At most 27 length inequalities define Maskit’s fundamental domain for the modular group in genus 2

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Abstract

In recently published work Maskit constructs a fundamental domain $D_g$ for the Teichmüller modular group of a closed surface $S$ of genus $g \geq 2$. Maskit’s technique is to demand that a certain set of $2g$ non-dividing geodesics $C_{2g}$ on $S$ satisfies certain shortness criteria. This gives an a priori infinite set of length inequalities that the geodesics in $C_{2g}$ must satisfy. Maskit shows that this set of inequalities is finite and that for genus $g = 2$ there are at most 45. In this paper we improve this number to 27. Each of these inequalities compares distances between Weierstrass points in the fundamental domain $\mathbb{S} \cap C_4$ for $S$; and is realised (as an equality) on one or other of two special surfaces.

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0 Introduction and preliminaries

In this paper we consider a fundamental domain defined by Maskit in [8] for the action of the Teichmüller modular group on the Teichmüller space of a closed surface of genus $g \geq 2$ in the special case of genus $g = 2$. McCarthy and Papadopoulos [9] have also defined such a fundamental domain, modelled on a Dirichlet region; for punctured surfaces there is the celebrated cell decomposition and associated fundamental domain due to Penner [10]. For genus $g = 2$ Semmler [11] has defined a fundamental domain based on locating the shortest dividing geodesic. Also for low signature surfaces the reader is referred to the papers of Keen [3] and of Maskit [7], [8].

Throughout $S$ will denote a closed orientable surface of genus $g = 2$, with some fixed hyperbolic metric. We say that a simple closed geodesic $\gamma$ on $S$
is: dividing if $S \cap \gamma$ has two components; or non-dividing if $S \cap \gamma$ has one component. By non-dividing geodesic we shall always mean simple closed non-dividing geodesic. We denote the length of $\gamma$ with respect to the hyperbolic metric on $S$ by $l(\gamma)$. Let $j \gamma$ denote the number of intersection points of two distinct geodesics.

We define a chain $C_n = \gamma_1; \ldots; \gamma_n$ to be an ordered set of non-dividing geodesics such that:

1. $j \gamma_i \gamma_{i+1} = 1$ for $1 \leq i < n$ and $\gamma_1 \gamma_0 = 1$; otherwise.
2. $j \gamma_n \gamma_1 = 1$ and $\gamma_1 \gamma_0 = 1$; otherwise.

Again we say that $B_n$ has length $n$, where $3 \leq n \leq 6$. Following Maskit, we call a bracelet of length 6 a necklace.

For $n = 4$ a chain of length $n$ can be always be extended to a chain of length $n + 1$. For $n = 4$ this extension is unique. Likewise a chain of length 5 extends uniquely to a necklace. So chains of length 4 or 5 and necklaces can be considered equivalent. We shall usually work with length 4 chains, which we call standard. (Maskit, for genus $g$, usually works with chains of length $2g + 1$, which he calls standard.)

As Maskit shows in [8] each surface, standard chain pair $S; C_4$ gives a canonical choice of generators for the Fuchsian group $F$ such that $\mathbb{H}^2/F = S$ and hence a point in $DF(1(S); PSL(2; \mathbb{R}))$, the set of discrete faithful representations of $1(S)$ into $PSL(2; \mathbb{R})$. Essentially this representation corresponds to the fundamental domain $S \cap C_4$ together with orientations for its side pairing elements. As Maskit observes, it is well known that $DF(1(S); PSL(2; \mathbb{R}))$ is real analytically equivalent to Teichmüller space. So, we define the Teichmüller space of closed orientable genus $g = 2$ surfaces $T_2$ to be the set of pairs $S; C_4$.

We say that a standard chain $C_4 = \gamma_1; \ldots; \gamma_4$ is minimal if for any chain $C_{m} = \gamma_1; \ldots; \gamma_{m-1}$, we have $l(\gamma_m) = l(\gamma_{m-1})$ for $1 \leq m \leq 4$. We then define the Maskit domain $D_2 \subset T_2$ to be the set of surface, standard chain pairs $S; C_4$ with $C_4$ minimal.

For $C_4$ to be minimal the geodesics $\gamma_1; \ldots; \gamma_4$ must satisfy an a priori infinite set of length inequalities. For genus $g$, Maskit gives an algorithm using cut-and-paste to show that only a finite number $N_g$ of length inequalities need to be satisfied. Applying his algorithm to genus $g = 2$, Maskit showed that $N_2 = 45$. We establish an independent proof that $N_2 = 27$. We could have shown that 18 of Maskit's 45 inequalities follow from the other 27. However, by tailoring all our techniques to the special case of genus 2, we are able to produce a much shorter proof.

