, of essential Seifert Klein bottles for a knot with a fixed boundary slope; we will regard any two such surfaces as equivalent if they are ambient isotopic. Our main result states that any crosscap number two hyperbolic knot admits at most 6 nonequivalent Seifert Klein bottles with a given slope; before stating our results in full we will need some definitions.

We work in the PL category; all 3-manifolds are assumed to be compact and orientable. We refer to ambient isotopies simply as isotopies. Let \( M^3 \) be a 3-manifold with boundary. The pair \((M^3, \partial M^3)\) is irreducible if \( M^3 \) is irreducible and \( \partial M^3 \) is incompressible in \( M^3 \). Any embedded circle in a once punctured Klein bottle is either a meridian (orientation preserving and nonseparating), a longitude (orientation preserving and separating), or a center (orientation reversing); any two meridians are isotopic within the surface, but there are infinitely many isotopy classes of longitudes and centers (cf Lemma 3.1). For a knot \( K \) in \( S^3 \) with exterior \( X_K = S^3 \setminus \text{int}(N(K)) \) and a nontrivial slope \( r \) in \( \partial X_K \), \( K(r) = X_K(r) \) denotes the manifold obtained by Dehn-filling \( X_K \) along \( r \), that is, the result of surgering \( K \) along \( r \). We denote by \( \mathcal{SK}(K, r) \) the collection of equivalence classes of essential Seifert Klein bottles in \( X_K \) with boundary slope \( r \); as pointed out above, \( \mathcal{SK}(K, r) \) is nonempty for at most two distinct integral slopes \( r_1, r_2 \). If \( |\mathcal{SK}(K, r)| \geq 2 \), we will say that the collection \( \mathcal{SK}(K, r) \) is meridional if any two distinct elements can be isotoped so as to intersect transversely in a common meridian, and that it is central if there is a link \( c_1 \cup c_2 \) in \( X_K \) such that any two distinct elements of \( \mathcal{SK}(K, r) \) can be isotoped so as to intersect transversely in \( c_1 \cup c_2 \), and \( c_1, c_2 \) are disjoint centers in each element. For \( P \in \mathcal{SK}(K, r) \), \( N(P) \) denotes a small regular neighborhood of \( P \), and \( H(P) = X_K \setminus \text{int}(N(P)) \) denotes the exterior of \( P \) in \( X_K \). We say \( P \) is unknotted if \( H(P) \) is a handlebody, and knotted otherwise; if the pair \((H(P), \partial H(P))\) is irreducible, we say \( P \) is strongly knotted. If \( \mu, \lambda \) is a standard meridian–longitude pair for a knot \( L \), and \( K \) is a circle embedded in \( \partial X_L \) representing \( p\mu + q\lambda \) for some relatively prime integers \( p, q \) with \( |q| \geq 2 \), we say \( K \) is a \((p, q)\) cable of \( L \); we also call \( K \) a \( q \)-cable knot, or simply a cable knot. In particular, the torus knot \( T(p, q) \) is the \((p, q)\)-cable of the trivial knot.

*Geometry & Topology Monographs, Volume 7 (2004)*
If $X$ is a finite set or a topological space, $|X|$ denotes its number of elements or of connected components.

**Theorem 1.1** Let $K$ be a hyperbolic knot in $S^3$ and $r$ a slope in $\partial X_K$ for which $SK(K, r)$ is nonempty. Then any element of $SK(K, r)$ is either unknotted or strongly knotted, and

(a) if $|SK(K, r)| \geq 2$ then $SK(K, r)$ is either central or meridional; in the first case the link $c_1 \cup c_2$ is unique up to isotopy in $X_K$ and $|SK(K, r)| \leq 3$; in the latter case $|SK(K, r)| \leq 6$;

(b) if $SK(K, r)$ is central then each of its elements is strongly knotted; if some element of $SK(K, r)$ is unknotted then $|SK(K, r)| \leq 2$, and $SK(K, r)$ is meridional if $|SK(K, r)| = 2$;

(c) if some element of $SK(K, r)$ is not $\pi_1$–injective then it is unknotted, $K$ has tunnel number one, and $K(r)$ is a Seifert fibered space over $S^2$ with at most 3 singular fibers of indices $2, 2, n$ and finite fundamental group;

(d) if some element of $SK(K, r)$ is $\pi_1$–injective and unknotted then $K(r)$ is irreducible and toroidal.

**Corollary 1.2** Let $K \subset S^3$ be a hyperbolic knot and $P$ an unknotted element of $SK(K, r)$. Then $\pi_1(K(r))$ is finite iff $P$ is not $\pi_1$–injective. In particular, if $r = 0$, then $P$ is $\pi_1$–injective.

In the case when $SK(K, r)$ is meridional and contains an unknotted element we give examples in Section 6.1 realizing the bound $|SK(K, r)| = 2$ for $K$ a hyperbolic knot. Such examples are obtained from direct variations on the knots constructed by Lyon in [16]; M. Teragaito (personal communication) has constructed more examples along similar lines. It is not known if the other bounds given in Theorem 1.1 are optimal, but see the remark after Lemma 6.8 for a discussion on possible ways of realizing the bound $|SK(K, r)| = 4$, and Section 6.2 for a construction of possible examples of central families $SK(K, r)$ with $|SK(K, r)| = 2$. On the other hand, hyperbolic knots which span a unique Seifert Klein bottle per slope are not hard to find: if we call *algorithmic* any black or white surface obtained from some regular projection of a knot, then the algorithmic Seifert Klein bottles in minimal projections of hyperbolic pretzel knots provide the simplest examples.

**Theorem 1.3** Let $K \subset S^3$ be a hyperbolic pretzel knot, and let $P$ be any algorithmic Seifert Klein bottle in a minimal projection of $K$. Then $P$ is unknotted, $\pi_1$–injective, and unique up to equivalence; moreover, $K(\partial P)$ is irreducible and toroidal.
We remark that any crosscap number two 2–bridge knot is a hyperbolic pretzel knot. Also, as mentioned before, the only hyperbolic knots which bound Seifert Klein bottles of distinct boundary slope are the $(2,1,1)$ and $(-2,3,7)$ pretzel knots. The standard projection of the $(2,1,1)$ pretzel knot simultaneously realizes two algorithmic Seifert Klein bottles of distinct slopes (in fact, this is the only nontrivial knot in $S^3$ with such property), so they are handled by Theorem 1.3. The $(-2,3,7)$ pretzel knot has both an algorithmic and a non algorithmic Seifert Klein bottle; it can be proved that the non algorithmic surface also satisfies the conclusion of Theorem 1.3, but we omit the details.

In contrast, no universal bound for $|SK(K,r)|$ exists for satellite knots:

**Theorem 1.4** For any positive integer $N$, there are satellite knots $K \subset S^3$ with $|SK(K,0)| \geq N$.

Among non hyperbolic knots, the families of cable or composite knots are of particular interest; we classify the crosscap number two cable knots, and find information about the Seifert Klein bottles bounded by composite knots.

**Theorem 1.5** Let $K$ be a knot in $S^3$ whose exterior contains an essential annulus and an essential Seifert Klein bottle. Then either

(a) $K$ is a $(2(2m+1)n \pm 1,4n)$–cable for some integers $m,n$, $n \neq 0$,

(b) $K$ is a $(2(2m+1)(2n+1) \pm 2,2n+1)$–cable of a $(2m+1,2)$–cable knot for some integers $m,n$,

(c) $K$ is a $(6(2n+1) \pm 1,3)$–cable of some $(2n+1,2)$–cable knot for some integer $n$, or

(d) $K$ is a connected sum of two 2–cable knots.

Any Seifert Klein bottle bounded by any composite knot is $\pi_1$–injective, while the opposite holds for any cable knot; in case (a), such a surface is unique up to equivalence.

The next result follows from the proof of Theorem 1.5:

**Corollary 1.6** The crosscap number two torus knots are $T(\pm 5,3)$, $T(\pm 7,3)$, and $T(2(2m+1)n \pm 1,4n)$ for some $m,n$, $n \neq 0$; each bounds a unique Seifert Klein bottle, which is unknotted and not $\pi_1$–injective.
The crosscap numbers of torus knots have been determined in [23]; our classification of these knots follows from Theorem 1.5, whose proof gives detailed topological information about the construction of Seifert Klein bottles for cable knots in general. The classification of crosscap number two composite knots also follows from [22] (where many-punctured Klein bottles are considered); these knots serve as examples of satellite knots any of whose Seifert Klein bottles is $\pi_1$–injective. The Seifert Klein bottles for the knots in Theorem 1.5(a),(b) are disjoint from the cabling annulus, and in (b) one expects the number of Seifert Klein bottles bounded by the knot to depend on the nature of its companions, as in [15, Corollary D]; in (c),(d) the Seifert Klein bottles intersect the cabling or splitting annulus, so in (d) one would expect there to be infinitely nonequivalent Seifert Klein bottles bounded by such knots, as in [7].

The paper is organized as follows. Section 2 collects a few more definitions and some general properties of Seifert Klein bottles. In Section 3 we look at a certain family of crosscap number two satellite knots which bound Seifert Klein bottles with zero boundary slope, and use it to prove Theorem 1.4. In Section 4 we first identify the minimal intersection between an essential annulus and an essential Seifert Klein bottle in a knot exterior, which is the starting point of the proof of Theorem 1.5 and its corollary. Section 5 contains some results on non boundary parallel separating annuli and pairs of pants contained in 3–manifolds with boundary, from both algebraic and geometric points of view. These results have direct applications to the case of unknotted Seifert Klein bottles, but we will see in Lemma 6.3 that if $K$ is any hyperbolic knot and $P,Q \in SK(K,r)$ are any two distinct elements, then $Q$ splits the exterior $H(P)$ of $P$ into two pieces, at least one of which is a genus two handlebody; this observation eventually leads to the proof of Theorem 1.1 in Section 6. After these developments, a proof of Theorem 1.3 is given within a mostly algebraic setting in Section 7.

Acknowledgements

I want to thank Francisco González-Acuña, Enrique Ramírez-Losada and Víctor Núñez for many useful conversations, and Masakazu Teragaito and the Hiroshima University Department of Mathematics for their hospitality while visiting them in the Spring of 2003, when part of this work was completed. Finally, I would like to thank the referee for his many useful suggestions and corrections to the original text.
2 Preliminaries

In this section we set some more notation we will use in the sequel, and establish some general properties of essential Seifert Klein bottles. Let $M^3$ be a 3–manifold with boundary. For any surface $F$ properly embedded in $M^3$ and $c$ the union of some components of $\partial F$; $\tilde{F}$ denotes the surface in $M^3(c) = M^3 \cup \{2\text{–handles along } c\}$ obtained by capping off the circles of $\partial F$ isotopic to $c$ in $\partial M^3$ suitably with disjoint disks in $M^3(c)$. If $G$ is a second surface properly embedded in $M^3$ which intersects $F$ transversely with $\partial F \cap \partial G = \emptyset$, let $N(F \cap G)$ be a small regular neighborhood of $F \cap G$ in $M^3$, and let $\mathcal{A}$ be the collection of annuli obtained as the closures of the components of $\partial N(F \cap G) \setminus (F \cup G)$. We use the notation $F \triangleright G$ to represent the surface obtained by capping off the boundary components in int $M^3$ of $F \cup G \setminus \text{int}(N(F \cap G))$ with suitable annuli from $\mathcal{A}$. As usual, $\Delta(\alpha, \beta)$ denotes the minimal geometric intersection number between circles of slopes $\alpha$, $\beta$ embedded in a torus.

Now let $P$ be a Seifert Klein bottle for a knot $K \subset S^3$, and let $N(P)$ be a small regular neighborhood of $P$ in $X_K$; $N(P)$ is an $I$–bundle over $P$, topologically a genus two handlebody. Let $H(P) = X_K \setminus \text{int}(N(P))$ be the exterior of $P$ in $X_K$. Let $A_K, A'_K$ denote the annuli $N(P) \cap \partial X_K, H(P) \cap \partial X_K$, respectively, so that $A_K \cup A'_K = \partial X_K$ and $A_K \cap A'_K = \partial A_K = \partial A'_K$. Then $\partial P$ is a core of $A_K$, and we denote the core of $A'_K$ by $K'$. Finally, let $T_P$ denote the frontier of $N(P)$ in $X_K$. $T_P$ is a twice-punctured torus such that $N(P) \cap H(P) = T_P$; since $N(P)$ is an $I$–bundle over $P$, $P$ is $\pi_1$–injective in $N(P)$ and $T_P$ is incompressible in $N(P)$.

For any meridian circle $m$ of $P$, there is an annulus $A(m)$ properly embedded in $N(P)$ with $P \cap A(m) = m$ and $\partial A(m) \subset T_P$; we call the circles $\partial A(m) = m_1 \cup m_2$ the lifts of $m$ (to $T_P$). Similarly, for any center circle $c$ of $P$, there is a Moebius band $B(c)$ properly embedded in $N(P)$ with $P \cap B(c) = c$ and $\partial B(c) \subset T_P$; we call $l = \partial B(c)$ the lift of $c$ (to $T_P$). For a pair of disjoint centers in $P$, similar disjoint Moebius bands can be found in $N(P)$. Since the meridian of $P$ is unique up to isotopy, the lifts of a meridian of $P$ are also unique up to isotopy in $T_P$; the lift of a center circle of $P$ depends only on the isotopy class of the center circle in $P$.

We denote the linking form in $S^3$ by $lk(\cdot, \cdot)$.

**Lemma 2.1** Let $P$ be a Seifert Klein bottle for a knot $K \subset S^3$. If $m$ is the meridian circle of $P$, then the boundary slope of $P$ is $\pm 2 \, lk(K, m)$. 

