4. Cohomological symbol for henselian discrete valuation fields of mixed characteristic

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4.1. Cohomological symbol map

Let $K$ be a field. If $m$ is prime to the characteristic of $K$, there exists an isomorphism

$$ h_{1,K}: K^*/m \to H^1(K, \mu_m) $$

supplied by Kummer theory. Taking the cup product we get

$$ (K^*/m)^q \to H^q(K, \mathbb{Z}/m(q)) $$

and this factors through (by [T])

$$ h_{q,K}: K_q(K)/m \to H^q(K, \mathbb{Z}/m(q)). $$

This is called the cohomological symbol or norm residue homomorphism.

**Milnor–Bloch–Kato Conjecture.** For every field $K$ and every positive integer $m$ which is prime to the characteristic of $K$ the homomorphism $h_{q,K}$ is an isomorphism.

This conjecture is shown to be true in the following cases:

(i) $K$ is an algebraic number field or a function field of one variable over a finite field and $q = 2$, by Tate [T].

(ii) Arbitrary $K$ and $q = 2$, by Merkur’ev and Suslin [MS1].

(iii) $q = 3$ and $m$ is a power of 2, by Rost [R], independently by Merkur’ev and Suslin [MS2].

(iv) $K$ is a henselian discrete valuation field of mixed characteristic $(0, p)$ and $m$ is a power of $p$, by Bloch and Kato [BK].

(v) $(K, q)$ arbitrary and $m$ is a power of 2, by Voevodsky [V].

For higher dimensional local fields theory Bloch–Kato’s theorem is very important and the aim of this text is to review its proof.

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Theorem (Bloch–Kato). Let $K$ be a henselian discrete valuation fields of mixed characteristic $(0, p)$ (i.e., the characteristic of $K$ is zero and that of the residue field of $K$ is $p > 0$), then
\[ h_{q,K} : K_q(K)/p^n \longrightarrow H^q(K, \mathbb{Z}/p^n(q)) \]
is an isomorphism for all $n$.

Till the end of this section let $K$ be as above, $k = k_K$ the residue field of $K$.

4.2. Filtration on $K_q(K)$

Fix a prime element $\pi$ of $K$.

Definition.
\[
U_mK_q(K) = \begin{cases} K_q(K), & m = 0 \\ \{ 1 + M^m_K \} \cdot K_{q-1}(K), & m > 0. \end{cases}
\]

Put $\text{gr}_mK_q(K) = U_mK_q(K)/U_{m+1}K_q(K)$.

Then we get an isomorphism by [FV, Ch. IX sect. 2]
\[
K_q(k) \oplus K_{q-1}(k) \overset{\rho_0}{\longrightarrow} \text{gr}_qK_q(K)
\]
\[
\rho_0 \left( \{ x_1, \ldots, x_q \}, \{ y_1, \ldots, y_{q-1} \} \right) = \{ \tilde{x}_1, \ldots, \tilde{x}_q \} + \{ \tilde{y}_1, \ldots, \tilde{y}_{q-1}, \pi \}
\]
where $\tilde{x}$ is a lifting of $x$. This map $\rho_0$ depends on the choice of a prime element $\pi$ of $K$.

For $m \geq 1$ there is a surjection
\[
\Omega_k^{q-1} \oplus \Omega_k^{q-2} \overset{\rho_m}{\longrightarrow} \text{gr}_mK_q(K)
\]
defined by
\[
\left( x \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_{q-1}}{y_{q-1}}, 0 \right) \mapsto \{ 1 + \pi^m \tilde{x}, \tilde{y}_1, \ldots, \tilde{y}_{q-1} \},
\]
\[
\left( 0, x \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_{q-2}}{y_{q-2}} \right) \mapsto \{ 1 + \pi^m \tilde{x}, \tilde{y}_1, \ldots, \tilde{y}_{q-2}, \pi \}.
\]