The fact that 18 of Maskit's 45 inequalities follow from the other 27 follows from applications of Theorem 2.2 (which appeared as Theorem 1.1 in [4]) and of Corollary 2.5. The latter follows immediately from Theorem 2.4, for which we give a proof in this paper. This is a characterisation of the octahedral surface $\text{Oct}$ (the well known genus two surface of maximal symmetry group) in terms of a finite set of length inequalities.

The 27 length inequalities have the properties that: each is realised on one or other of two special surfaces (for all but 2 this special surface is $\text{Oct}$); and each compares distances between Weierstrass points in the fundamental domain $S \cap C_4$ for $S$.

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1 The hyperelliptic involution and the main result

It is well known that every closed genus two surface without boundary $S$ admits a uniquely determined hyperelliptic involution, an isometry of order two with six fixed points, which we denote by $J$. The fixed points of $J$ are known as Weierstrass points. Every simple closed geodesic $\gamma$ of $S$ is setwise fixed by $J$, and the restriction of $J$ to $\gamma$ has no fixed points if $\gamma$ is dividing and two fixed points if $\gamma$ is non-dividing (see Haas-Susskind [2]). So every non-dividing geodesic on $S$ passes through two Weierstrass points. It is a simple consequence that sequential geodesics in a chain intersect at Weierstrass points. We say that two non-dividing geodesics $\gamma$ and $\delta$ contain a point that is not a Weierstrass point.

The quotient orbifold $O = S/J$ is a sphere with six order two cone points, endowed with a fixed hyperbolic metric. Each cone point on $O$ is the image of a Weierstrass point under the projection $J : S \to O$ and each non-dividing geodesic on $S$ projects to a simple geodesic between distinct cone points on $O$ (what we shall call an arc). Definitions of chains, bracelets and crossing all pass naturally to the quotient.

Let $C_4$ be a standard chain on $S$, which extends to a necklace $N$. We number Weierstrass points on $N$ so that $!_i = \gamma_{i-1} \setminus \gamma_i$ for $2 \leq i \leq 6$ and $!_1 = \gamma_6 \setminus \gamma_1$. 

*Geometry and Topology Monographs, Volume 1 (1998)*
Choose an orientation upon \( \mathbb{S} \) and project to the quotient orbifold \( \mathbb{O} = \mathbb{S}/\Gamma \). For the rest of the paper we shall work on the quotient orbifold \( \mathbb{O} \). We label the components of \( \mathbb{O} \nabla \mathbb{N} \) by \( \mathbb{H} \); \( \mathbb{H} \) so that \( \gamma_1; \gamma_2; \cdots; \gamma_n \) lie anticlockwise around \( \mathbb{H} \). Label by \( i_1; i_2; \cdots; i_n \) (respectively \( j_1; j_2; \cdots; j_n \)) the arc between the cone points \( !_j; !_k \) (\( j < k \)) crossing the sequence of arcs \( \gamma_1; \gamma_2; \cdots; \gamma_n \) and having the subarc between \( !_j; \gamma_{i_1} \) lying in \( \mathbb{H} \) (respectively \( \mathbb{H} \)).

Our main result is then the following. (We abuse notation so that \( 1:6 = 1:6 = \gamma_6 \) and \( 2:3 = 2:3 = \gamma_2 \). We then have repetitions, \( l(\gamma_2) = l(\gamma_6) \) twice, and redundancies, \( l(\gamma_2) = l(\gamma_2) \) also twice.)

**Theorem 1.1** The standard chain \( \mathbb{C}_4 \) is minimal if the following are satisfied:

1. \( l(\gamma_1); l(\gamma_1); i \in \{ 2, 3, 4, 5 \} \)
2. \( l(\gamma_2); l(\gamma_2); l(\gamma_2); l(\gamma_2); i \in \{ 2, 3, 4, 5, 6 \} \)
3. \( l(\gamma_3); l(\gamma_3); l(\gamma_3); l(\gamma_3); j \in \{ 2, 3, 4, 5 \} \)
4. \( l(\gamma_4); l(\gamma_4); l(\gamma_4); j \in \{ 4, 6 \} \).

Each length \( l(\gamma_i) \) or \( l(\gamma_j; \gamma_k) \) (respectively \( l(\gamma_j; \gamma_k) \)) is a distance between cone points in \( \mathbb{H} \) (respectively \( \mathbb{H} \)). Likewise each length \( l(\gamma_j; \gamma_k), l(\gamma_j; \gamma_k) \) is a distance between cone points in \( \mathbb{O} \nabla \mathbb{G} \). So each length inequality in Theorem 1.1 compares distances between cone points in \( \mathbb{O} \nabla \mathbb{G} \) (and hence distances between Weierstrass points in \( \mathbb{S} \nabla \mathbb{G} \)).