*Geometry & Topology Monographs, Volume 7 (2004)*
Proof  For $m$ a meridian circle of $P$ let $P'$ be the pair of pants $P \setminus \text{int} N(m)$ and $A'$ the annulus $P \cap N(m)$, where $N$ is a small regular neighborhood of $m$ in $X_K$; thus $P = P' \cup A'$, and $\partial P' = \partial P \cup \partial A'$. Fixing an orientation of $P'$ induces an orientation on $\partial P'$ such that the circles $\partial A'$ become coherently oriented in $A'$. In this way an orientation on $m$ is induced, coherent with that of $\partial A'$, such that $\ell k(K, m_1) = \ell k(K, m_2) = \ell k(K, m)$. As the slope of $\partial P$ is integral, hence equal to $\pm \ell k(K, \partial P)$, and $\ell k(K, m_1 \cup m_2 \cup \partial P) = 0$, the lemma follows.

Lemma 2.2  If $P \in SK(K, r)$ then $\tilde{P}$ is incompressible in $K(r)$.

Proof  If $\tilde{P}$ compresses in $K(r)$ along a circle $\gamma$ then $\gamma$ must be orientation-preserving in $\tilde{P}$. Thus, surgering $\tilde{P}$ along a compression disk $D$ with $\partial D = \gamma$ produces either a nonseparating 2-sphere (if $\gamma$ is a meridian) or two disjoint projective planes (if $\gamma$ is a longitude) in $K(r)$, neither of which is possible as $K$ satisfies Property R \cite{8} and $K(r)$ has cyclic integral homology.

Lemma 2.3  Let $m$ be a meridian circle and $c_1, c_2$ be two disjoint center circles of $P$; let $m_1, m_2$ and $l_1, l_2$ be the lifts of $m$ and $c_1, c_2$, respectively. Then,

(a) neither circle $K', l_1, l_2$ bounds a surface in $H(P)$,
(b) neither pair $m_1, m_2$ nor $l_1, l_2$ cobound a surface in $H(P)$,
(c) none of the circles $m_i, l_i$ cobounds an annulus in $H(P)$ with $K'$, and
(d) if $A$ is an annulus in $H(P)$ with $\partial A = \partial A'_K$, then $A$ is not parallel in $X_K$ into $A_K$.

Proof  Let $B_i$ be a Moebius band in $N(P)$ bounded by $l_i$. If $K'$ or $l_i$ bounds a surface $F$ in $H(P)$ then out of the surfaces $P, B_i, F$ it is possible to construct a nonorientable closed surface in $S^3$, which is impossible; thus (a) holds.

Consider the circles $K', \alpha_1, \alpha_2$ in $\partial H(P)$, where $\alpha_1, \alpha_2 = m_1, m_2$ or $l_1, l_2$; such circles are mutually disjoint. Let $P_1$ be the closure of some component of $\partial H(P) \setminus (K' \cup \alpha_1 \cup \alpha_2)$; $P_1$ is a pair of pants. If $\alpha_1, \alpha_2$ cobound a surface $F$ properly embedded in $H(P)$ then $F \cup_{\alpha_1 \cup \alpha_2} P_1$ is a surface in $H(P)$ bounded by $K'$, which can not be the case by (a). Hence $\alpha_1, \alpha_2$ do not cobound a surface in $H(P)$ and so (b) holds.

If any circle $m_1, m_2, l_1, l_2$ cobounds an annulus in $H(P)$ with $K'$ then $\tilde{P}$ compresses in $K(\partial P)$, which is not the case by Lemma 2.2; thus (c) holds.

Finally, if $A \subset H(P)$ is parallel to $A_K$ then the region $V$ cobounded by $A \cup A_K$ is a solid torus with $P \subset V$ and $\partial P$ a core of $A_K$, hence $V(\partial P) = S^3$ contains the closed Klein bottle $\tilde{P}$, which is impossible. This proves (d).
The next result follows from [10, Theorem 1.3] and [19, Lemma 4.2]:

**Lemma 2.4** Let $K$ be a nontrivial knot in $S^3$. If $P$ is a once-punctured Klein bottle properly embedded in $X_K$ then $P$ is essential iff $K$ is not a 2-cable knot, and in such case $P$ has integral boundary slope.

### 3 The size of $\text{SK}(K, 0)$

In this section we consider a special family of crosscap number two satellite knots $\{\mathcal{K}_n\}$, which generalizes the example of W.R. Alford in [1]; the knots in this family are constructed as follows. For each $i \geq 1$ let $K_i$ be a nontrivial prime knot in $S^3$. Figure 1 shows a pair of pants $F_n = A' \cup B_n \cup A''$ and the knot $\mathcal{K}_n \subset \partial F_n$.

![Figure 1: The pair of pants $F_n = A' \cup B_n \cup A''$ and the knot $\mathcal{K}_n \subset \partial F_n$](image)

For each $1 \leq s < n$, let $P_s$ be the Seifert Klein bottle bounded by $\mathcal{K}_n$ constructed by attaching an annulus $A_s$ to the two boundary circles of $F_n$ other than $\mathcal{K}_n$, which swallows the factors $K_1, \ldots, K_s$ and follows the factors $K_{s+1}, \ldots, K_n$, as indicated in Figure 2.

Notice that the core of $A_s$, which has linking number zero with $\mathcal{K}_n$, is a meridian circle of $P_s$, so the boundary slope of $P_s$ is zero by Lemma 2.1. Our goal is to show that $P_r$ and $P_s$ are not equivalent for $r \neq s$, so that $|\text{SK}(\mathcal{K}_n, 0)| \geq n - 1$,
which will prove Theorem 1.4. In fact, we will prove the stronger statement that $P_r$ and $P_s$ are not equivalent for $r \neq s$ even under homeomorphisms of $S^3$.

The following elementary result on intersection properties between essential circles in a once-punctured Klein bottle will be useful in determining all centers of any of the above Seifert Klein bottles of $K_n$; we include its proof for the convenience of the reader.

**Lemma 3.1** Let $P = A \cup B$ be a once-punctured Klein bottle, where $A$ is an annulus and $B$ is a rectangle with $A \cap B = \partial A \cap \partial B$ consisting of two opposite edges of $\partial B$, one in each component of $\partial A$. Let $m$ be a core of $A$ (ie a meridian of $P$) and $b$ a core arc in $B$ parallel to $A \cap B$. If $\omega$ is any nontrivial circle embedded in $P$ and not parallel to $\partial P$ which has been isotoped so as to intersect $m \cup b$ transversely and minimally, then $\omega$ is a meridian, center, or longitude circle of $P$ iff $(|\omega \cap m|, |\omega \cap b|) = (0, 0)$, $(1, 1)$, or $(2, 2)$, respectively.

**Proof** Let $I_1, I_2$ denote the components of $A \cap B = \partial A \cap \partial B$. After isotoping $\omega$ so as to intersect $m \cup b$ transversely and minimally, either $\omega$ lies in int $A$ and is parallel to $m$, or $\omega \cap B$ consists of disjoint spanning arcs of $B$ with endpoints in $I_1 \cup I_2$, while $\omega \cap A$ consists of disjoint arcs which may split into at most 4 parallelism classes, denoted $\alpha, \beta, \gamma, \delta$; the situation is represented in Figure 3.

Suppose we are in the latter case. As $\omega$ is connected and necessarily $|\alpha| = |\delta|$, if $|\alpha| > 0$ then $\omega \cap A$ must consist of one arc of type $\alpha$ and one arc of type $\delta$; but then $\omega$ is parallel to $\partial P$, which is not the case. Thus we must have $|\alpha| = |\delta| = 0$, in which case $|\beta| + |\gamma| = |\omega \cap b| = n$ for some integer $n \geq 1$. Notice that $\omega$ is a center if $n = 1$ and a longitude if $n = 2$; in the first case
\(|\beta|, |\gamma|\} = \{0, 1\}$$, while in the latter $$\{ |\beta|, |\gamma| \} = \{0, 2\}$$. These are the only possible options for $$\omega$$ and $$n$$ whenever $$|\beta| = 0$$ or $$|\gamma| = 0$$.

Assume that $$|\beta|, |\gamma| \geq 1$$, so $$n \geq 3$$, and label the endpoints of the arcs $$\omega \cap A$$ and $$\omega \cap B$$ in $$I_1, I_2$$ consecutively with 1, 2, \ldots, $$n$$, as in Figure 3. We assume, as we may, that $$|\beta| \leq |\gamma|$$. We start traversing $$\omega$$ from the point labelled 1 in $$I_1 \subset \partial B$$ in the direction of $$I_2 \subset \partial B$$, within $$B$$, then reach the endpoint of an arc component of $$\omega \cap A$$ in $$I_2 \subset \partial A$$, then continue within $$A$$ to an endpoint in $$I_1 \subset \partial A$$, and so on until traversing all of $$\omega$$. The arc components of $$\omega \cap B$$ and $$\omega \cap A$$ traversed in this way give rise to permutations $$\sigma, \tau$$ of 1, 2, \ldots, $$n$$, respectively, given by

$$\sigma(x) = n - x + 1$$ for $$1 \leq x \leq n$$, and $$\tau(x) = n + x - |\beta|$$ for $$1 \leq x \leq |\beta|$$, $$\tau(x) = x - |\beta|$$ for $$|\beta| < x \leq n$$.

Clearly, the number of components of $$\omega$$ equals the number of orbits of the permutation $$\tau \circ \sigma$$. But, since $$|\beta| \leq |\gamma|$$, the orbit of $$\tau \circ \sigma$$ generated by 1 consists only of the numbers 1 and $$n - |\beta|$$; as $$\omega$$ is connected, we must then have $$n \leq 2$$, which is not the case. Therefore, the only possibilities for the pair $$(|\omega \cap m|, |\omega \cap b|)$$ are the ones listed in the lemma.

For a knot $$L \subset S^3$$, we will use the notation $$C_2(L)$$ to generically denote any 2–cable of $$L$$; observe that any nontrivial cable knot is prime.

**Lemma 3.2** For $$1 \leq s < n$$, any center of $$P_s$$ is a knot of type

$$K_1 \# \cdots \# K_s \# C_2(K_{s+1} \# \cdots \# K_n)$$.

**Proof** By Lemma 3.1, any center $$c_s \subset P_s$$ can be constructed as the union of two arcs: one that runs along the band $$B_n$$ and the other any spanning arc of $$A_s$$. Since the annulus ‘swallows’ the factor $$K_1 \# \cdots \# K_s$$ and ‘follows’
Seifert Klein bottles for knots with common boundary slopes

\[ K_{s+1} \# \cdots \# K_n, \] any such center circle \( c_s \) (shown in Figure 2) isotopes into a knot of the type represented in Figure 4, which has the given form.

\[ \square \]

**Lemma 3.3** For \( 1 \leq r < s < n \), there is no homeomorphism \( f: S^3 \to S^3 \) which maps \( P_r \) onto \( P_s \).

**Proof** Suppose there is a homeomorphism \( f: S^3 \to S^3 \) with \( f(P_r) = P_s \); then for any center \( c_r \) of \( P_r \), \( c_s = f(c_r) \) is a center of \( P_s \). But then \( c_r \) and \( c_s \) have the same knot type in \( S^3 \), which by Lemma 3.2 can not be the case since \( c_r \) has \( r + 1 \) prime factors while \( c_s \) has \( s + 1 \) prime factors.

\[ \square \]

**Proof of Theorem 1.4** By Lemma 3.3, the Seifert Klein bottles \( P_r \) and \( P_s \) for \( K_n \) are not equivalent for \( 1 \leq r < s < n \), hence \( |\text{SK}(K_n, 0)| \geq n - 1 \) and the theorem follows.

\[ \square \]

4 Cable and composite knots

In this section we assume that \( K \) is a nontrivial knot in \( S^3 \) whose exterior \( X_K \) contains an essential annulus \( A \) and an essential Seifert Klein bottle \( P \); that is, \( K \) is a crosscap number two cable or composite knot. We assume that \( A \) and \( P \) have been isotoped so as to intersect transversely with \( |A \cap P| \) minimal, and denote by \( G_A = A \cap P \subset A \) and \( G_P = A \cap P \subset P \) their graphs of intersection. We classify these graphs in the next lemma; the case when \( A \) has meridional boundary slope is treated in full generality in [7, Lemma 7.1].

**Lemma 4.1** Either \( A \cap P = \emptyset \) or \( \Delta(\partial A, \partial P) = 1 \) and \( A \cap P \) consists of a single arc which is spanning in \( A \) and separates \( P \) into two Möbius bands.
**Proof** Suppose first that $\Delta(\partial A, \partial P) = 0$, so that $\partial A \cap \partial P = \emptyset$; in particular, the boundary slopes of $A$ and $P$ are integral and $K$ is a cable knot with cabling annulus $A$. If $A \cap P \neq \emptyset$, then $A \cap P$ consists of nontrivial orientation preserving circles in $A$ and $P$. If any such circle $\gamma \subset A \cap P$ is parallel to $\partial P$ in $P$, we may assume it cobounds an annulus $A_\gamma$ with $\partial P$ in $P$ such that $A \cap \text{int} A_\gamma = \emptyset$, so, by minimality of $|A \cap P|$, $A_\gamma$ must be an essential annulus in the closure $V$ of the component of $X_K \setminus A$ containing it; as $K$ is a cable knot, $V$ must be the exterior of some nontrivial knot of whom $K$ is a $q$–cable for some $q \geq 2$. But then the boundary slope of $A_\gamma$ in $V$ is of the form $a/q$, that is, nonintegral nor $1$, contradicting the fact that $A_\gamma$ is essential. Hence no component of $A \setminus P$ is parallel to $\partial P$ in $P$ and so $\hat{P}$ compresses in $K(\partial P)$ along a subdisk of $\hat{A}$, which is not possible by Lemma 2.2. Therefore $A$ and $P$ are disjoint in this case.

Suppose now that $\Delta(\partial A, \partial P) \neq 0$; by minimality of $|A \cap P|$, $A \cap P$ consists only of arcs which are essential in both $A$ and $P$. If $\alpha$ is one such arc then, as $\alpha$ is a spanning arc of $A$, $|\partial P| = 1$, and $X_K$ is orientable, it is not hard to see that $\alpha$ must be a positive arc in $P$, in the sense of [13] (this fact does not follow directly from the parity rule in [13] since $|\partial A| \neq 1$, but its proof is equally direct). Thus all the arcs of $G_P$ are positive in $P$.