Definition.
\[
k_q(K) = K_q(K)/p, h_q(K) = H^q(K, \mathbb{Z}/p(q)),
\]
\[
U_mk_q(K) = \text{im}(U_mK_q(K)) \text{ in } k_q(K), \quad U_mh_q(K) = h_{q,K}(U_mk_q(K)),
\]
\[
\text{gr}_mh_q(K) = U_mh_q(K)/U_{m+1}h_q(K).
\]
Proposition. Denote \( \nu_q(k) = \ker(\Omega^q_k \xrightarrow{1-C^{-1}} \Omega^q_k/d\Omega^q_k) \) where \( C^{-1} \) is the inverse Cartier operator:

\[
x^p \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_q}{y_q} \rightarrow x^p \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_q}{y_q}.
\]

Put \( e' = pe/(p - 1) \), where \( e = v_K(p) \).

(i) There exist isomorphisms \( \nu_q(k) \rightarrow k_q(k) \) for any \( q \); and the composite map denoted by \( \tilde{\rho}_0 \)

\[
\tilde{\rho}_0: \nu_q(k) \oplus \nu_{q-1}(k) \rightarrow k_q(k) \oplus k_{q-1}(k) \rightarrow \text{gr}_0 k_q(K)
\]

is also an isomorphism.

(ii) If \( 1 \leq m < e' \) and \( p \nmid m \), then \( \rho_m \) induces a surjection

\[
\tilde{\rho}_m: \Omega^q_k -1 \rightarrow \text{gr}_m k_q(K).
\]

(iii) If \( 1 \leq m < e' \) and \( p \mid m \), then \( \rho_m \) factors through

\[
\tilde{\rho}_m: \Omega^q_k -1/Z^q_1 \oplus \Omega^q_k -2/Z^q_1 -2 \rightarrow \text{gr}_m k_q(K)
\]

and \( \tilde{\rho}_m \) is a surjection. Here we denote \( Z^q_1 = Z_1 \Omega^q_k = \ker(d: \Omega^q_k \rightarrow \Omega^{q+1}_k) \).

(iv) If \( m = e' \in \mathbb{Z} \), then \( \rho_{e'} \) factors through

\[
\tilde{\rho}_{e'}: \Omega^q_k -1/(1 + aC) Z^q_1 -1 \oplus \Omega^q_k -2/(1 + aC) Z^q_1 -2 \rightarrow \text{gr}_{e'} k_q(K)
\]

and \( \tilde{\rho}_{e'} \) is a surjection.

Here \( a \) is the residue class of \( p^{e} \), and \( C \) is the Cartier operator

\[
x^p \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_q}{y_q} \rightarrow x \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_q}{y_q}, \quad d\Omega^q_k -1 \rightarrow 0.
\]

(v) If \( m > e' \), then \( \text{gr}_m k_q(K) = 0 \).

Proof. (i) follows from Bloch–Gabber–Kato’s theorem (subsection 2.4). The other claims follow from calculations of symbols. \( \Box \)

Definition. Denote the left hand side in the definition of \( \tilde{\rho}_m \) by \( G^q_m \). We denote the composite map \( G^q_m \xrightarrow{\tilde{\rho}_m} \text{gr}_m k_q(K) \xrightarrow{h_{q,K}} \text{gr}_m h^q(K) \) by \( \tilde{\rho}_m \); the latter is also surjective.

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In this and next section we outline the proof of Bloch–Kato’s theorem.