![Figure 1: How the length inequalities in Theorem 1.1 are realized on Oct and E](image)

Theorem 1.1 gives a sufficient list of inequalities. As to the necessity each inequality, we make the following observation. Each inequality is realised (as an equality) on either Oct or E. { cf Theorem 1.1 in [5]. The octahedral orbifold Oct is the well known orbifold of maximal conformal symmetry group. Any minimal standard chain on Oct lies in its set of shortest arcs. This arc set has the combinatorial edge pattern of the Platonic solid. The exceptional orbifold E, which was constructed in [5], has conformal symmetry group \( \mathbb{Z}_2 \times \mathbb{Z}_2 \). However...
it is not needed by the action of its symmetry group alone, it also requires a certain length inequality to be satisfied. Any minimal standard chain on $E$ lies in its set of shortest and second shortest arcs.

In Figure 1 we have illustrated necklaces on $O_c$ and $E$ that are the extensions of minimal standard chains. As with other figures in this paper, we use wireframe diagrams to illustrate the orbifolds. Solid (respectively dashed) lines represent arcs in front (respectively behind) the figure. Thick lines represent arcs in the necklace $N$. The minimal standard chain on $E$ in Figure 1 has: $l(y_1) = l(y_5); l(y_2) = l(\frac{1}{3}; 1) = l(1; 4) = l(2; 4); l(y_3) = l(\frac{6}{3}; 4) = l(\frac{3}{5}) = l(\frac{1}{5}); l(y_4) = l(\frac{4}{6})$. Making such a list for all the orbifolds in Figure 1, together with their mirror images, we see that all the inequalities in Therem 1.1 are realised as equalities on either $O_c$ or $E$.

2 Length inequalities for systems of arcs

In order to prove Theorem 1.1 we need a number of length inequality results for systems of arcs. Let $K_{4} = 0;1;\ldots;30$ denote a length 4 bracelet such that each component of $O_{n}$ contains an interior cone point. Using mod 4 addition throughout, label cone points: on $K_{4}$ by $c_{k} = k-1:k \backslash k:k+1$ for $k \geq 0; \ldots; 3g$; and on $K_{4}$ by $a_{l}$ for $l = 2f; 4g$. Label by $O_{l}$ the component of $O_{n}$ containing $c_{k}$ and label arcs in $O_{l}$ so that $k+l$ is between $c_{k}; a_{l}$. Let $k$ denote the arc between $c_{k}; c_{k+1}$ crossing only $a_{k}$.

The following two results appeared as Lemma 2.3 in [5] (in Maskit’s terminology this is a cut-and-paste) and as Theorem 1.1 in [4] respectively.

Lemma 2.1 (i) $2l(0; 4) < l(0) + l(3)$ (ii) $2l(3; 0) < l(0) + l(2)$.

Theorem 2.2 If $l(3; 4)$, $l(0; 4)$, $l(3; 5)$, $l(0; 5)$, $l(0)$, $l(2)$ then $l(0; 4)$, $l(0; 5)$, $l(3; 5) = l(0; 5)$, $l(0) = l(2)$.

Corollary 2.3 If $l(3; 4)$, $l(0; 4)$, $l(3; 5)$, $l(0; 5)$, $l(1; 4)$, $l(2; 4)$ then $l(1; 5); l(2; 5)$.

Proof of Corollary 2.3 Since $l(3; 4), l(0; 4), l(3; 5), l(0; 5)$ Theorem 2.2 implies that $l(0) \leq l(2)$. Moreover $l(1; 4) \leq l(2; 4)$ and so again, by Theorem 2.2, $l(1; 5) \leq l(2; 5)$.

**Theorem 2.4** Suppose \( l(2,3) \), \( l(3,0) \), \( l(1,2) \), \( fl(0,1) \), \( l(1,3) \)g and \( l(0,1) \), \( fl(0,1) \), \( l(3,1) \)g then \( l(k,1) = l(k,k+1) \) for each \( k;1 \) and \( O \) is the octahedral orbifold.

**Proof of Theorem 2.4** We postpone this until Section 3. \( \square \)

**Corollary 2.5** Suppose \( l(2,3) \), \( l(3,0) \), \( l(1,2) \), \( fl(0,1) \), \( l(1,3) \)g and \( l(0,1) \), \( fl(0,1) \), \( l(3,1) \)g then \( l(3,0) = l(1,2) \).