Suppose $a, b$ are arcs of $G_P$ which are parallel and adjacent in $P$, and let $R$ be the closure of the disk component of $P \setminus (a \cup b)$. Then $R$ lies in the closure of some component of $X_K \setminus A$, and, by minimality of $|A \cap P|$, the algebraic intersection number $\partial R \cdot \text{core}(A)$ must be $\pm 2$. But then $K$ must be a 2–cable knot, with cabling annulus $A$, contradicting Lemma 2.4 since $P$ is essential. Therefore no two arcs of $G_P$ are parallel.

Since any two disjoint positive arcs in $P$ are mutually parallel, and $A$ separates $X_K$, the above arguments show that $G_P$ consists of exactly one essential arc which is separating in $P$. The lemma follows.

**Proof of Theorem 1.5** By Lemma 4.1, $A \cap P = \emptyset$ or $|A \cap P| = 1$. Throughout the proof, none of the knots considered will be a 2–cable, hence any Seifert Klein bottle constructed for them will be essential by Lemma 2.4. Let $V, W$ be the closures of the components of $X_K \setminus A$.

**Case 1** $A \cap P = \emptyset$

Here $K$ must be a cable knot: for the slope of $\partial A$ must be integral or $\infty$, the latter case being impossible since otherwise $\hat{P}$ is a closed Klein bottle in

*Geometry & Topology Monographs, Volume 7 (2004)*
Seifert Klein bottles for knots with common boundary slopes

$K(\infty) = S^3$. Hence $\partial V$ and $\partial W$ are parallel tori in $S^3$, and we may regard them as identical for framing purposes. We will assume $P \subset V$.

Suppose that $V$ is a solid torus; then $P$ is not $\pi_1$–injective in $X_K$. Let $\mu, \lambda$ be a standard meridian–longitude pair for $\partial V$, framed as the boundary of the exterior of a core of $V$ in $S^3$. Since $V$ is a solid torus and $P$ is essential in $X_K$, $P$ must boundary compress in $V$ to some Moebius band $B$ in $V$ with $\Delta(\partial P, \partial B) = 2$. Suppose the slope of $\partial P$ in $V$ is $p\mu + q\lambda$; as the slope of $\partial B$ is of the form $(2m+1)\mu + 2\lambda$ for some integer $m$, we must have $(2m+1)q - 2p = \pm 2$, hence $q \equiv 0 \mod 4$ and $p$ is odd. Therefore, the slope of $\partial P$ in $V$ must be of the form $(2(2m+1)\mu + 4n\lambda)$ for some $n \neq 0$. Conversely, it is not hard to see that any such slope bounds an incompressible once-punctured Klein bottle in $V$, which is easily seen to be unique up to equivalence. Therefore, $K$ is a $(2(2m+1)n \pm 1, 4n)$ cable of the core of $V$ for some integers $m, n$, and (a) holds.

If $V$ is not a solid torus then $W$ is a solid torus and, since $\partial V$ and $\partial W$ are parallel in $S^3$, we can frame $\partial V$ as the boundary of the exterior of a core of $W$ in $S^3$ via a standard meridian–longitude pair. Then a core of $A$ has slope $p\mu + q\lambda$ in $\partial V$ with $|q| \geq 2$, so $P$ has nonintegral boundary slope in $V$ and hence must boundary compress in $V$ by Lemmas 2.2 and 2.4 to an essential Moebius band $B$ in $V$ with $\Delta(\partial P, \partial B) = 2$; in particular, $P$ is not $\pi_1$–injective in $V$, hence neither in $X_K$. The slope of $\partial B$ in $\partial V$ must be of the form $(2(2m+1)\mu + 4n\lambda)$ for some integer $m$, and so the slope of $\partial P$ in $\partial V$ is of the form $(2(2m+1)\mu + 4n\lambda)$ for some integer $n$. It follows that $K$ is a $(2(2m+1)\mu + 4n\lambda)$ cable of some $(2m+1, 2n+1)$ cable knot for some integers $m, n$, and (b) holds.

Case 2 $|A \cap P| = 1$

By Lemma 4.1, $B_1 = P \cap V$ and $B_2 = P \cap W$ are Moebius bands whose boundaries intersect $A$ in a single spanning arc.

If the slope of $\partial A$ in $X_K$ is $\infty$ then $K = K_1 \# K_2$ for some nontrivial knots $K_1, K_2$ with $V = X_{K_1}$ and $W = X_{K_2}$. Moreover, as $B_i \subset X_{K_i}$, the $K_i$’s are 2–cable knots and (d) holds.

Suppose now that $\partial A$ has integral slope in $X_K$; then at least one of $V, W$, say $V$, is a solid torus. As in Case 1, we may regard $\partial V$ and $\partial W$ as identical tori for framing purposes. If, say, $W$ is not a solid torus, frame $\partial V$ and $\partial W$ via a standard meridian–longitude pair $\mu, \lambda$ as the boundary of the exterior of a core of $V$ in $S^3$. Then $\partial A$ has slope $p\mu + q\lambda$ for some integers $p, q$ with $|q| \geq 2$ in...
The situation is somewhat different if both \( V \) and \( W \) are solid tori. Here one can find meridian–longitude framings \( \mu, \lambda \) for \( \partial V \) and \( \partial W \), such that \( \mu \) bounds a disk in \( V \) and \( \lambda \) bounds a disk in \( W \). Now, the slope of \( \partial A \) in \( \partial V \) is of the form \( p\mu + q\lambda \) for some integers \( p,q \) with \( |p|,|q| \geq 2 \). The slope of \( \partial B \) in \( V \) is of the form \( a\mu + 2\lambda \), while that of \( \partial B \) in \( W \) is of the form \( 2\mu + b\lambda \). As \( \Delta(\partial A, \partial B) = 1 \) for \( i = 1,2 \), it follows that \( 2p - aq = \varepsilon \) and \( 2q - bp = \delta \) for some \( \varepsilon, \delta = \pm 1 \). Assuming, without loss of generality, that \( |p| > |q| \geq 2 \), the only solutions to the above equations can be easily shown to be \( (|p|,|q|) = (5,3) \) or \( (7,3) \). Hence \( K \) must be the torus knot \( T(\pm 5,3) \) or \( T(\pm 7,3) \). Notice that this case fits in Case (c) with \( n = -1,0 \).

Thus in each of the above cases the knot \( K \) admits an essential Seifert Klein bottle, which is unique in Case (a). We have also seen that in Cases (a) and (b) such a surface is never \( \pi_1 \)-injective.

In case (c), \( K \) is a cable knot with cabling annulus \( A \) such that \( X_K = V \cup_A W \), where \( V \) is a solid torus and \( W \) is the exterior of some (possibly trivial) knot in \( S^3 \), and \( B_1 = P \cap V, B_2 = P \cap W \) are Moebius bands. Using our notation for \( H(P), T_P, K' \) relative to the surface \( P \), as \( N(P) = N(B_1) \cup_{A \cap N(P)} N(B_2) \), we can see that

\[
H(P) = (V \setminus \text{int} N(B_1)) \cup_D (W \setminus \text{int} N(B_2))
\]

where \( D = \text{cl}(A \setminus N(P \cap A)) \subset A \) is a disk (a rectangle). Observe that

(i) \( V \setminus \text{int} N(B_1) = A_1 \times I \), where \( A_1 = A_1 \times 0 \) is the frontier annulus of \( N(B_1) \) in \( V \),

(ii) the rectangle \( D \subset A \) has one side along the boundary component of the annulus \( A'_1 = \partial(A_1 \times I) \setminus \text{int} A_1 \subset \partial(A_1 \times I) \) and the opposite side along the other boundary component, and

(iii) \( K' \cap (A_1 \times I) \) is an arc with one endpoint on each of the sides of the rectangle \( \partial D \) interior to \( A'_1 \).

Let \( \alpha \) be a spanning arc of \( A'_1 \) which is parallel and close to one of the arcs of \( \partial D \) interior to \( A'_1 \), and which is disjoint from \( D \); such an arc exists by (i)–(iii), and \( \alpha \) intersects \( K' \) transversely in one point by (iii). Therefore \( \alpha \times I \subset A_1 \times I \).
is a properly embedded disk in $H(P)$ which intersects $K'$ transversely in one point, so $T_P$ is boundary compressible in $X_K$ and hence $P$ is not $\pi_1$-injective.

In case (d), keeping the same notation as above, if $P$ is not $\pi_1$-injective then $T_P$ is not $\pi_1$-injective either, so $T_P$ compresses in $X_K$, and in fact in $H(P)$, producing a surface with at least one component an annulus $A_P \subset H(P)$ properly embedded in $X_K$ with the same boundary slope as $P$. Notice that $A_P$ can not be essential in $X_K$: for $A_P$ must separate $X_K$, and if it is essential then not both graphs of intersection $A \cap A_P \subset A$, $A \cap A_P \subset A_P$ can consist of only essential arcs by the Gordon–Luecke parity rule [6]. Therefore $A_P$ must be boundary parallel in $X_K$ and the region of parallelism must lie in $H(P)$ by Lemma 2.3(d), so $T_P$ is boundary compressible and there is a disk $D'$ in $H(P)$ intersecting $K'$ transversely in one point. As before,

$$H(P) = (V \setminus \text{int } N(B_1)) \cup_D (W \setminus \text{int } N(B_2)),$$

where $V = X_{K_1}$ and $W = X_{K_2}$. Isotope $D'$ so as to intersect $D$ transversely with $|D \cap D'|$ minimal. If $|D \cap D'| > 0$ and $E'$ is an outermost disk component of $D' \setminus D$, then, as $|D' \cap K'| = 1$, $E'$ is a nontrivial disk in, say, $V \setminus \text{int } N(B_1)$ with $|E' \cap K'| = 0$ or 1. Hence $V \setminus \text{int } N(B_1)$ is a solid torus whose boundary intersects $K'$ in a single arc $K_1'$ with endpoints on $D \subset \partial V$, and using $E'$ it is possible to construct a meridian disk $E''$ of $V \setminus \text{int } N(B_1)$ disjoint from $D$ which intersects $K'$ coherently and transversely in one or two points. If $|D \cap D'| = 0$ we set $E'' = D'$.

Let $L_1$ be the trivial knot whose exterior $X_{L_1}$ is the solid torus $V \setminus \text{int } N(B_1)$, so that $K_1$ is a 2-cable of $L_1$. As $X_{K_1} = X_{L_1} \cup N(B_1)$, where the gluing annulus $X_{L_1} \cap N(B_1)$ is disjoint from the arc $K' \cap X_{L_1}$ in $\partial X_{L_1}$, $E''$ must also intersect $\partial B_1$ coherently and transversely in one or two points. In the first case $K_1$ must be a trivial knot, while in the second case $X_{K_1} \subset S^3$ contains a closed Klein bottle. As neither option is possible, $P$ must be $\pi_1$-injective.

**Proof of Corollary 1.6** That any crosscap number two torus knot is of the given form follows from the proof of Theorem 1.5. The uniqueness of the slope bounded by a Seifert Klein bottle in each case follows from [19], and that no such surface is $\pi_1$-injective also follows from the proof of Theorem 1.5. For a knot $K$ of the form $T(\cdot, 4n)$, any Seifert Klein bottle $P$ is disjoint from the cabling annulus and can be constructed on only one side of the cabling annulus. Since $P$ is not $\pi_1$-injective, $T_P$ compresses in $H(P)$ giving rise to the cabling annulus of $K$; thus uniqueness and unknottedness follows. For the knots $T(\pm 5, 3), T(\pm 7, 3)$ any Seifert Klein bottle $P$ is separated by the cabling annulus into two Moebius bands; as a Moebius band in a solid torus is
unique up to ambient isotopy fixing its boundary, uniqueness of $P$ again follows. Since, in the notation of the proof of Theorem 1.5, $H(P) = (V \setminus \text{int } N(B_1)) \cup_D (W \setminus \text{int } N(B_2))$, and $V \setminus \text{int } N(B_1)$, $W \setminus \text{int } N(B_2)$ are solid tori, $H(P)$ is a handlebody and so any Seifert Klein bottle in these last cases is unknotted.

5 Primitives, powers, and companion annuli

Let $M^3$ be a compact orientable 3-manifold with boundary, and let $A$ be an annulus embedded in $\partial M^3$. We say that a separating annulus $A_0$ properly embedded in $M^3$ is a companion of $A$ if $\partial A_0 = \partial A$ and $A_0$ is not parallel into $\partial M^3$; we also say that $A'$ is a companion of any circle $c$ embedded in $\partial M^3$ which is isotopic to a core of $A$. Notice that the requirement of a companion annulus being separating is automatically met whenever $M^3 \subset S^3$, and that if $\partial M^3$ has no torus component then we only have to check that $A'$ is not parallel into $A$. The following general result will be useful in the sequel.

Lemma 5.1 Let $M^3$ be an irreducible and atoroidal 3-manifold with connected boundary, and let $A', B'$ be companion annuli of some annuli $A, B \subset \partial M^3$. Let $R, S$ be the regions in $M^3$ cobounded by $A, A'$ and $B, B'$, respectively. If $A$ is incompressible in $M^3$ then $R$ is a solid torus, and if $A$ and $B$ are isotopic in $\partial M^3$ then $R \cap S \neq \emptyset$. In particular, $A'$ is unique in $M^3$ up to isotopy.

Proof Let $c$ be a core of $A$; push $R$ slightly into int $M^3$ via a small collar $\partial M^3 \times I$ of $\partial M^3 = \partial M^3 \times 0$, and let $A'' = c \times I$. Observe the annulus $A''$ has its boundary components $c \times 0$ on $\partial M^3$ and $c \times 1$ on $\partial R$. Also, $\partial R$ can not be parallel into $\partial M^3$, for otherwise $A'$ would be parallel into $\partial M^3$; $M^3$ being atoroidal, $\partial R$ must compress in $M^3$.