4.3.1. Norm argument.

We may assume $\zeta_p \in K$ to prove Bloch–Kato’s theorem. Indeed, $|K(\zeta_p) : K|$ is a divisor of $p - 1$ and therefore is prime to $p$. There exists a norm homomorphism $N_{L/K}: K_q(L) \to K_q(K)$ (see [BT, Sect. 5]) such that the following diagram is commutative:

\[
\begin{array}{cccc}
K_q(K)/p^n & \longrightarrow & K_q(L)/p^n & \longrightarrow & K_q(K)/p^n \\
\downarrow h_{q,K} & & \downarrow h_{q,L} & & \downarrow h_{q,K} \\
H^q(K, \mathbb{Z}/p^n(q)) & \overset{\text{res}}{\longrightarrow} & H^q(L, \mathbb{Z}/p^n(q)) & \overset{\text{cor}}{\longrightarrow} & H^q(K, \mathbb{Z}/p^n(q))
\end{array}
\]

where the left horizontal arrow of the top row is the natural map, and res (resp. cor.) is the restriction (resp. the corestriction). The top row and the bottom row are both multiplication by $j_{L/K}$, thus they are isomorphisms. Hence the bijectivity of $h_{q,K}$ follows from the bijectivity of $h_{q,L}$ and we may assume $\zeta_p \in K$.

4.3.2. Tate’s argument.

To prove Bloch–Kato’s theorem we may assume that $n = 1$. Indeed, consider the cohomological long exact sequence

\[
\cdots \to H^{q-1}(K, \mathbb{Z}/p(q)) \overset{\delta}{\longrightarrow} H^q(K, \mathbb{Z}/p^{n-1}(q)) \overset{\partial}{\longrightarrow} H^q(K, \mathbb{Z}/p^n(q)) \to \cdots
\]

which comes from the Bockstein sequence

\[
0 \longrightarrow \mathbb{Z}/p^{n-1} \overset{P}{\longrightarrow} \mathbb{Z}/p^n \overset{\mod p}{\longrightarrow} \mathbb{Z}/p \longrightarrow 0.
\]

We may assume $\zeta_p \in K$, so $H^{q-1}(K, \mathbb{Z}/p(q)) \simeq h_{q-1}(K)$ and the following diagram is commutative (cf. [T, §2]):

\[
\begin{array}{cccc}
k_{q-1}(K) & \overset{\{s, \zeta_p\}}{\longrightarrow} & K_q(K)/p^{n-1} & \overset{p}{\longrightarrow} & K_q(K)/p^n & \overset{\mod p}{\longrightarrow} & k_q(K) \\
\downarrow h_{q-1,K} & & \downarrow h_{q,K} & & \downarrow h_{q,K} & & \downarrow h_{q,K} \\
h^{q-1}(K) & \overset{\cup \zeta_p}{\longrightarrow} & H^q(K, \mathbb{Z}/p^{n-1}(q)) & \overset{p}{\longrightarrow} & H^q(K, \mathbb{Z}/p^n(q)) & \overset{\mod p}{\longrightarrow} & h^q(K).
\end{array}
\]

The top row is exact except at $K_q(K)/p^{n-1}$ and the bottom row is exact. By induction on $n$, we have only to show the bijectivity of $h_{q,K}: k_q(K) \to h^q(K)$ for all $q$ in order to prove Bloch–Kato’s theorem.

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4.4. Bloch–Kato’s Theorem

We review the proof of Bloch–Kato’s theorem in the following four steps.

I \( \overline{\pi}_m: \text{gr}_m k_q(K) \to \text{gr}_m h^q(K) \) is injective for \( 1 \leq m < e' \).

II \( \overline{\pi}_0: \text{gr}_0 k_q(K) \to \text{gr}_0 h^q(K) \) is injective.

III \( h^q(K) = U_0 h^q(K) \) if \( k \) is separably closed.

IV \( h^q(K) = U_0 h^q(K) \) for general \( k \).

4.4.1. Step I.

Injectivity of \( \overline{\pi}_m \) is preserved by taking inductive limit of \( k \). Thus we may assume \( k \) is finitely generated over \( \mathbb{F}_p \) of transcendence degree \( r < \infty \). We also assume \( \mathbb{F}_p \subseteq K \).