**Proof of Corollary 2.5** If \( l(3,0) \) \( l(1,2) \) then by Theorem 2.4 \( l(1,1) = l(1,k+1) \) for each \( k;1 \). In particular \( l(3,0) = l(1,2) \). So \( l(3,0) = l(1,2) \). \( \square \)

3 The proofs

**Proof of Theorem 1.1** Let \( m \) denote an arc such that \( C_m = V_1;\cdots;V_{m-1};V_m \). \( m \) is a chain, for \( 1 \leq m \leq 4 \); \( m \in V_m \). We will show that \( l(V_m) = l(m) \) for arcs of the form \( \frac{W_1;\cdots;W_n}{i;k} \). The same arguments work for arcs of the form \( \frac{W_1;\cdots;W_n}{j;k} \). Let \( X(\ ;\ ) \) denote the number of crossing points of a distinct pair of arcs \( \{ \} \) is the number of intersection points of \( \{ \} \) that are not cone points. Let \( n = 1 \), if \( X(V_m; i) = 0 \) for \( i \leq 2 \) \( f_1;\cdots;f_6 \); otherwise let \( n = \min i 2 f_1;\cdots;f_6 \) such that \( X(V_n; i) > 0 \). We note that \( n \geq m \).

Let \( P_{m,n,p} \) be the proposition that \( l(V_m) = l(m) \) for \( X(m;V_n) = p \). Clearly, if \( n = 1 \) then \( p = 0 \). For \( n \leq 2 \) \( f_1;\cdots;f_4 \) we consider \( p = 1 \) and \( p > 1 \). We order the propositions as follows:

- \( P_{4;1;6} \) followed by \( P_{4;4;1} \) followed by \( P_{4;4;4;p>1} \).
- \( P_{3;3;1} \) followed by \( P_{3;3;3;p>1} \) followed by \( P_{3;4;1} \) followed by \( P_{3;4;4;p>1} \).
- \( P_{2;2;1} \) followed by \( P_{2;2;p>1} \) followed by \( P_{1;4;1} \).

Suppose \( n = 1 \); \( m \) does not cross \( N \). If \( m > 1 \) then \( P_{m;1;0} \) is a hypothesis.

If \( m = 1 \) then either \( P_{1;1;0} \) is a hypothesis, \( 1 = \gamma_i \) for some \( i \leq 2 \) \( f_2;\cdots;f_5 \),

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*Figure 2: Arc sets for Lemma 2.1, for Theorem 2.2 and for Corollary 2.3*
or $P_{1;1;0}$ follows from the hypotheses, $l(y_1) \leq l(y_i); l(y_i) \leq l(y_4)$ for some $i \neq 2 \neq 3 \neq 4$.

Suppose $n = 2 f 5; 6g$: $m$ crosses $N$ but does not cross $C_4$.

For $m = 4$, by inspection, $4 = \frac{6}{4}; 5$. So $m; y_m$ share endpoints, $n > m + 1$ and we can apply the argument (i) below. So we have $P_{4;1;1}$ for $n = 2 f 5; 6g$.

In Figures 3, 4, 5 we illustrate applications of length inequalities results to the proof. As above we use wire frame figures of the octahedral orbifold, with the necklace $N$ in thick black. Other arcs are in thick grey. Figures have been drawn so arcs in the application correspond to arcs in the length inequality result.

![Figure 3: Application (i) for $a = \frac{6}{4}; 5$, $3 = \frac{5}{3}; 4$ and of Theorem 2.2, (ii) for $3 = \frac{6}{3}; 5$](image)

For $m = 3$. By inspection, $3$ is one of $\frac{5}{3}; 4; \frac{6}{3}; 4; \frac{6}{5}; 4; \frac{6}{3}; 5$. For $\frac{5}{3}; 4; \frac{6}{3}; 4; \frac{6}{5}; 4; \frac{6}{3}; 5$, $m$ share endpoints, $n > m + 1$ and so we can apply either argument (i) or (ii) below. For $\frac{6}{3}; 5$ we can apply Theorem 2.2 in conjunction with argument (i): by hypothesis $l(y_4) \leq l(y_6)$ and by argument (ii) $l(y_3) \leq l(y_3)$ and so $l(\frac{6}{3}; 3)$ and of Theorem 2.2. Again by hypothesis $l(y_3) \leq l(\frac{3}{6})$ and so $l(y_3) \leq l(\frac{3}{6})$. This gives $P_{3;1;1}$ for $n = 2 f 5; 6g$.