Let $D$ be a nontrivial compression disk of $\partial R$ in $M^3$. If $D$ lies in $N^3 = M^3 \setminus \text{int } R$ then $\partial D$ compresses in $N^3 \cup (2\text{-handle along } c \times 0)$ along the circles $\partial D$ and $c \times 1 \subset \partial D$, hence $\partial D$ and $c \times 1$ are isotopic in $\partial R$, which implies that $A$ compresses in $M^3$, contradicting our hypothesis. Thus $D$ lies in $R$, so $R$ is a solid torus.

Suppose now that $R \cap S = \emptyset$, so that $A \cap B = \emptyset = A' \cap B'$, and that $A, B$ are isotopic in $\partial M^3$. Then one boundary component of $A$ and one of $B$ cobound an annulus $A^*$ in $\partial M^3$ with interior disjoint from $A \cup B$ (see Figure 5(a)). Since, by the above argument, $R$ and $S$ are solid tori with $A, B$ running more than once around $R, S$, respectively, the region $N(R \cup S \cup A^*)$ is a Seifert fibered
space over a disk with two singular fibers, hence not a solid torus, contradicting our initial argument. Therefore $R \cap S \neq \emptyset$.

Finally, isotope $B$ into the interior of $A$, carrying $B'$ along the way so that $A'$ and $B'$ intersect transversely and minimally. If $A' \cap B' = \emptyset$ then $B'$ lies in the solid torus $R$ and so must be parallel to $A'$ in $R$. Otherwise, any component $B''$ of $B' \cap (M^3 \setminus \text{int} R)$ is an annulus that cobounds a region $V \subset M^3 \setminus \text{int} R$ with some subannulus $A''$ of $A'$ (see Figure 5(b)). Since $|A' \cap B'|$ is minimal, $A''$ and $B''$ are not parallel within $V$ and so $R \cup A'' V$ is not a solid torus. But then the frontier of $R \cup A'' V$ is a companion annulus of $A$, contradicting our initial argument. The lemma follows.

**Remark** In the context of Lemma 5.1, if $A \subset \partial M^3$ compresses in $M^3$ and $M^3$ is irreducible, then $A$ has a companion if the core of $A$ bounds a nonseparating disk in $M^3$, in which case $A$ has infinitely many nonisotopic companion annuli.

In the special case when $H$ is a genus two handlebody, an algebraic characterization of circles in $\partial H$ that admit companion annuli will be useful in the sequel, particularly in Section 7; we introduce some terminology in this regard.

For $H$ a handlebody and $c$ a circle embedded in $\partial H$, we say $c$ is *algebraically primitive* if $c$ represents a primitive element in $\pi_1(H)$ (relative to some basepoint), and we say $c$ is *geometrically primitive* if there is a disk $D$ properly embedded in $H$ which intersects $c$ transversely in one point. It is well known that these two notions of primitivity for circles in $\partial H$ coincide, so we will refer to such a circle $c$ as being simply *primitive* in $H$. We say that $c$ is a *power* in $H$ if $c$ represents a proper power of some nontrivial element of $\pi_1(H)$.

The next result follows essentially from [3, Theorem 4.1]; we include a short version of the argument for the convenience of the reader.

**Lemma 5.2** Let $H$ be a genus two handlebody and $c$ a circle embedded in $\partial H$ which is nontrivial in $H$. Then,
(a) $\partial H \setminus c$ compresses in $H$ iff $c$ is primitive or a proper power in $H$, and
(b) $c$ has a companion annulus in $H$ iff $c$ is a power in $H$.

Proof For (a), let $D \subset H$ be a compression disk of $\partial H \setminus c$. If $D$ does not separate $H$ then, since $c$ and $\partial D$ can not be parallel in $\partial H$, there is a circle $\alpha$ embedded in $\partial H \setminus c$ which intersects $\partial D$ transversely in one point, and so the frontier of a regular neighborhood in $H$ of $D \cup \alpha$ is a separating compression disk of $\partial H \setminus c$. Thus, we may assume that $D$ separates $H$ into two solid tori $V, W$ with $D = V \cap W = \partial V \cap \partial W$ and, say, $c \subset V$. Therefore $c$ is either primitive or a power in $V$ and hence in $H$. Conversely, suppose $c$ is primitive or a power in $H$. Then $\pi_1(H(c))$ is either free cyclic or has nontrivial torsion by [17, Theorems N3 and 4.12] and so the pair $(H(c), \partial H(c))$ is not irreducible by [12, Theorem 9.8]. Therefore, by the 2-handle addition theorem (cf [3]), the surface $\partial H \setminus c$ must be compressible in $H$.

For (b), let $A$ be an annulus neighborhood of $c$ in $\partial H$. If $A'$ is a companion annulus of $A$ then $A'$ must boundary compress in $H$ into a compression disk for $\partial H \setminus A$; as in (a), we can assume that $\partial H \setminus A$ compresses along a disk $D$ which separates $H$ into two solid tori $V, W$ with $D = V \cap W = \partial V \cap \partial W$ and $c, A, A' \subset V$. Since $A'$ is parallel in $V$ into $\partial V$ but not into $A$, it follows that $c$ is a power in $V$, hence in $H$. The converse follows in a similar way.

A special family of incompressible pairs of pants properly embedded in a 3–manifold $H \subset S^3$ with $\partial H$ a genus two surface, which are not parallel into $\partial H$, appear naturally in Section 6. We will establish some of their properties in the next lemma; the following construction will be useful in this regard. If $F$ is a proper subsurface of $\partial H$, $c$ is a component of $\partial F$, and $A$ is a companion annulus of $c$ in $H$ with $\partial A = \partial_1 A \cup \partial_2 A$, we isotope $A$ so that, say, $\partial_1 A = c$ and $\partial_2 A \subset \partial H \setminus F$, and denote by $F \oplus A$ the surface $F \cup A$, isotoped slightly so as to lie properly embedded in $H$.

For $c_1, c_2, c_3$ disjoint circles embedded in $\partial H$ and nontrivial in $H$, we say $c_1, c_2$ are simultaneously primitive away from $c_3$ if there is some disk $D$ in $H$ disjoint from $c_3$ which transversely intersects $c_1$ and $c_2$ each in one point; notice that if $H$ is a handlebody and $c_1, c_2$ are simultaneously primitive away from some other circle, then $c_1, c_2$ are indeed primitive in $H$. We also say $c_1, c_2$ are coannular if they cobound an annulus in $H$.

Lemma 5.3 Let $H \subset S^3$ be a connected atoroidal 3–manifold with connected boundary of genus two. Let $c_1, c_2, c_3$ be disjoint nonseparating circles embedded
in $\partial H$ which are nontrivial in $H$, no two are coannular, and separate $\partial H$ into two pairs of pants $P_1, P_2$. Let $Q_0$ be a pair of pants properly embedded in $H$ with $\partial Q_0 = c_1 \cup c_2 \cup c_3$ and not parallel into $\partial H$. Then $Q_0$ is incompressible and separates $H$ into two components with closures $H_1, H_2$ and $\partial H_i = Q_0 \cup P_i$, and if $Q_0$ boundary compresses in $H$ the following hold:

(a) $c_i$ has a companion annulus in $H$ for some $i = 1, 2, 3$;
(b) if only $c_1$ has a companion annulus in $H$, say $A'_1$, then,
   (i) for $i, j = 1, 2$, $P_i \oplus A'_i$ is isotopic to $P_j$ in $H$ iff $H$ is a handlebody and $c_2, c_3$ are simultaneously primitive in $H$ away from $c_1$;
   (ii) if $Q_0$ boundary compresses in $H_i$ then it boundary compresses into a companion annulus of $c_1$ in $H_i$, $Q_0 = P_i \oplus A'_i$ in $H_i$; $H_1$ is a handlebody, and any pair of pants in $H_i$ with boundary $\partial Q_0$ is parallel into $\partial H_i$,
   (iii) if $R_0$ is a boundary compressible pair of pants in $H$ disjoint from $Q_0$ with $\partial R_0$ isotopic to $\partial Q_0$ in $\partial H$, then $R_0$ is parallel to $Q_0$ or $\partial H$.

**Proof** Observe $H$ is irreducible and $Q_0$ is incompressible in $H$; since $H \subset S^3$, $Q_0$ must separate $H$, otherwise $Q_0 \cup P_i$ is a closed nonseparating surface in $S^3$, which is impossible. Also, by Lemma 5.1, a companion annulus of any $c_i$ is unique up to isotopy and cobounds a solid torus with $\partial H$. Let $H_1, H_2$ be the closures of the components of $H \setminus Q_0$, with $\partial H_i = Q_0 \cup P_i$; as $Q_0$ is incompressible, both $H_1$ and $H_2$ are irreducible and atoroidal, so again companion annuli in $H_i$ are unique up to isotopy and cobound solid tori with $\partial H_i$. Let $D$ be a boundary compression disk for $Q_0$, say $D \subset H_1$. We consider three cases.

**Case 1** The arc $Q_0 \cap \partial D$ does not separate $Q_0$.

Then $D$ is a nonseparating disk in $H_1$, and we may assume the arc $Q_0 \cap \partial D$ has one endpoint in $c_1$ and the other in $c_2$, so that $|c_1 \cap D| = 1 = |c_2 \cap D|$ and $c_3 \cap D = \emptyset$. Hence the frontier $D'$ of a small regular neighborhood of $c_1 \cup D$ is a properly embedded separating disk in $H_1$ which intersects $c_2$ in two points and whose boundary separates $c_1$ from $c_3$ in $\partial H$; the situation is represented in Figure 6, with $c_1 = u, c_2 = v, c_3 = w$ and $D = \Sigma, D' = \Sigma'$. Clearly, boundary compressing $Q_0$ along $D$ produces an annulus $A'_3$ in $H_1$ with boundaries parallel to $c_3$ in $\partial H_1$, and if $A'_3$ is parallel into $\partial H_1$ then $Q_0$ itself must be parallel into $P_1 \subset \partial H_1$, which is not the case. Therefore $c_3$ has a companion annulus in $H_1$, hence in $H$.
Notice that $H_1$ must be a handlebody in this case since the region cobounded by $A'_2$ and $\partial H_1$ is a solid torus by Lemma 5.1, and that $Q_0 = P_1 \oplus A'_3$ in $H$ and $c_1, c_2$ are simultaneously primitive in $H_1$ away from $c_3$.

**Case 2** The arc $Q_0 \cap \partial D$ separates $Q_0$ and $D$ separates $H_1$.

Here the endpoints of the arc $Q_0 \cap \partial D$ lie in the same component of $\partial Q_0$, so we may assume that $|c_1 \cap D| = 2$ with $c_1 \cdot D = 0$ while $c_2, c_3$ are disjoint from and separated by $D$; the situation is represented in Figure 6, with $c_1 = v, c_2 = u, c_3 = w$, and $D = \Sigma'$. Thus, boundary compressing $Q_0$ along $D$ produces two annuli $A'_2, A'_3$ in $H_1 \setminus D$ with boundaries parallel to $c_2, c_3$, respectively. Since $Q_0$ is not parallel into $P_1 \subset \partial H_1$, at least one of these annuli must be a companion annulus.

Notice that if only one such annulus, say $A'_2$, is a companion annulus then, as in Case 1, $H_1$ is a handlebody, $Q_0 = P_1 \oplus A'_2$ in $H$, and $c_1, c_3$ are simultaneously primitive in $H_1$ away from $c_2$.

**Case 3** The arc $Q_0 \cap \partial D$ separates $Q_0$ and $D$ does not separate $H_1$.

As in Case 2, we may assume that $|c_1 \cap D| = 2$ with $c_1 \cdot D = 0$ while $c_2, c_3$ are disjoint from $D$. Since $D$ does not separate $H_1$, boundary compressing $Q_0$ along $D$ produces two nonseparating annuli in $H_1$, each with one boundary parallel to $c_2$ and the other parallel to $c_3$. Since $c_2, c_3$ are not coannular in $H$, this case does not arise. Therefore (a) holds.

For (b)(i), let $V, H'$ be the closures of the components of $H \setminus A'_1$, with $V$ a solid torus and $P_j \subset \partial H'$; observe that $\partial H'$ can be viewed as $(P_1 \oplus A'_1) \cup P_j$. If $P_1 \oplus A'_1$ and $P_j$ are isotopic in $H$ then $H' \approx P_j \times I$ with $P_j$ corresponding to $P_j \times 0$, from which it follows that $c_2, c_3$ are simultaneously primitive in $H$ away from $c_1$. Moreover, $c_1$ is primitive in the handlebody $H'$, and so $H = H' \cup A'_1 V$.
is also a handlebody. Conversely, suppose $H$ is a handlebody and $c_2, c_3$ are simultaneously primitive away from $c_1$; notice that $c_2, c_3, A'_1 \subset \partial H'$. Suppose $D$ is a disk in $H$ realizing the simultaneous primitivity of $c_2, c_3$ away from $c_1$; we assume, as we may, that $D$ lies in $H'$ (see Figure 6 with $D = \Sigma$, $c_2 = u$, $c_3 = v$, $\text{core}(A'_1) = w$). Compressing $\partial H' \setminus \text{int} A'_1$ in $H'$ along $D$ gives rise to an annulus $A''_1$ in $H$ which, due to the presence of $A'_1$, is necessarily a companion annulus of $c_1$ in $H$. It follows from Lemma 5.1 that $A''_1$ and $A'_1$ are parallel in $H'$, hence that $P_i \oplus A'_1$ and $P_j$ are parallel in $H'$ and so in $H$.

For (b)(ii), if $Q_0$ boundary compresses into, say, $H_1$, then it follows immediately from the proof of (a) that $H_1$ is a handlebody, $Q_0 = P_i \oplus A'_1$, and $c_2, c_3$ are simultaneously primitive in $H_1$ away from $c_1$, so $Q_0$ compresses into a companion annulus of $c_1$ in $H_1$. If $R_0$ is any pair of pants in $H_1$ with boundary $\partial Q_0$ then by the same argument $R_0$ must be isotopic in $H_1$ to $Q_0 \oplus A'_1$ or $P_1 \oplus A'_1$, hence by (b)(i) $R_0$ is parallel into $\partial H_1$.