Then we get

\[
\text{gr}_{e'} h^{r+2}(K) = U_{e'} h^{r+2}(K) \neq 0.
\]

For instance, if \( r = 0 \), then \( K \) is a local field and \( U_{e'} h^2(K) = \text{Br}(K) = \mathbb{Z}/p \).

If \( r \geq 1 \), one can use a cohomological residue to reduce to the case of \( r = 0 \). For more details see [K1, Sect. 1.4] and [K2, Sect. 3].

For \( 1 \leq m < e' \), consider the following diagram:

\[
\begin{array}{ccc}
G^q_m \times G^{r+2-q}_{e'-m} & \xrightarrow{\pi_m \times \pi_{e'-m}} & \text{gr}_m h^q(K) \oplus \text{gr}_{e'-m} h^{r+2-q}(K) \\
\varphi_m & \downarrow \text{cup product} & \\
\Omega^r_k/d\Omega^{r-1}_k & \xrightarrow{\overline{\pi}_{e'}} & \text{gr}_{e'} h^{r+2}(K)
\end{array}
\]

where \( \varphi_m \) is, if \( p \nmid m \), induced by the wedge product

\[
\Omega^{q-1}_k \times \Omega^{r+1-q}_k \to \Omega^r_k/d\Omega^{r-1}_k,
\]

and if \( p \mid m \),

\[
\frac{\Omega^{q-1}_k}{Z^{q-1}_1} \oplus \frac{\Omega^{q-2}_k}{Z^{q-2}_1} \times \frac{\Omega^{r+1-q}_k}{Z^{r+1-q}_1} \oplus \frac{\Omega^{r-q}_k}{Z^{r-q}_1} \xrightarrow{\varphi_m} \Omega^q_k/d\Omega^{q-1}_k
\]

\[
(x_1, x_2, y_1, y_2) \longmapsto x_1 \wedge dy_2 + x_2 \wedge dy_1,
\]

and the first horizontal arrow of the bottom row is the projection

\[
\Omega^q_k/d\Omega^{q-1}_k \to \Omega^r_k/(1 + a C)Z^r_1 = G^{r+2}_{e'}
\]

since \( \Omega^{r+1}_k = 0 \) and \( d\Omega^{r-1}_k \subseteq (1 + a C)Z^r_1 \).

The diagram is commutative, \( \Omega^r_k/d\Omega^{r-1}_k \) is a one-dimensional \( k^p \)-vector space and \( \varphi_m \) is a perfect pairing, the arrows in the bottom row are both surjective and \( \text{gr}_{e'} h^{r+2}(K) \neq 0 \), thus we get the injectivity of \( \overline{\pi}_m \).
4.4.2. Step II.

Let $K'$ be a henselian discrete valuation field such that $K \subset K'$, $e(K'|K) = 1$ and $k_{K'} = k(t)$ where $t$ is an indeterminate. Consider
\[ \gr_0 h_q(K) \xrightarrow{\cup 1 + \pi t} \gr_1 h^{q+1}(K'). \]

The right hand side is equal to $\Omega^q_{K(t)}$ by (I). Let $\psi$ be the composite
\[ \nu_q(k) \oplus \nu_{q-1}(k) \xrightarrow{\varphi_0} \gr_0 h_q(K) \xrightarrow{\cup 1 + \pi t} \gr_1 h^{q+1}(K') \simeq \Omega^q_{K(t)}. \]

Then
\[ \psi \left( \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_q}{x_q}, 0 \right) = t \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_q}{x_q}, \]
\[ \psi \left( 0, \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_{q-1}}{x_{q-1}} \right) = \pm dt \wedge \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_{q-1}}{x_{q-1}}. \]

Since $t$ is transcendental over $k$, $\psi$ is an injection and hence $\varphi_0$ is also an injection.