For $m = 2$: $2$ is one of $\frac{5}{2}; 3; \frac{6}{2}; 3$ or one of $\frac{5}{2}; 4; \frac{6}{2}; 4; \frac{6}{5}; 4; \frac{6}{2}; 5; \frac{5}{2}; 1; 3; \frac{5}{1}; 4$. By hypothesis $l(y_2) \leq l(y_2)$. For $\frac{5}{2}; 3; \frac{6}{2}; 3$ we can again apply either argument (i) or (ii). For $\frac{5}{2}; 4; \frac{6}{2}; 4; \frac{6}{5}; 4; \frac{5}{2}; 1; 3$ we apply Theorem 2.2 in conjunction with argument (i). We give the argument for $\frac{5}{2}; 4$. By argument (ii), we have $l(y_2) < l(\frac{5}{2}; 3)$. Also, by hypothesis, $l(y_3) \leq l(\frac{3}{5})$ and so by Theorem 2.2 $l(\frac{2}{5}) < l(\frac{5}{2}; 4)$. Again, by hypothesis, $l(y_2) \leq l(\frac{2}{5})$ and so $l(y_2) \leq l(\frac{5}{2}; 4)$.

For $2 = \frac{5}{1}; 4$ we argue as follows. By hypothesis we have $l(y_3) \leq l(\frac{3}{5}; 1); l(\frac{3}{6})$ and $l(y_2) \leq l(\frac{1}{5}; 1); l(y_6); l(\frac{2}{5})$ and $l(y_1) \leq l(\frac{1}{5}; 1); l(y_6); l(y_4) \leq l(\frac{4}{6})$. By Corollary 2.5: $l(\frac{5}{1}; 4) \leq l(y_2)$. Hence $P_{2;1;1}$ for $n = 2 f 5; 6g$.
For $m = 1$. If $f_1; kg \in f1; 2g$ or $f_1; kg \in f5; 6g$ then $l(y_1), l(y_2), l(\gamma_1)$ are hypotheses, or preceding propositions, for some $i \in \{2, 3, 4\}$. If $f_1; kg = f1; 2g$ then, by inspection, $\gamma_1 = 2$ we can again apply argument (i). By inspection there is no such $f_1; kg = f5; 6g$. This completes $P_{mn;1}$ for $n \in \{2, 5, 6\}$.

We now give the arguments for: $\gamma_m$ share endpoints and $n > m + 1$. The arc set $\Gamma := \bigcap_{m} \gamma_m$ divides $O$ into two components. Either: (i) $\Gamma$ divides one cone point (c) from three, or (ii) $\Gamma$ divides two cone points from two. For (i) we let $O_c; \ O_c^0$ denote the components of $O \cap \Gamma$ so that $c \in O_c$ and we let $m$ (respectively $m^0$) denote the arc between $\gamma_m; c^0$ (respectively between $\gamma_{m+1}; c$) in $O_c$.

First $m = 4$, (i), $n = 3$. None of $\gamma_1; \gamma_2; \gamma_3$ crosses $\Gamma = 4 \bigcap \gamma_4$, so $C_3 = \gamma_1; \gamma_2; \gamma_3$ lies in one or other component of $O \cap \Gamma$. Now $C_3$ contains three cone points disjoint from $\Gamma$, so $C_3 = O_3^0$. So $c = 1$ or $c = 5$ and $C_3^0 = \gamma_1; \gamma_2; \gamma_3$. $\gamma_3$ is a chain. We observe { see Figure 3 } that $\gamma_3 = 4, 6$ and hence $l(\gamma_4), l(\gamma_5) \in \gamma_5$ is a hypothesis. By Lemma 2.1(i): $l(\gamma_4) < l(\gamma_5) + l(\gamma_6)$ and so $l(\gamma_4) < l(\gamma_3) + l(\gamma_6)$.

Second $m = 3$, (i), $n = 2$. $f_1, f_3, f_5$. Neither $\gamma_1$ nor $\gamma_2$ cross $\Gamma = 3 \bigcap \gamma_3$, so $C_2 = \gamma_1; \gamma_2$ lies in one or other component of $O \cap \Gamma$. Now $C_2$ contains two cone points disjoint from $\Gamma$, so $C_2 = O_3^0$. $c = 1$ or $c = 5$ and $C_3^0 = \gamma_1; \gamma_2; \gamma_3$. $\gamma_3$ is a chain. We observe { see Figure 3 } that $\gamma_3 = 3, 5$ or $\gamma_3 = 3, 4$ and hence $l(\gamma_3) - l(\gamma_3) \in \gamma_3$ is hypothesis. Again, by Lemma 2.1(i): $l(\gamma_3) < l(\gamma_3) + l(\gamma_3)$. 

Figure 4: Applications of (i) or (ii) for $z = \gamma_3, \gamma_4$, and $\gamma_5$.
so \( l(y_3) \leq l(\gamma) < l(\gamma_3) \). For (ii) we have that \( \gamma = \frac{6}{3} \) and \( l(y_3) \leq l(\gamma) < l(\gamma_3) \) is a hypothesis.