For (b)(iii), consider the disjoint pairs of pants $Q_0, R_0$ and suppose $R_0$ is also not parallel into $\partial H$. By (b)(ii), $Q_0 = P_i \oplus A'_1$ and $R_0 = P_j \oplus A'_1$ for some $i, j \in \{1, 2\}$. Thus, if $i \neq j$, $Q_0$ boundary compresses in the direction of $P_i$, away from $R_0$, into a companion annulus of $c_1$; a similar statement holds for $R_0$, and so such companion annuli of $c_1$ are separated by $Q_0 \cup R_0$, contradicting Lemma 5.1. Therefore $i = j$, so $Q_0$ is parallel to $R_0$. The lemma follows.

**Remark** If any two of the circles $c_1, c_2, c_3$ in Lemma 5.3 are coannular in $H$ then part (a) need not hold; an example of this situation can be constructed as follows. Let $H$ be a genus two handlebody and let $c_1, c_2$ be disjoint, nonparallel, nonseparating circles in $\partial H$ which are nontrivial in $H$ and cobound an annulus $A$ in $H$. Let $\alpha \subset \partial H$ be an arc with one endpoint in $c_1$ and the other in $c_2$ which is otherwise disjoint from $A$. We then take the pair of pants $Q_0$ to be the frontier of $H_1 = N(A \cup \alpha)$ in $H$ so that, up to isotopy, $\partial Q_0 = c_1 \cup c_2 \cup K$, where $K \subset \partial H$ is the sum of $c_1$ and $c_2$ along $\alpha$. Observe $Q_0$ boundary compresses in $H_1$ as in Case 3 of Lemma 5.3(a). Thus, if $c_1$ and $c_2$ are primitive in $H$ then, by Lemma 5.2, no component of $\partial Q_0$ has a companion in $H$ and $Q_0$ need not be parallel into $\partial H$, as illustrated by the example in Figure 7. Moreover, if $P$ is the once-punctured Klein bottle $A \cup B$, where $B \subset \partial H$ is a band with core $\alpha$, pushed slightly off $\partial H$ so as to properly embed in $H_1$, then $\partial P$ is isotopic to $K$, $H_1$ is a regular neighborhood of $P$, and the two components of $\partial Q_0$ isotopic to $c_1, c_2$ are lifts of the meridian of $P$. 

*Geometry & Topology Monographs, Volume 7 (2004)*
6 Hyperbolic knots

In this section we fix our notation and let $K$ be a hyperbolic knot in $S^3$ with exterior $X_K$. If $P, Q$ are distinct elements of $SK(K, r)$ which have been isotoped so as to intersect transversely and minimally, then $|P \cap Q| > 0$, $\partial P \cap \partial Q = \emptyset$, and each circle component of $P \cap Q$ is nontrivial in $P$ and $Q$. Notice that any circle component $\gamma$ of $P \cap Q$ must be orientation preserving in both $P, Q$, or orientation reversing in both $P, Q$. If $\gamma$ is a meridian (longitude, center) in both $P$ and $Q$, we will say that $\gamma$ is a simultaneous meridian (longitude, center, respectively) in $P, Q$. Recall our notation for $H(P), T_P, K_0, A_0$ relative to the surface $P$.

**Lemma 6.1** Let $m_1, m_2 \subset T_P$ and $l \subset T_P$ be the lifts of a meridian circle and a center circle of $P$, respectively. Then $m_1, m_2$ can not both have companions in $H(P)$, and neither $l$ nor $K' \subset \partial H(P)$ have companions.

**Proof** Let $A_1, A_2, A$ be annular neighborhoods of $m_1, m_2, l$ in $T_P$, respectively, with $A_1 \cap A_2 = \emptyset$. Let $A'_1, A'_2$ be companions of $A_1, A_2$ in $H(P)$, respectively, and suppose they intersect transversely and minimally; then $A'_1 \cap A'_2 = \emptyset$ by Lemma 2.3(b). Let $V_i$ be the region in $H(P)$ cobounded by $A_i, A'_i$ for $i = 1, 2$. Now, the lifts $m_1, m_2$ cobound the annulus $A(m)$ in $N(P)$ for some meridian circle $m \subset P$. Since $A'_1$ and $A_i$ are not parallel in $V_i$, it follows that, for a small regular neighborhood $N = A(m) \times I$ of $A(m)$ in $N(P)$, $V_i \cup N \cup V_2 \subset X_K$ is the exterior $X_L$ of some nontrivial knot $L$ in $S^3$ with $A_1 \subset X_L$ an essential annulus (see Figure 8(a)). Observe $H(P)$ is irreducible.
and atoroidal since \( K \) is hyperbolic and \( P \) is essential. As \( A_1, A_2 \) are incompressible in \( H(P) \), the \( V_i \)'s are solid tori by Lemma 5.1 and so \( L \) is a nontrivial torus knot with cabling annulus \( A_1 \). However, as the pair of pants \( P_0 = P \setminus \text{int} X_L \) has two boundary components coherently oriented in \( \partial X_L \) with the same slope as \( \partial A_1 \), \( K \) must be a satellite of \( L \) of winding number two, contradicting the hyperbolicity of \( K \).

Suppose \( A' \) is a companion of \( A \) in \( H(P) \), and let \( V \) be the region in \( H(P) \) cobounded by \( A, A' \). The circle \( l \) bounds a Moebius band \( B(c) \) in \( N(P) \) with \( B(c) \cap P \) some center circle \( c \) of \( P \). If \( M \) is a small regular neighborhood of \( B(c) \) in \( N(P) \) then, as \( A, A' \) are not parallel in \( V \), \( V \cup M \) is the exterior \( X_L \) of some nontrivial knot \( L \) in \( S^3 \) (see Figure 8(b)), and \( A \) is an essential annulus in \( X_L \) which, since \( M \) is a solid torus, necessarily has integral boundary slope in \( \partial X_L \). This time, \( P_0 = P \setminus \text{int} X_L \) is a once-punctured Moebius band with one boundary component in \( \partial X_L \) having the same slope as \( \partial A \), so \( K \) is a nontrivial satellite of \( L \) with odd winding number, again contradicting the hyperbolicity of \( K \).

Finally, if \( A' \) has a companion annulus \( B' \) then, as \( K \) is hyperbolic, \( B' \) must be boundary parallel in \( X_K \) in the direction of \( P \), contradicting Lemma 2.3(d). The lemma follows.

Given distinct elements \( P, Q \in \mathcal{SK}(K,r) \), if \( P \cap Q \) is a single simultaneous meridian or some pair of disjoint simultaneous centers, we will say that \( P \) and \( Q \) intersect meridionally or centrally, respectively.

**Lemma 6.2** If \( P, Q \in \mathcal{SK}(K,r) \) are distinct elements which intersect transversely and minimally, then \( P, Q \) intersect meridionally or centrally.

**Proof** By minimality of \( |P \cap Q| \), any component \( \gamma \) of \( P \cap Q \) must be nontrivial in both \( P \) and \( Q \), hence, in \( P \) or \( Q \), \( \gamma \) is either a circle parallel to the boundary,
a meridian, a longitude, or a center circle. Observe that if $\gamma$ is a center in $P$ then, as it is orientation reversing in $P$ it must be orientation reversing in $Q$, and hence $\gamma$ must also be a center in $Q$.

Suppose $\gamma$ is parallel to $\partial P$ in $P$; without loss of generality, we may assume that $\partial P$ and $\gamma$ cobound an annulus $A_P$ in $P$ with $Q \cap \text{int} A_P = \emptyset$. Since $\gamma$ preserves orientation in $P$ it must also preserve orientation in $Q$, hence $\gamma$ is either a meridian or longitude of $Q$, or parallel to $\partial Q$. In the first two cases, $\hat{Q}$ would compress in $K(\partial Q)$ via the disk $\hat{A}_P$, which is not the case by Lemma 2.2; thus, $\gamma$ and $\partial Q$ cobound an annulus $A_Q$ in $Q$. As $K$ is hyperbolic, the annulus $A_P \cup \gamma A_Q \subset H(P)$ is boundary parallel in $X_K$ by Lemma 6.1; but then $|P \cap Q|$ is not minimal, which is not the case. Therefore no component of $P \cap Q$ is parallel to $\partial P, \partial Q$ in $P, Q$, respectively.

Suppose now that $\gamma$ is a meridian in $P$ and a longitude in $Q$. Then $P \cap Q$ consists only of meridians of $P$ and longitudes of $Q$. If $\gamma$ is the only component of $P \cap Q$ then $P \simeq Q$ is a nonorientable (connected) surface properly embedded in $X_K$ with two boundary components, which is impossible. Thus we must have $|P \cap Q| \geq 2$; as the circles $P \cap Q$ are mutually parallel meridians in $P$, it follows that $P \cap H(Q)$ consists of a pair of pants and at least one annulus component $A$. But $P \cap N(Q)$ consists of a disjoint collection of annuli $\{A_i\}$ with $\{A_i \cap Q\}$ disjoint longitude circles of $Q$, and so the circles $\partial A_i$ form at most two parallelism classes in $T_Q \subset \partial H(Q)$, corresponding to the lifts of some disjoint pair of centers of $Q$. Since the circles $\partial A_i$ are among those in $\cup \partial A_i$, and $A$ is not parallel into $\partial H(Q)$ by minimality of $|P \cap Q|$, we contradict Lemmas 2.3 and 6.1.

Therefore, each component of $P \cap Q$ is a simultaneous meridian, longitude, or center of $P, Q$. There are now two cases left to consider.

Case 1 $P \cap Q$ consists of simultaneous meridians.

Suppose $|P \cap Q| = k + 1$, $k \geq 0$. Then $Q \cap N(P)$ consists of disjoint parallel annuli $A_0, \ldots, A_k$, each intersecting $P$ in a meridian circle, and $Q \cap H(P) = Q_0 \cup A_1' \cup \cdots \cup A_k'$, where $Q_0$ is a pair of pants with $\partial Q \subset \partial Q_0$ and the $A_i'$’s are annuli, none of which is parallel into $\partial H(P)$. The circles $\partial A_i = \partial A_i'$ consist of two parallelism classes in $\partial H$, denoted I and II, corresponding to the two distinct lifts of a meridian circle of $P$ to $\partial N(P)$.

By Lemma 2.3, the circles $\partial A_i'$ are both of type I or both of type II, for each $i$. Also, the components of $\partial Q_0$ are $\partial Q$ and two circles $\partial_1Q_0, \partial_2Q_0$ of type I or II. If the circles $\partial_1Q_0, \partial_2Q_0$ are both of type I or both of type II, then the union of
$Q_0$ and an annulus in $T_P$ cobounded by $\partial_1 Q_0$ and $\partial_2 Q_0$ is a once-punctured surface in $X_K$ disjoint from $P$, contradicting Lemma 2.3(b). Therefore, one of the circles $\partial_1 Q_0, \partial_2 Q_0$ is of type I and the other of type II. Thus, if $k > 0$ then some annulus $A'_i$ has boundaries of type I and some annulus $A'_j$ has boundaries of type II, which contradicts Lemma 6.1. Therefore $k = 0$, and so $P \cap Q$ consists of a single simultaneous meridian.

**Case 2** $P \cap Q$ consists of simultaneous longitudes or centers.

$P \cap Q$ can have at most two simultaneous centers; if it has at most one simultaneous center then $P \cong Q$ is a nonorientable surface properly embedded in $X_K$ with two boundary components, which is not possible. Therefore $P \cap Q$ contains a pair $c_1, c_2$ of disjoint center circles, so $Q \cap N(P)$ consists of two Moebius bands and, perhaps, some annuli, while $Q \cap H(P)$ consists of one pair of pants $Q_0$ and, perhaps, some annuli. Thus the circles $Q \cap \partial H(P)$ are divided into two parallelism classes, corresponding to the lifts of $c_1$ and $c_2$, and we may proceed as in Case 1 to show that $Q \cap H(P)$ has no annulus components. Hence $P \cap Q = c_1 \cup c_2$.

**Remark** Notice that if $P, Q$ are any two distinct elements of $SK(K, r)$, so $P \cap Q$ is central or meridional, and $Q_0 = Q \cap H(P)$, then by Lemma 2.3, since $K$ is hyperbolic, $H(P)$ and $\partial Q_0$ satisfy the hypothesis of Lemma 5.3; however, $Q_0$ need not boundary compress in $H(P)$.

The following result gives constraints on the exteriors of distinct elements of $SK(K, r)$.

**Lemma 6.3** Suppose $P, Q \in SK(K, r)$ are distinct elements which intersect centrally or meridionally; let $Q_0 = Q \cap H(P)$, and let $V, W$ be the closures of the components of $H(P) \setminus Q_0$. Then $Q_0$ is not parallel into $\partial H(P)$, and

- (a) for $X = V, W$, or $H(P)$, either $X$ is a handlebody or the pair $(X, \partial X)$ is irreducible and atoroidal;

- (b) if $P \cap Q$ is central then $(H(P), \partial H(P))$ is irreducible and $Q_0$ is boundary incompressible in $H(P)$, and

- (c) if the pair $(V, \partial V)$ is irreducible then $W$ is a handlebody.

In particular, if $P$ is not $\pi_1$–injective then $P$ is unknotted, $P \cap Q$ is meridional, and $K'$ is primitive in $H(P)$.
Figure 9: The manifold $\tilde{M}^3 = \tilde{N}(P) \cup_{P_1 \cup (V \cap N(K))} V$

**Proof** Let $P_1, P_2$ be the closures of the components of $\partial H(P) \setminus \partial Q_0$. If $Q_0$ is parallel to, say $P_1$, then $Q$ is isotopic in $X_K$ to $P_1 \cup (Q \cap N(P)) \subset N(P)$, which is clearly isotopic to $P$ in $N(P)$ (see Figure 9 and the proof of Lemma 6.7); thus $Q_0$ is not parallel into $\partial H(P)$.

Let $R$ be the maximal compression body of $\partial H(P) = \partial_+ R$ in $H(P)$ (notation as in \cite{2, 3}). Since $H(P)$ is irreducible and atoroidal, either $\partial_- R$ is empty and $H(P)$ is a handlebody or $R$ is a trivial compression body and $(H(P), \partial H(P))$ is irreducible and atoroidal. As $Q_0$ is incompressible in $H(P)$, a similar argument shows that either $V$ ($W$) is a handlebody or the pair $(V, \partial V)$ ($((W, \partial W)$, respectively) is irreducible and atoroidal; thus (a) holds.