4.4.3. Step III.

Denote $sh^q(K) = U_0 h^q(K)$ (the letter $s$ means the symbolic part) and put
\[ C(K) = h^q(K)/sh^q(K). \]

Assume $q \geq 2$. The purpose of this step is to show $C(K) = 0$. Let $\widehat{K}$ be a henselian discrete valuation field with algebraically closed residue field $k_{\widehat{K}}$ such that $K \subset \widehat{K}$, $k \subset k_{\widehat{K}}$ and the valuation of $K$ is the induced valuation from $\widehat{K}$. By Lang [L], $\widehat{K}$ is a $C_1$-field in the terminology of [S]. This means that the cohomological dimension of $\widehat{K}$ is one, hence $C(\widehat{K}) = 0$. If the restriction $C(K) \to C(\widehat{K})$ is injective then we get $C(K) = 0$. To prove this, we only have to show the injectivity of the restriction $C(K) \to C(L)$ for any $L = K(b^{1/p})$ such that $b \in \mathcal{O}_{K}^*$ and $b \notin k_{\widehat{K}}^{p}$.

We need the following lemmas.

**Lemma 1.** Let $K$ and $L$ be as above. Let $G = \text{Gal}(L/K)$ and let $sh^q(L)^G$ (resp. $sh^q(L)_G$) be $G$-invariants (resp. $G$-coinvariants). Then
(i) $sh^q(K) \xrightarrow{\text{res}} sh^q(L)^G \xrightarrow{\text{cor}} sh^q(K)$ is exact.
(ii) $sh^q(K) \xrightarrow{\text{res}} sh^q(L)_G \xrightarrow{\text{cor}} sh^q(K)$ is exact.

**Proof.** A nontrivial calculation with symbols, for more details see ([BK, Prop. 5.4].

**Lemma 2.** Let $K$ and $L$ be as above. The following conditions are equivalent:
(i) $h^{q-1}(K) \xrightarrow{\text{res}} h^{q-1}(L)^G \xrightarrow{\text{cor}} h^{q-1}(K)$ is exact.
(ii) $h^{q-1}(K) \xrightarrow{\text{cor}} h^q(K) \xrightarrow{\text{res}} h^q(L)$ is exact.
Proof. This is a property of the cup product of Galois cohomologies for \( L/K \). For more details see [BK, Lemma 3.2].

By induction on \( q \) we assume \( sh^{q-1}(K) = h^{q-1}(K) \). Consider the following diagram with exact rows:

\[
\begin{array}{cccccc}
& h^{q-1}(K) & \downarrow h^{q-1}(K) & \downarrow h^{q-1}(K) & \downarrow h^{q-1}(K) & \downarrow 0 \\
0 & \longrightarrow & sh^{q}(K) & \longrightarrow & h^{q}(K) & \longrightarrow & C(K) & \longrightarrow & 0 \\
& \downarrow res & \downarrow res & \downarrow res & \downarrow res & \downarrow \\
0 & \longrightarrow & sh^{q}(L)^G & \longrightarrow & h^{q}(L)^G & \longrightarrow & C(L)^G & \longrightarrow & 0 \\
& \downarrow cor & \downarrow cor & \downarrow cor & \downarrow \\
0 & \longrightarrow & sh^{q}(K) & \longrightarrow & h^{q}(K) \\
\end{array}
\]

By Lemma 1 (i) the left column is exact. Furthermore, due to the exactness of the sequence of Lemma 1 (ii) and the inductional assumption we have an exact sequence

\[
h^{q-1}(K) \xrightarrow{\text{res}} h^{q-1}(L)^G \longrightarrow h^{q-1}(K).\]

So by Lemma 2

\[
h^{q-1}(K) \xrightarrow{h} h^{q}(K) \xrightarrow{\text{res}} h^{q}(L)
\]

is exact. Thus, the upper half of the middle column is exact. Note that the lower half of the middle column is at least a complex because the composite map \( \text{cor} \circ \text{res} \) is equal to multiplication by \( |L : K| = p \). Chasing the diagram, one can deduce that all elements of the kernel of \( C(K) \to C(L)^G \) come from \( h^{q-1}(K) \) of the top group of the middle column. Now \( h^{q-1}(K) = sh^{q-1}(K) \), and the image of

\[
sh^{q-1}(K) \xrightarrow{h} h^{q}(K)
\]

is also included in the symbolic part \( sh^{q}(K) \) in \( h^{q}(K) \). Hence \( C(K) \to C(L)^G \) is an injection. The claim is proved.