Next \( m = 2 \), (i), \( n < 2 \) f 4; 5; 6g. The arc \( y_1 \) does not cross \( \Gamma = 2 \{ y_2 \), so \( y_1 \) \( \mathbb{O} \), and \( c \) f 4; 5; 6g (respectively \( y_1 \) \( \mathbb{O} \), and \( c = \Gamma_1 \)). For \( n = 2 \) f 5; 6g \{ see Figure 4 \} we have that \( \frac{d}{2} = \frac{2}{5} \) (respectively \( \frac{d}{2} = \frac{1}{5} \)). For \( n = 4 \) \{ see Figure 5 \} we have that \( \frac{d}{2} = \frac{2}{4} \) or \( \frac{2}{5} \) (respectively \( \frac{2}{d} \) there is no such \( \frac{2}{d} \)). So \( l(y_2) < l(\gamma) \) (respectively \( l(y_2) < l(\gamma) \)) is a hypothesis. By Lemma 2.1(i): \( 2l(\gamma) < l(y_2) + l(\gamma) \) and so \( l(y_2) < l(\gamma) \) (respectively \( l(y_2) < l(\gamma) \)).

For (ii), again, \( y_1 \) lies in one component of \( O \) n \( \Gamma \). Let \( \gamma \) denote the unique arc disjoint from \( \Gamma \) in this component of \( O \) n \( \Gamma \). For \( n = 2 \) f 5; 6g \{ again see Figure 4 \} we have that \( \frac{d}{2} = \frac{6}{5} \). For \( n = 4 \) { again see Figure 5 \} we have \( \frac{d}{2} = \frac{4}{5} \) or \( \frac{4}{5} \). So \( l(y_2) < l(\gamma) \) is a hypothesis. By Lemma 2.1(ii): \( 2l(\gamma) < l(y_2) + l(\gamma) \) and so \( l(y_2) < l(\gamma) \).

Finally, \( m = 1 \), (i), \( n < 2 \) f 3; 4; 5; 6g. For \( n = 2 \) f 5; 6g: \( \gamma = \frac{5}{2} \) and \( l(y_2) < l(\gamma) \) is a hypothesis. For \( n = 2 \) f 3; 4g: \( l(y_2) < l(\gamma) \) is a proceeding proposition. Since \( l(y_1) \leq l(y_2) \) is a hypothesis, we have that \( l(y_1) < l(y_2) < l(\gamma) \). By Lemma 2.1(i): \( 2l(\gamma) < l(y_1) + l(\gamma) \) and so \( l(y_1) < l(\gamma) \).

For (ii), \( n = 2 \) f 5; 6g, there is no such \( \gamma \). For \( n = 2 \) f 3; 4g, we let \( \gamma \) denote the unique arc disjoint from \( \Gamma \) in the same component of \( O \) n \( \Gamma \) as \( y_2 \). Here \( C_\gamma = \gamma_1; \gamma_2 \); \( \gamma \) is a chain and so \( l(y_3) < l(\gamma) \) is a proceeding proposition. Since \( l(y_1) \leq l(y_3) \) is a hypothesis, we have that \( l(y_1) < l(y_3) < l(\gamma) \). By Lemma 2.1(ii): \( 2l(\gamma) < l(y_1) + l(\gamma) \) and so \( l(y_1) < l(\gamma) \).

Now suppose \( n = 2 \) f 1; 4; 5; 6g; \( m = 4 \) crosses \( C_\gamma \).

**Lemma 3.1** Suppose that either \( X(m; \gamma) > 1 \) or \( \gamma \) share an endpoint. Then there exist arcs \( \gamma_0 \) between the same respective endpoints as \( m; \gamma \) such that \( l(\gamma_0) < l(m) \) or \( l(\gamma_0) < l(\gamma) \); \( X(\gamma_0; \gamma) < X(\gamma_0; \gamma) \); \( X(\gamma_0; \gamma) < X(\gamma_0; \gamma) \); and \( X(\gamma_0; \gamma) = X(\gamma_0; \gamma) = 0 \) for \( i < n - 1 \). In particular \( C_\gamma = \gamma_1; \gamma_2; \gamma_3; \gamma_4; \gamma_5; \gamma_6 \) are both chains.

**Proof** This result is essentially Proposition 3.1 in [5], with additional observations upon the number of crossing points. However, upon going through the proof, these observations become clear.

The following argument gives \( P_{m; n; p} \): it uses induction on \( p \), the first induction step being the set of propositions that precede \( P_{m; n; p} \).