If $P \cap Q$ is central and either $H(P)$ is a handlebody or $Q_0$ is boundary compressible, then at least one of the circles $K', l_1, l_2$, where $l_1, l_2$ are the lifts of the simultaneous centers $P \cap Q$, has a companion annulus by Lemma 5.3(a); this contradicts Lemma 6.1, so (b) now follows from (a).

For part (c), let $\tilde{N}(P) = N(P) \cup A_K N(K) = S^3 \setminus \text{int} H(P)$, the extended regular neighborhood of $P$ in $S^3$; notice that $K' \subset \partial \tilde{N}(P)$ and, since $P$ has integral boundary slope, that $A_K$ and $A'_K$ are parallel in $N(K)$, so $\tilde{N}(P)$ and $N(P)$ are homeomorphic in a very simple way.

Suppose the pair $(V, \partial V)$ is irreducible; without loss of generality, we may assume $P_1 \subset \partial V$ and $P_2 \subset \partial W$. As none of the circles $\partial P_1$ bounds a disk in $N(P)$, the pair of pants $P_1$ is incompressible in $N(P)$ and hence in $\tilde{N}(P)$; thus, since $(V, \partial V)$ is irreducible, it is not hard to see that, for the manifold $\tilde{M}^3 = \tilde{N}(P) \cup_{P_1 \cup (V \cap N(K))} V$ (see Figure 9), the pair $(\tilde{M}^3, \partial \tilde{M}^3)$ is irreducible. As $S^3 = \tilde{M}^3 \cup W$, $\partial W$ must compress in $W$, so $W$ is a handlebody by (a).

Finally, if $P$ is not $\pi_1$-injective then $T_P$ compresses in $H(P)$, so $H(P)$ is a handlebody by (a), $K'$ is primitive in $H(P)$ by Lemmas 5.2 and 6.1, and $P \cap Q$ is meridional by (b).

---

*Geometry \\& Topology Monographs, Volume 7 (2004)*
Lemma 6.4 If $P, Q, R \in SK(K, r)$ are distinct elements and each intersection $P \cap Q, P \cap R$ is central or meridional, then $P \cap Q$ and $P \cap R$ are isotopic in $P$.

Proof We will assume that $P \cap Q$ and $P \cap R$ are not isotopic in $P$ and obtain a contradiction in all possible cases. Since any two meridian circles of $P$ are isotopic in $P$, we may assume that $P \cap Q = m$ or $c_1 \cup c_2$ and $P \cap R = c'_1 \cup c'_2$, where $m$ is the meridian of $P$ and $c_1, c_2$ and $c'_1, c'_2$ are two non isotopic pairs of disjoint centers of $P$; we write $\partial Q_0 = \partial Q \cup \alpha_1 \cup \alpha_2$ and $\partial R_0 = \partial R \cup l'_1 \cup l'_2$, where $\alpha_1, \alpha_2 \subset T_P$ are the lifts $m_1, m_2$ of $m$ or $l_1, l_2$ of $c_1, c_2$, and $l'_1, l'_2 \subset T_P$ are the lifts of $c'_1, c'_2$. Isotope $P \cap Q$ and $P \cap R$ in $P$ so as to intersect transversely and minimally; then their lifts $\alpha_1 \cup \alpha_2$ and $l'_1 \cup l'_2$ will also intersect minimally in $\partial H(P)$. Finally, isotope $Q_0, R_0$ in $H(P)$ so as to intersect transversely with $|Q_0 \cap R_0|$ minimal; necessarily, $|Q_0 \cap R_0| > 0$, and any circle component of $Q_0 \cap R_0$ is nontrivial in $Q_0$ and $R_0$.

Let $G_{Q_0} = Q_0 \cap R_0 \subset Q_0$, $G_{R_0} = Q_0 \cap R_0 \subset R_0$ be the graphs of intersection of $Q_0, R_0$. Following [6], we think of the components of $\partial Q_0$ as fat vertices of $G_{Q_0}$, and label each endpoint of an arc of $G_{Q_0}$ with $1'$ or $2'$ depending on whether such endpoint arises from an intersection involving $l'_1$ or $l'_2$, respectively; the graph $G_{R_0}$ is labelled with $1, 2$ in a similar way. Such a graph is essential if each of its components is essential in the corresponding surface. As $P \cap R$ is central, $R_0$ is boundary incompressible in $H(P)$ by Lemma 6.3(b) and so $G_{Q_0}$ is essential; similarly, $G_{R_0}$ is essential if $P \cap Q$ is central. Thus, if $G_{R_0}$ has inessential arcs then $P \cap Q$ is meridional and, by minimality of $|Q_0 \cap R_0|$, $Q_0$ boundary compresses along an essential arc of $G_{Q_0}$. By Lemma 3.1, since the meridian $m$ and the centers $c'_1, c'_2$ can be isotoped so that $|m \cap c'_j| = 1$, it follows by minimality of $|\partial Q_0 \cap \partial R_0|$ that $|m_i \cap l'_j| = 1$ for $i, j = 1, 2$. Hence any inessential arc of $G_{R_0}$ has one endpoint in $m_1$ and the other in $m_2$ and so, by Case 1 of Lemma 5.3(a), since $Q_0$ is not parallel into $\partial H(P)$ by Lemma 6.3, $Q_0$ boundary compresses to a companion annulus of $K'$ in $H(P)$, contradicting Lemma 6.1.

Therefore the graphs $G_{Q_0}, G_{R_0}$ are always essential. Since $P \cap Q$ and $P \cap R$ are not isotopic in $\partial H(P)$, any circle component of $Q_0 \cap R_0$ must be parallel to $\partial Q, \partial R$ in $Q_0, R_0$, respectively, so, by minimality of $|Q_0 \cap R_0|$, $K'$ must have a companion annulus in $H(P)$, contradicting Lemma 6.1. Thus $Q_0 \cap R_0$ has no circle components. We consider two cases.

Case 1 $P \cap Q = m$

In this case, we have seen that $|m_i \cap l'_j| = 1$ for $i, j = 1, 2$. Therefore the $m_1, m_2$ and $l'_1, l'_2$ fat vertices of $G_{Q_0}$ and $G_{R_0}$, respectively, each have valence
2, and so, by essentiality, each graph $G_{Q_0}, G_{R_0}$ consist of two parallel arcs, one annulus face, and one disk face $D$. The situation is represented in Figure 10(a), where only $G_{Q_0}$ is shown; notice that the labels $l_1', l_2'$ must alternate around $\partial D$.

Let $V, W$ be the closures of the components of $H(P) \setminus R_0$, and suppose $Q_0 \cap V$ contains the disk face $D$ of $G_{Q_0}$. By minimality of $|Q_0 \cap R_0|$ and $|\partial Q_0 \cap \partial R_0|$, $\partial D$ is nontrivial in $\partial V$, hence $V$ is a handlebody by Lemma 6.3 and $K'$ is primitive in $V$ by Lemmas 5.2 and 6.1. Since the labels $l_1', l_2'$ alternate around $\partial D$, $D$ must be nonseparating in $V$; thus there is an essential disk $D'$ in $V$ disjoint from $D$ such that $D, D'$ form a complete disk system for $V$. Now, for $i = 1, 2$, $|\partial D \cap l_i'| = 2$ and so $D \cdot l_i'$ is even. If the intersection number $D' \cdot l_i'$ is even for some $i = 1, 2$ then $l_i'$ is homologically trivial mod 2 in $V$, while if $D' \cdot l_i'$ is odd for $i = 1, 2$ then $l_1' \cup l_2'$ is homologically trivial mod 2 in $V$. Thus one of $l_1', l_2'$, or $l_1' \cup l_2'$ bounds a surface in $V$, hence in $H(P)$, contradicting Lemma 2.3. Therefore this case does not arise.

Case 2 $P \cap Q = c_1 \cup c_2$

Recall that $c_1 \cup c_2$ and $c_1' \cup c_2'$ intersect transversely and minimally in $P$, so their lifts $l_1 \cup l_2$ and $l_1' \cup l_2'$ also intersect minimally in $\partial H(P)$. Moreover, after exchanging the roles of $Q_0, R_0$ or relabelling the pairs $c_1, c_2$ and $c_1', c_2'$, if necessary, we must have that $|c_1 \cap (c_1' \cup c_2')| = 2n + 1$ and $|c_2 \cap (c_1' \cup c_2')| = 2n - 1$ for some integer $n \geq 0$: this can be easily seen by viewing $P$ as the union of an annulus $A$ with core $m$ and a rectangle $B$ with core $b$, as in Lemma 3.1, and isotoping $c_1 \cup c_2$ and $c_1' \cup c_2'$ so as to intersect $m \cup b$ minimally. Figure 11
represents two pairs $c_1 \cup c_2$ and $c_1' \cup c_2'$ with minimal intersections; in fact, all examples of such pairs can be obtained from Figure 11(a) by choosing the rectangle $B$ appropriately so that the arcs of $(c_1 \cup c_2) \cap A$ are as shown, and suitably Dehn-twisting the pair $c_0^1 \cup c_0^2$ along $m$.

Thus $|l_1 \cap (l_1' \cup l_2')| = 4n + 2$ and $|l_2 \cap (l_1' \cup l_2')| = |4n - 2|$, and so the essential graphs $G_{Q_0}, G_{R_0}$ must both be of the type shown in Figure 10(b) (which is in fact produced by the intersection pattern of Figure 11(b), where $n = 1$). Let $V, W$ be the closures of the components of $H(P) \setminus R_0$. If $n > 1$ then the graph $G_{Q_0}$ has the two disk components $D_1, D_2$ labelled $\ast$ and $\ast\ast$ in Figure 10(b), respectively, as well as an annulus face $A$ with $\partial_1 A = \partial Q$, all lying in, say, $V$. Thus $V$ is a handlebody by Lemma 6.3 and, by minimality of $|\partial Q_0 \cap \partial R_0|$ and the essentiality of $G_{Q_0}, G_{R_0}$, the four disjoint circles $\partial_1 A, \partial_2 A, \partial D_1, \partial D_2$ are all essential in $\partial V$ and intersect $\partial R_0$ minimally. Given that $|\partial_1 A \cap (l_1' \cup l_2')| = 0$, $|\partial_2 A \cap (l_1' \cup l_2')| = 2$, $|\partial D_1 \cap (l_1' \cup l_2')| = 6$, and $|\partial D_2 \cap (l_1' \cup l_2')| = 4$, no two of such four circles can be isotopic in $\partial V$, an impossibility since $\partial V$ has genus two. The case $n = 0$ is similar to Case 1 and yields the same contradiction.

Finally, for $n = 1$ the intersection pattern in $P$ between $c_1 \cup c_2$ and $c_1' \cup c_2'$ must be the one shown in Figure 11(b), and it is not hard to see that only two labelled graphs $G_{Q_0}$ are produced, up to combinatorial isomorphism. The first possible labelled graph is shown in Figure 12(a); capping off $l_1, l_2$ and $l_1', l_2'$ with the corresponding Moebius bands $Q \cap N(P)$, we can see that $Q \cap R$ consists of a single circle component (shown in Figure 12(a) as the union of the broken and solid lines), which must be a meridian by Lemma 6.2, contradicting Case 1.
since \( Q \cap P \) is central. In the case of Figure 12(b), the disk faces \( D_1, D_2 \) of \( G_{Q_0} \) lie both in, say, \( V \), so \( V \) is a handlebody by Lemma 6.3 and \( K' \) is primitive in \( V \) by Lemmas 5.2 and 6.1. Moreover, since \( D_1 \) and \( D_2 \) do not intersect \( l_1' \cup l_2' \) in the same pattern, \( D_1 \) and \( D_2 \) are nonisotopic compression disks of \( \partial V \setminus K' \) in \( V \), so at least one of the disks \( D_1, D_2 \) must be nonseparating in \( V \).

As either disk \( D_1, D_2 \) intersects each circle \( l_0 \) an even number of times, we get a contradiction as in Case 1. Therefore this case does not arise either. \( \square \)

The next corollary summarizes some of our results so far.

**Corollary 6.5** If \( K \) is hyperbolic and \( |SK(K,r)| \geq 2 \) then \( SK(K,r) \) is either meridional or central; in the latter case, the link \( c_1 \cup c_2 \) obtained as the intersection of any two distinct elements of \( SK(K,r) \) is unique in \( X_K \) up to isotopy.

If \( |SK(K,r)| \geq 2 \) and \( P \in SK(K,r) \), let \( \alpha_1, \alpha_2 \subset H(P) \) be the lifts of the common meridian or pair of disjoint centers of \( P \) which, by Corollary 6.5, are determined by the elements of \( SK(K,r) \), and define \( P(K,r) \) as the collection of all pairs of pants \( X \subset H(P) \) with \( \partial X = K' \cup \alpha_1 \cup \alpha_2 \) and not parallel into \( \partial H(P) \), modulo isotopy. Notice that \( SK(K,r) \setminus \{P\} \) embeds in \( P(K,r) \) by Corollary 6.5, and so \( |SK(K,r)| \leq |P(K,r)| + 1 \). Our strategy for bounding \( |SK(K,r)| \) in the next lemmas will be to bound \( |P(K,r)| \).

**Lemma 6.6** Let \( K \) be a hyperbolic knot with \( |SK(K,r)| \geq 2 \). If \( SK(K,r) \) is central then \( |SK(K,r)| \leq 3 \).

**Proof** Fix \( P \in SK(K,r) \), and suppose \( Q_0, R_0, S_0 \) are distinct elements of \( P(K,r) \), each with boundary isotopic to \( K' \cup l_1 \cup l_2 \), where \( l_1, l_2 \) are the lifts of...
Let some lift of the meridian of \( P \) has an unknotted element \( P \) meridional, and reaching in the process, the end surface of the isotopy is \( P \), and we may assume that \( R_0 \) separates \( Q_0 \) from \( S_0 \) in \( H(P) \).