4.4.4. Step IV.

We use the Hochschild–Serre spectral sequence

\[
H^r(G_k, h^q(K_{ur})) \Rightarrow h^{q+r}(K).
\]

For any \( q \),

\[
\Omega_{k,\text{sep}}^q \cong \Omega_k^q \otimes_k k^{\text{sep}}, \quad Z_1\Omega_{k,\text{sep}}^q \cong Z_1\Omega_k^q \otimes_k (k^{\text{sep}})^p.
\]
Thus, $\text{gr}_m h^q(K_\text{ur}) \simeq \text{gr}_m h^q(K) \otimes_{k^p} (k^\text{sep})^p$ for $1 \leq m < e'$. This is a direct sum of copies of $k^\text{sep}$, hence we have

$$H^0(G_k, U_1 h^q(K_\text{ur})) \simeq U_1 h^q(K)/U_{e'} h^q(K),$$

$$H^r(G_k, U_1 h^q(K_\text{ur})) = 0$$

for $r \geq 1$ because $H^r(G_k, k^\text{sep}) = 0$ for $r \geq 1$. Furthermore, taking cohomologies of the following two exact sequences

$$0 \rightarrow U_1 h^q(K_\text{ur}) \rightarrow h^q(K_\text{ur}) \rightarrow \nu_{k^\text{sep}}^q \oplus \nu_{k^\text{sep}}^{q-1} \rightarrow 0,$$

$$0 \rightarrow \nu_{k^\text{sep}}^q \rightarrow Z_1 \Omega_{k^\text{sep}}^{q-1} \rightarrow \Omega_{k^\text{sep}}^q \rightarrow 0,$$

we have

$$H^0(G_k, h^q(K_\text{ur})) \simeq sh^q(K)/U_{e'} h^q(K) \simeq k^q(K)/U_{e'} k^q(K),$$

$$H^1(G_k, h^q(K_\text{ur})) \simeq H^1(G_k, \nu_{k^\text{sep}}^q \oplus \nu_{k^\text{sep}}^{q-1})$$

$$\simeq (\Omega_{k^\text{sep}}^q/(1 - C)Z_1 \Omega_{k^\text{sep}}^{q-1}) \oplus (\Omega_{k^\text{sep}}^{q-2}/(1 - C)Z_1 \Omega_{k^\text{sep}}^{q-1}),$$

$$H^r(G_k, h^q(K_\text{ur})) = 0$$

for $r \geq 2$, since the cohomological $p$-dimension of $G_k$ is less than or equal to one (cf. [S, II-2.2]). By the above spectral sequence, we have the following exact sequence

$$0 \rightarrow \Omega_{k}^{q-1}/(1 - C)Z_1^{q-1} \oplus \Omega_{k}^{q-2}/(1 - C)Z_1^{q-2} \rightarrow h^q(K)$$

$$\rightarrow k_q(K)/U_{e'} k_q(K) \rightarrow 0.$$

Multiplication by the residue class of $(1 - \zeta_p^p)/\pi^{e'}$ gives an isomorphism

$$(\Omega_{k}^{q-1}/(1 - C)Z_1^{q-1}) \oplus \Omega_{k}^{q-2}/(1 - C)Z_1^{q-2})$$

$$\rightarrow (\Omega_{k}^{q-1}/(1 + a C)Z_1^{q-1}) \oplus \Omega_{k}^{q-2}/(1 + a C)Z_1^{q-2}) = \text{gr}_{e'} k_q(K),$$

hence we get $h^q(K) \simeq k_q(K)$.

References


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