*Geometry and Topology Monographs, Volume 1 (1998)*
Let $X ( m; Y_n ) = p > 1$ and so by Lemma 3.1 there exist arcs $0_m; Y_n^0$ as stated. Let $p^0 = X ( 0_m; Y_n ) < p$; $p^0 = X ( Y_n^0; Y_n ) < p$. We note that $l(Y_n) = l(0_m)$ is either: $P_{m,n;p^0>1}$ if $p^0 > 1$; or a preceding proposition if $p^0 = 1$. Likewise, $l(Y_n) = l(Y_n^0)$ is either: $P_{m,n;p^0>1}$ if $n = m$ and $p^0 > 1$; or a preceding proposition if $n > m$ or $p^0 = 1$. Since $l(0_m) < l(0_m)$ or $l(Y_n^0) < l(Y_n)$ it follows, by induction on $p$, that $l(Y_n) = l(0_m) < l(0_m)$. So, for the rest of the proof, we may suppose that $X ( m; Y_n ) = 1$.

**Lemma 3.2** Suppose that $m; Y_n$ have distinct endpoints and that $k > n + 1$. Then there exist arcs $0_m; Y_n^0$ between $!j; !n+1$ and $!n !k$ such that $l(0_m) < l(0_m)$ or $l(Y_n^0) < l(Y_n)$ and $X ( 0_m; Y_n ) = X ( Y_n^0; Y_n ) = 0$ for $i$. In particular $C_0 = Y_1; \ldots ; Y_{n-1}; 0_m; C_0 = Y_1; \ldots ; Y_{n-1}; Y_n$ are both chains.

**Proof** This is essentially Lemma 3.3 in [5], again with additional observations upon the number of crossing points. Again, these observations are clear. □

We now give two general arguments using these two lemmas.

Suppose: (1) $m; Y_n$ share an endpoint. Again we can apply Lemma 3.1; there exist arcs $0_m; Y_n^0$ as stated. In particular $X ( 0_m; Y_n ) = X ( Y_n^0; Y_n ) = 0$ for $i$. So $l(Y_n) = l(0_m)$; $l(Y_n) = l(Y_n^0)$ are both preceding propositions. Since $l(0_m) < l(0_m)$ or $l(Y_n^0) < l(Y_n)$, it follows that $l(Y_n) = l(0_m) < l(0_m)$.

Suppose: (2) $m; Y_n$ have distinct endpoints and $k > n + 1$. By Lemma 3.2 there exist arcs $0_m; Y_n^0$ as stated. Again $l(Y_n) = l(0_m); l(Y_n) = l(Y_n^0)$ are both preceding propositions. As $l(0_m) < l(0_m)$ or $l(Y_n^0) < l(Y_n)$, we have that $l(Y_n) = l(0_m) < l(0_m)$.

For $m = 4; j = 4; k 2 f 5; 6g$ and $n = 4$: $4; Y_4$ share the endpoint $! 4 (1)$.

For $m = 3; j = 3; k 2 f 4; 5; 6g$. For $n = 4$ if $k 2 f 4; 5g$ then $3; Y_4$ share the endpoint $! k (1)$; if $k = 6$ then $3; Y_4$ have distinct endpoints and $k > n + 1$ (2). For $n = 3$: $3; Y_3$ share the endpoint $! 3 (1)$.

For $m = 2; j = 2; f 1; 2g; k 2 f 3; \ldots ; 6g$. For $n = 4$ if $k = 3$ then, by inspection, $2$ is one of $\frac{5}{4}, \frac{4}{5}; 2; \frac{3}{2}, 2; \frac{3}{2}; 2; \frac{4}{5}$, and we can apply argument (i) or (ii), or is one of $\frac{1}{2}, \frac{4}{5}; \frac{1}{3}, \frac{2}{3}$, and we apply Theorem 2.2 in conjunction with argument (ii) { see Figure 5. If $k 2 f 4; 5g (1)$; if $k = 6$ (2). For $n = 3$ if $k 2 f 3; 4g (1)$; if $k 2 f 5; 6g (2)$. For $n = 2$ if $k 2 f 3; 4g (1)$; if $k 2 f 3; 4g (2)$. Finally $m = 1$. Suppose $n = 4$. If $f j; k g \notin f 1; 2g$ or $f j; k g \notin f 5; 6g$ then $l(Y_1) = l(Y_1); l(Y_1) = l(Y_1)$ are both preceding propositions for some
Proof of Theorem 2.4

As $l(3;0); l(0;5); l(2;3); l(2;5); l(0;1); l(0;4)$, by Corollary 2.3, we have that $l(1;2) = l(2;4)$. Likewise, since $l(3;0); l(0;4); l(2;3); l(2;4); l(0;1); l(0;5)$ we have that $l(1;2) = l(2;5)$. That is $l(1;2) = l(2;1)$.