Let \( V, W \) be the closures of the components of \( H(P) \setminus R_0 \), with \( Q_0 \subset V, S_0 \subset W \); then \( V \), say, is a handlebody by Lemma 6.3. As \( Q_0 \) is not parallel into \( \partial V \) by Lemma 6.3, it follows from Lemma 5.3(a) that one of the circles \( \partial Q_0 \) has a companion annulus in \( V \), hence in \( H(P) \), contradicting Lemma 6.1. Therefore \(|P(K, r)| \leq 2\), and so \(|SK(K, r)| \leq 3\).

**Lemma 6.7** Let \( K \) be a hyperbolic knot with \(|SK(K, r)| \geq 2\). If \( SK(K, r) \) has an unknotted element \( P \) then \( SK(K, r) \) is meridional, \(|SK(K, r)| = 2\), and some lift of the meridian of \( P \) has a companion annulus in \( H(P) \).

**Proof** Let \( P, Q \in SK(K, r) \) be distinct elements with \( P \) unknotted, and let \( Q_0 = Q \cap H(P) \). Then \( P \cap Q \) is meridional by Lemma 6.3(b), so \( SK(K, r) \) is meridional, and \( Q_0 \) boundary compresses into a companion annulus \( A \) of exactly one of the lifts \( m_1 \) or \( m_2 \) of the meridian circle \( m \) of \( P \) by Lemmas 5.3(a) and 6.1. Now, if \( P_1, P_2 \) are the closures of the components of \( \partial H(P) \setminus \partial Q_0 \) then, by Lemma 5.3(b)(ii), \( Q_0 = P_i \circ A \) for some \( i = 1, 2 \), so if \( A(m) \) is an annulus in \( N(P) \) cobounded by \( m_1, m_2 \) then \( Q \) is equivalent to one of the Seifert Klein bottles \( R_i = P_i \circ A \cup A(m), i = 1, 2 \). But \( R_1 \) and \( R_2 \) are isotopic in \( X_K \): this isotopy is described in Figure 13, where a regular neighborhood \( N(P) \) of \( P \) is shown (as a box) along with the lifts of the meridian \( m \) of \( P \) (as two of the solid dots in the boundary of \( N(P) \)); regarding \( P_1 \) as the closure of a component of \( \partial N(P) \setminus (N(A(m) \cup P) \setminus \partial H(P)) \), the idea is to construct \( P_i \circ A \cup A(m) \), start pushing \( P_1 \) onto \( P \) using the product structure of \( N(P) \setminus A(m) \), and continue until reaching \( P_2 \) on the ‘other side’ of \( P \); as the annulus \( A \cup A(m) \) is ‘carried along’ in the process, the end surface of the isotopy is \( P_2 \circ A \cup A(m) \). The lemma follows.

**Remark** For a hyperbolic knot \( K \subset S^3 \) with Seifert Klein bottle \( P \) and meridian lifts \( m_1, m_2 \), if there is a pair of pants \( Q_0 \subset H(P) \) with \( \partial Q_0 = K' \cup m_1 \cup m_2 \) which is not parallel into \( \partial H(P) \), it may still be the case that the Seifert Klein bottle \( Q = Q_0 \cup A(m) \) is equivalent to \( P \) in \( X_K \). An example of this situation is provided by the hyperbolic 2–bridge knots with crosscap number two; see Section 7 and the proof of Theorem 1.3 for more details.

**Lemma 6.8** Let \( K \) be a hyperbolic knot with \(|SK(K, r)| \geq 2\). If \( SK(K, r) \) is meridional then \(|SK(K, r)| \leq 6\).
Figure 13: An isotopy between \((P_1 \oplus A) \cup A(m)\) and \((P_2 \oplus A) \cup A(m)\)

**Proof** Fix \(P \in SK(K,r)\), and suppose \(Q_0, R_0, S_0, T_0\) are distinct elements of \(P(K,r)\) which can be isotoped in \(H(P)\) so as to be mutually disjoint; we may assume \(R_0\) separates \(Q_0\) from \(S_0 \cup T_0\), while \(S_0\) separates \(Q_0 \cup R_0\) from \(T_0\). Let \(V, W\) be the closure of the components of \(H(P) \setminus R_0\), with \(Q_0 \subset V\) and \(S_0 \cup T_0 \subset W\). Then \(W\) can not be a handlebody by Lemma 5.3(b)(iii), hence, by Lemmas 5.3(a) and 6.3(c), \(V\) is a handlebody and \(Q_0\) gives rise to a companion annulus in \(V\) of some lift of the meridian of \(P\). However, a similar argument shows that \(T_0\) gives rise to a companion annulus of some lift of the meridian of \(P\) in \(W\), contradicting Lemmas 5.1 and 6.1.

Therefore at most three distinct elements of \(P(K,r)\) can be isotoped at a time so as to be mutually disjoint in \(H(P)\). Consider the case when three such disjoint elements exist, say \(Q_0, R_0, S_0\). Let \(E, F, G, H\) be the closure of the components of \(H(P) \setminus (Q_0 \cup R_0 \cup S_0)\), and let \(P_1, P_2\) be the closure of the pair of pants components of \(\partial H(P) \setminus \partial (Q_0 \cup R_0 \cup S_0)\), as shown (abstractly) in Figure 14.

Suppose \(T_0 \in P(K,r) \setminus \{Q_0, R_0, S_0\}\) has been isotoped in \(H(P)\) so as to intersect \(Q_0 \cup R_0 \cup S_0\) transversely and minimally. Then \(|T_0 \cap (Q_0 \cup R_0 \cup S_0)| > 0\) by the above argument. Since \(T_0 \cap (Q_0 \cup R_0 \cup S_0)\) consists of circles parallel to \(\partial T_0\), at least one component of \(\partial T_0\) must have a companion annulus \(A_1\) in \(H(P)\); the annulus \(A_1\) may be constructed from an outermost annulus component of \(T_0 \setminus (Q_0 \cup R_0 \cup S_0)\), and hence can be assumed to lie in one of the regions \(E, F, G, H\). By Lemma 6.1, \(A_1\) is a companion of some lift \(m_1\) of the meridian of \(P\), and by Lemma 2.3(b)(c) the components of \(T_0 \cap (Q_0 \cup R_0 \cup S_0)\) are all mutually parallel in each of \(T_0, Q_0, R_0, S_0\). So, if, say, \(|T_0 \cap Q_0| \geq 2\), then \(m_1\) gets at least one companion annulus on either side of \(Q_0\), which is not possible by Lemma 5.1; thus \(|T_0 \cap X| \leq 1\) for \(X = Q_0, R_0, S_0\).
Let $T'_0$ be the closure of the pants component of $T_0 \setminus (Q_0 \cup R_0 \cup S_0)$; then $T'_0$ lies in one of $E, F, G,$ or $H$, and so $T'_0$ must be isotopic to $P_1, P_2, Q_0, R_0,$ or $S_0$. Also, since $A_1$ is unique up to isotopy by Lemma 5.1, we can write $T_0 = T'_0 \oplus A_1$. Therefore, the only choices for $T'_0$ are $X \oplus A_1$ for $X = P_1, Q_0, R_0, S_0$; here we exclude $X = P_2$ since $P_1 \oplus A_1$ and $P_2 \oplus A_1$ give rise to isotopic once-punctured Klein bottles in $X_K$ by the proof of Lemma 6.7.

If $A_1 \subset E$ (the case $A_1 \subset H$ is similar) then $P_1 \oplus A_1 = Q_0$ and $Q_0 \oplus A_1 = P_1$ (see Figure 14(a)), hence $T_0 = R_0 \oplus A_1$ or $S_0 \oplus A_1$. If $A_1 \subset F$ (the case $A_1 \subset G$ is similar) then $R_0 \oplus A_1 = Q_0$ and $Q_0 \oplus A_1 = R_0$ (see Figure 14(b)), hence $T_0 = P_1 \oplus A_1$ or $S_0 \oplus A_1$. In either case we have $|P(K, r)| \leq 5$, and hence $|SK(K, r)| \leq 6$. Finally, if at most two distinct elements of $P(K, r)$ can be isotoped so as to be disjoint in $H(P)$, it is not hard to see by an argument similar to the above one that in fact the smaller bound $|SK(K, r)| \leq 4$ holds.  

**Remark**  It is possible to realize the bound $|P(K, r)| = 3$, so $|SK(K, r)| \leq 4$, as follows. By [18, Theorem 1.1], any unknotted solid torus $S^1 \times D^2$ in $S^3$ contains an excellent properly embedded arc whose exterior $V \subset S^1 \times D^2$ is an excellent manifold with boundary of genus two; in particular, $(V, \partial V)$ is irreducible, $V$ is atoroidal and anannular, and $S^3 \setminus \text{int} V$ is a handlebody. Let $H, Q_0, P$ be the genus two handlebody, pair of pants, and once-punctured Klein bottle constructed in the remark just after Lemma 5.3 (see Figure 7). Let $H'$ be the manifold obtained by gluing a solid torus $U$ to $H$ along an annulus in $\partial U$ which runs at least twice along $U$ and which is a regular neighborhood of one of the components of $\partial Q_0$ which is a lift of the meridian of $P$; since such a component is primitive in $H$, $H'$ is a handlebody. Finally, glue $V$ and $H'$ together along their boundaries so that $V \cup H' = S^3$. Using our results so far

---

*Figure 14: The possible pairs of pants $T_0$ (in broken lines) in $H(P)$*
in this section it can be proved (cf proof of Lemma 6.9) that $K = \partial P$ becomes a hyperbolic knot in $S^3$, $H(P)$ contains two disjoint nonparallel pair of pants with boundary isotopic to $K'$ and the lifts of a meridian $m$ of $P$, and one of the lifts of $m$ has a companion annulus in $H(P)$, so $|P(K, r)| = 3$. The bound $|P(K, r)| = 5$ could then be realized if $V$ contained a pair of pants not parallel into $\partial V$ with the correct boundary.

**Proof of Theorem 1.1** That $SK(K, r)$ is either central or meridional and parts (a),(b) follow from Corollary 6.5 and Lemmas 6.3, 6.6, 6.7, and 6.8.

Let $P \in SK(K, r)$. If $P$ is not $\pi_1$–injective then $P$ is unknotted and $K'$ is primitive in $H(P)$ by Lemma 6.3. Thus, there is a nonseparating compression disk $D$ of $T_P$ in $H(P)$; since $K'$ has no companion annuli in $H(P)$, it follows that $N(P) \cup N(D)$ is homeomorphic to $X_K$, hence $K$ has tunnel number one. Moreover, if $H(P)(K')$ is the manifold obtained from $H(P)$ by attaching a 2–handle along $K'$, then $H(P)(K')$ is a solid torus and so $K(r) = N(\hat{P}) \cup_\partial H(P)(K')$ is a Seifert fibered space over $S^2$ with at most three singular fibers of indices 2, 2, $n$. As the only such spaces with infinite fundamental group are $S^1 \times S^2$ and $RP^3 \# RP^3$, that $\pi_1((K(r)))$ is finite follows from Property R [8] and the fact that $K(r)$ has cyclic integral first homology. Thus (c) holds.

If $P$ is unknotted and $\pi_1$–injective then $T_P$ is incompressible in $H(P)$, hence, by the 2–handle addition theorem [3], the pair $(H(P)(K'), \partial H(P)(K'))$ is irreducible. As $K(r) = N(\hat{P}) \cup_\partial H(P)(K')$, (d) follows.

We discuss now two constructions of crosscap number two hyperbolic knots. The first construction produces examples of meridional families $SK(K, r)$ with $|SK(K, r)| = 2$. The second one gives examples of knots $K$ and surfaces $P, Q$ in $SK(K, r)$ which intersect centrally and such that $|SK(K, r)| \leq 2$.

### 6.1 Meridional families

It is not hard to produce examples of hyperbolic knots $K$ bounding nonequivalent Seifert Klein bottles $P, Q$ which intersect meridionally: for in this case one of the surfaces can be unknotted and the other knotted, making the surfaces clearly non isotopic. This is the strategy followed by Lyon in [16] (thanks to V. Núñez for pointing out this fact) to construct nonequivalent Seifert tori for knots, and his construction can be easily modified to provide infinitely many examples of hyperbolic knots $K$ with $|SK(K, r)| = 2$, bounding an unknotted Seifert Klein bottle and a strongly knotted one along the same slope.

The construction of these knots goes as follows. As in [16], let \( V \) be a solid torus standardly embedded in \( S^3 \), let \( A \) be an annulus embedded in \( \partial V \) whose core is a \((\pm 4, 3)\) cable of the core of \( V \), and let \( A' \) be the closure of \( \partial V \setminus A \). We glue a rectangular band \( B \) to \( \partial A \) on the outside of \( V \), as in Figure 15, with an odd number of half-twists (\(-3\) are shown). Then the knot \( K = \partial(A \cup B) \) bounds the Seifert Klein bottles \( P = A \cup B \) and \( Q = A' \cup B \) with common boundary slopes; clearly, \( P \) and \( Q \) can be isotoped so that \( P \cap Q = A \cap A' \) is a simultaneous meridian. As in [16], \( P \) is unknotted and \( Q \) is knotted; this is clear since \( B \) is a tunnel for the core of \( A \) but not for the core of \( A' \). It is not hard to check that if \( m_1, m_2 \) are the lifts of the meridian of \( P \) then \( m_1 \), say, is a power (a cube) in \( H(P) \) while neither \( K', m_2 \) is primitive nor a power in \( H(P) \). That \( K \) has the desired properties now follows from the next general result.

**Lemma 6.9** Let \( K \) be a knot in \( S^3 \) which spans two Seifert Klein bottles \( P, Q \) with common boundary slope \( r \), such that \( P \) is unknotted, \( Q \) is knotted, and \( P \cap Q \) is meridional. For \( m_1, m_2 \) lifts of the meridian \( m \) of \( P \), suppose \( m_1 \) is a power in \( H(P) \) but neither \( K', m_2 \) is primitive nor a power in \( H(P) \). Then \( K \) is hyperbolic, \( P \) is \( \pi_1 \)-injective, and \( SK(K, r) = \{P, Q\} \).

**Proof** As \( K' \) is not primitive nor a power in \( H(P) \), \( T_P \) is incompressible in...
Suppose $T$ is an essential torus in $X_K$ which intersects $P$ transversely and minimally. Then $P \cap T$ is nonempty and $P \cap T \subset P$ consists of circles parallel to $\partial P$ and meridians or longitudes of $P$; by Lemmas 2.3 and 6.1, since $T$ is not parallel into $\partial X_K$, it is not hard to see that $P \cap T$ consists of only meridians of $P$ or only longitudes of $P$. Since the lift of a longitude of $P$ is also a lift of some center of $P$ then, by Lemma 2.3, either the lift $l$ of some center $c$ of $P$ or both lifts $m_1,m_2$ of the meridian of $P$ have companions in $H(P)$. The second option can not be the case by Lemma 5.2 since only $m_1$ is a power in $H(P)$. For the first option, observe that, since $m$ and $c$ can be isotoped in $P$ so as to intersect transversely in one point, $l$ can be isotoped in $T_P$ so as to transversely intersect $m_1,m_2$ each in one point.

Suppose $A^*$ is a companion annulus of $l$ in $H(P)$ with $\partial A^* = l_1 \cup l_2$, and let $Q_0 = Q \cap H(P)$; notice $Q_0$ is not parallel into $\partial H(P)$, since $Q$ and $P$ are not equivalent in $X_K$. Isotope $A^*,Q_0$ so as to intersect transversely and minimally, and let $G_{Q_0} = Q_0 \cap A^* \subset Q_0, G_{A^*} = Q_0 \cap A^* \subset A^*$ be their graphs of intersection; each graph has two arc components. If $G_{Q_0}$ is inessential then $A^*$ is either parallel into $\partial H(P)$ or boundary compresses in $H(P)$ into an essential disk disjoint from $K'$; the first option is not the case, while the latter can not be the case either by Lemma 5.2 since, by hypothesis, $K'$ is neither primitive nor a power in $H(P)$. If $G_{A^*}$ is inessential then $Q_0$ boundary compresses in $H(P)$ into a companion annulus for $K'$, which is also not the case by Lemma 5.2 since $K'$ is not a power in $H(P)$; for the same reason, $Q_0 \cap A^*$ has no circle components. Thus $G_{Q_0}$ and $G_{A^*}$ are essential graphs, as shown in Figure 16. But, due to the disk face $D_0$ of $G_{Q_0}$, it follows that $A^*$ runs twice around the solid torus region $R$ cobounded by $A^*$ and $T_P$; hence $R \cup N(B(c)) \subset X_K$ contains a closed Klein bottle, which is impossible. Thus $X_K$ is atoroidal.

6.2 Central families

Let $S$ be a closed genus two orientable surface embedded in $S^3$. Suppose there are curves $l_1,l_2 \subset S$ which bound disjoint Moebius bands $B_1,B_2$, respectively, embedded in $S^3$ so that $B_i \cap S = l_i$ for $i = 1,2$. Now let $K$ be an embedded circle in $S \setminus (l_1 \cup l_2)$ which is not parallel to either $l_1$ or $l_2$, does not separate $S$, and separates $S \setminus (l_1 \cup l_2)$ into two pairs of pants, each containing a copy of both $l_1$ and $l_2$ in its boundary. Let $P_0,Q_0$ be the closures of the components of $S$.
Seifert Klein bottles for knots with common boundary slopes

Then $K$ bounds the Seifert Klein bottles $P = P_0 \cup B_1 \cup B_2$ and $Q = Q_0 \cup B_1 \cup B_2$, which have common boundary slope and can be isotoped so as to intersect centrally in the cores of $B_1, B_2$. Any knot $K$ bounding two Seifert Klein bottles $P, Q$ with common boundary slope which intersect centrally can be constructed in this way, say via the surface $S = (P \times Q) \cup A$, where $A$ is a suitable annulus in $N(K)$ bounded by $\partial P \cup \partial Q$.

Specific examples can be constructed as follows; however checking the nonequivalence of two Seifert Klein surfaces will not be as simple as in the meridional case, as any two such surfaces are always strongly knotted. Let $S$ be a genus two Heegaard surface of $S^3$ splitting $S^3$ into genus 2 handlebodies $H, H'$. Let $l_1, l_2$ be disjoint circles embedded in $S$ which bound disjoint Moebius bands $B_1, B_2$ in $H$, and let $H_0 \subset H$ be the closure of $H \setminus N(B_1 \cup B_2)$. Finally, let $K$ be a circle in $\partial H \setminus (l_1 \cup l_2)$ as specified above, with $P, Q$ the Seifert Klein bottles induced by $K, l_1, l_2$. It is not hard to construct examples of $K, l_1, l_2$ satisfying the following conditions:

(C1) $l_1, l_2$ are not powers in $H'$,

(C2) $K$ is neither primitive nor a power in $H_0, H'$.

The simplest such example is shown in Figure 17; here $l_1, l_2$ are primitive in $H'$, and $K$ represents $y^2 x^{-2} y^{-2} x^{-2} y^{-2} x^{-2}, X Y X Y^{-1} X^{-1} Y^{-1}$ in $\pi_1(H), \pi_1(H')$, respectively, relative to the obvious (dual) bases shown in Figure 17. The properties of $K, P, Q$ are given in the next result.

**Lemma 6.10** If $K, l_1, l_2 \subset \partial H$ satisfy (C1) and (C2) and $r$ is the common boundary slope of $P, Q$, then $K$ is hyperbolic, $P$ and $Q$ are strongly knotted, and $|\mathcal{SK}(K, r)| \leq 2$; in particular, if $P$ and $Q$ are not equivalent then $\mathcal{SK}(K, r) = \{P, Q\}$.
Figure 17: The circles $K, l_1, l_2$ in $\partial H$

**Proof** Observe that $H(P) = H_0 \cup Q_0 H'$ and $H(Q) = H_0 \cup P_0 H'$. That $Q_0$ is incompressible and boundary incompressible in $H(P)$ follows from (C1) and (C2) along with the fact that $l_1, l_2$ are primitive in $H_0$. Thus $(H(P), \partial H(P))$ is irreducible, so $K$ is nontrivial and not a cable knot by Theorem 1.5, and $P$ (similarly $Q$) is strongly knotted. That $K$ is hyperbolic follows now from an argument similar to that of the proof of Lemma 6.9 and, since both $H_0$ and $H'$ are handlebodies, the bound $|SK(K, r)| \leq 2$ follows from the proof of Lemma 6.6.

7 Pretzel knots

We will denote a pretzel knot of length three with the standard projection shown in Figure 18 by $p(a, b, c)$, where the integers $a, b, c$, exactly one of which is even, count the number of signed half-twists of each tangle in the boxes. It is not hard to see that if \{a', b', c'\} = \{\varepsilon a, \varepsilon b, \varepsilon c\} for \(\varepsilon = \pm 1\) then $p(a, b, c)$ and $p(a', b', c')$ have the same knot type. For any pretzel knot $p(a, b, c)$ with $a$ even, the black surface of its standard projection shown in Figure 18 is an algorithmic Seifert Klein bottle with meridian circle $m$, which has integral boundary slope $\pm 2(b + c)$ by Lemma 2.1; an algorithmic Seifert surface is always unknotted. By [19], with the exception of the knots $p(2, 1, 1)$ (which is the only knot that has two algorithmic Seifert Klein bottles of distinct slopes produced by the
same projection diagram) and \( p(-2, 3, 7) \), this is the only slope of \( p(a, b, c) \) which bounds a Seifert Klein bottle. Finally, if at least one of \( a, b, c \) is \( \pm 1 \) then \( p(a, b, c) \) has bridge number at most 2, the only pretzels \( p(a, b, c) \) with \( |a|, |b|, |c| \geq 2 \) which are torus knots are \( p(-2, 3, 3) \) and \( p(-2, 3, 5) \), and if one of \( a, b, c \) is zero then \( p(a, b, c) \) is either a 2–torus knot or a connected sum of two 2–torus knots.

Now let \( F \) be the free group on \( x, y \). If \( w \) is a cyclically reduced word in \( x, y \) which is primitive in \( F \) then, by [5] (cf [9]), the exponents of one of \( x \) or \( y \), say \( x \), are all 1 or all \(-1\), and the exponents of \( y \) are all of the form \( n, n+1 \) for some integer \( n \). Finally, a word of the form \( x^m y^n \) is a proper power in \( F \) iff \( \{m, n\} = \{0, k\} \) for some \( |k| \geq 2 \).

**Proof of Theorem 1.3** Let \( K \) be the hyperbolic pretzel knot \( p(a, b, c) \) for some integers \( a, b, c \) with \( a \) even and \( b, c \) odd. Let \( P \) be the unknotted Seifert Klein bottle spanned by \( K \) in its standard projection.

We will show that \( K' \) is never primitive in \( H(P) \), so \( P \) is \( \pi_1 \)–injective by Lemma 6.3; thus \( K(r) \) is irreducible and toroidal by Theorem 1.1(d). We will also show that whenever a lift \( m_1, m_2 \) of the meridian \( m \) of \( P \) is a power in \( H(P) \) then \( K \) is a 2–bridge knot; in such case, by [11, Theorem 1], \( P \) is obtained as a plumbing of an annulus and a Moebius band (cf [20]) and \( P \) is unique up to isotopy. Along with Lemma 6.7, it will then follow that \( |SK(K, r)| = 1 \) in all cases.

The proof is divided into cases, depending on the relative signs of \( a, b, c \). Figure 19 shows the extended regular neighborhood \( \hat{N}(P) = N(P) \cup_{A_K} N(K) \) of \( P \), which is a standard unknotted handlebody in \( S^3 \), along with the circles \( K', m_1, m_2 \) with a given orientation. The disks \( D_x, D_y \) shown form a complete disk system for \( H(P) \), and give rise to a basis \( x, y \) for \( \pi_1(H(P)) \), oriented as indicated by the head (for \( x \)) and the tail (for \( y \)) of an arrow. Figure 19 depicts the knot \( p(2, 3, 3) \) and illustrates the general case when \( a, b, c > 0 \); we will
continue to use the same figure, with suitable modifications, in all other cases. By the remarks at the beginning of this section, the following cases suffice.

**Case 1**  \( a = 2n > 0, \ b = 2p + 1 > 0, \ c = 2q + 1 > 0 \)

In this case, up to cyclic order, the words for \( K', m_1, m_2 \) in \( \pi_1(H(P)) \) are:

\[ K' = y^{q+1}(xy)^{p+n+1}(yx)^{p+1}x^{n-1} \]
\[ m_1 = y^q(yx)^{p+1} \]
\[ m_2 = y^{q+1}(yx)^p \]

Since \( n > 0 \) and \( p, q \geq 0 \), the word for \( K' \) is cyclically reduced and both \( x \) and \( x^{-1} \) appear in \( K' \); thus \( K' \) is not primitive in \( H(P) \).

Consider now \( m_1 \) and \( m_2 \); as \( y \) and \( yx \) form a basis of \( \pi_1(H(P)) \), if \( m_1 \) is a power in \( H(P) \) then \( q = 0 \), while if \( m_2 \) is a power then \( p = 0 \). In either case \( K \) is a pretzel knot of the form \( p(\cdot, \cdot, 1) \), hence \( K \) is a 2-bridge knot.

**Case 2**  \( a = -2n < 0, \ b = 2p + 1 > 0, \ c = 2q + 1 > 0 \)

This time words for \( K', m_1, m_2 \) are, up to cyclic order,

\[ K' = y^{q+1}(xy)^{p+n+1}(yx)^{p+1}x^n \]
\[ m_1 = y^q(yx)^{p+1} \]
\[ m_2 = y^{q+1}(yx)^p. \]

If \( n > 1 \) then the word for \( K' \) is cyclically reduced and both \( x \) and \( x^{-1} \) appear in \( K' \) and so \( K' \) is not primitive in \( H(P) \). If \( n = 1 \) then, switching to the basis \( y, u = xy \) of \( \pi_1(H(P)) \), \( K' \) is represented up to cyclic order by the word...
Seifert Klein bottles for knots with common boundary slopes

$y^q u^p y u^p y^q u$. Observe that if $p = 0$ or $q = 0$ then $K = p(-2, 1, \cdot)$ which is a 2–torus knot, so $p, q > 0$. Thus, if $K'$ is primitive then necessarily $(p, q) = \{1, 2\}$ and so $K = p(-2, 3, 5)$ is a torus knot. Therefore, $K'$ is not primitive in $H(P)$. The analysis of the words $m_1$ and $m_2$ is identical to that of Case 1 and yields the same conclusion.

**Case 3** $a = 2n > 0$, $b = 2p + 1 > 0$, $c = -(2q + 1) < 0$

Up to cyclic order, words for $K', m_1, m_2$ are:

\[
K' = y^q(xy^{-1})^p x^{n+1}(y^{-1}x)^py^q x^{-n}
\]
\[
m_1 = y^{q+1}(xy^{-1})^{p+1}
\]
\[
m_2 = y^q(y^{-1}x)^p.
\]

If $p, q > 0$ then the word for $K'$ is cyclically reduced and contains all of $x, x^{-1}, y, y^{-1}$, so $K'$ is not primitive in $H(P)$. If $p = 0$ then $K' = y^q x^{n+1} y^q x^{-n}$, so $K'$ is primitive iff $q = 0$, in which case $K = p(2n, 1, -1)$ is a 2–torus knot. The case when $q = 0$ is similar, therefore $K'$ is not primitive in $H(P)$.

In this case $m_1$ cannot be a power in $H(P)$ for any values of $p, q \geq 0$, while if $m_2$ is a power then $p = 0$ or $q = 0$ and hence $K$ is a 2–bridge knot.

**References**


*Geometry & Topology Monographs, Volume 7 (2004)*

Department of Mathematical Sciences, University of Texas at El Paso
El Paso, TX 79968, USA

Email: valdez@math.utep.edu

Received: 10 November 2003 Revised: 10 March 2004