The arc set $K$ divides $O$ into eight triangles. We label these as follows: let $t_k$ (respectively $T_k$) denote the triangle with one edge $k:k+1$ and one vertex $c_4$ (respectively $c_5$). We shall use $\angle q_k$ to denote the angle at the $q$ {vertex of $t_k$, et cetera. Cut $O$ open along $3:0; 0:1; 1:4; 1:2; 1:5$ to obtain a domain $\Omega$.

We show that \( l(2;3) \), \( l(2;1) \), \( l(3;0) \), \( l(1;2) \), \( l(0;1) \), \( l(0;0) \) implies that \( \min l(3;1) \) \( l(0;1) \) with equality if and only if \( O \) is the octahedral orbifold. First we show that: \( \triangle c_2 t_2 \triangle c_4 t_0 \) or \( \triangle c_2 t_2 \triangle c_5 T_0 \).

Now \( l(1;2) \), \( l(1;1) \), \( l(3;0) \) \( l(0;1) \) so \( \angle c_2 t_1 \angle c_4 t_1 \angle c_2 T_1 \angle c_5 T_1 \angle c_3 t_3 \angle c_4 t_3 \angle c_3 T_3 \angle c_5 T_3 \), which imply

\[
\triangle c_2 t_1 + \triangle c_5 T_2 + \triangle c_3 t_3 + \triangle c_5 T_3 \quad \angle c_4 t_1 + \angle c_5 T_1 + \angle c_4 t_3 + \angle c_5 T_3
\]

\[
( - \angle c_2 t_1 - \angle c_2 T_1) + ( - \angle c_3 t_3 - \angle c_3 T_3)
\]

\[
( - \angle c_4 t_1 - \angle c_4 t_3) + ( - \angle c_5 T_1 - \angle c_5 T_3)
\]

and \( l(2;3) \), \( l(2;1) \) so \( \triangle c_2 t_2 \triangle c_4 t_2 \triangle c_5 T_2 \angle c_5 T_2 \angle c_2 T_2 \angle c_4 t_0 + \angle c_5 T_0 \) \( \triangle c_2 t_2 \triangle c_4 t_0 \) or \( \triangle c_2 T_2 \triangle c_5 T_0 \).

\[\text{Figure 6: The triangles } t_k: T_k \text{ in the domain } \Omega\]

Up to relabelling, we may suppose that \( \angle c_2 t_2 \angle c_4 t_0 \). We now show that \( l(3;4) \), \( l(0;1) \). There are two arguments. Firstly we show that if \( \angle c_3 t_3 = \) then \( l(0;4) < l(3;0) \) \{ contradicting a hypothesis. So \( \angle c_3 t_2 < \) and we then show that \( l(3;4) \), \( l(0;1) \). The angle is given as follows. Let \( I_2 \) be an isosceles triangle with vertices \( v_2; v_3; v_4 \) and edges "2:3;"2:4;"3:4 such that \( l(2;3) = l(2;4) = l(2;4) \) and \( l(2;4) \) \( \angle v_2 l_2 = \angle c_2 t_2 \). Then \( = \triangle v_3 l_2 = \angle v_4 l_2 \).

Let \( C_2; C_4 \) denote circles of radius \( l(2;4) \) about \( c_2; c_4 \) respectively. As in Figure 7 \( c_3 \) must lie inside \( C_2 \) since \( l(2;3) \), \( l(2;4) \). Likewise \( c_0 \) must lie outside \( C_4 \) since \( l(0;4) \), \( l(1;2) \), \( l(2;4) \). Similarly \( c_1 \) must lie outside \( C_4 \) since \( l(1;4) \), \( l(1;2) \), \( l(2;4) \). Moreover since the angle sum at any cone point is \( \angle c_3 t_2 + \angle c_3 t_3 < \). In Figure 6 we have also constructed the point \( x \) as

\[\text{Geometry and Topology Monographs, Volume 1 (1998)}\]
the intersection of the radius through \( c_3 \) and \( C_4 \). Let \( t_x \) denote the triangle spanning \( x; c_3; C_4 \).

Now \( \angle c_3 t_2 \) is equivalent to \( \angle c_3 t_x \). It follows that \( \angle c_4 t_x \neq \angle c_3 t_x \). By inspection \( \angle c_4 t_3 > \angle c_4 t_x \) and \( \angle c_3 t_x > \angle c_3 t_3 \). So \( \angle c_4 t_3 > \angle c_4 t_x \neq \angle c_3 t_3 \) or equivalently \( l(0,4) < l(0,3) \).

So \( \angle c_3 t_2 < \) and we will compare \( t_2; t_0 \). Firstly, \( \angle c_3 t_2 < \) implies that \( l(3,4) \neq l(3,4) \). (Recall that \( "3,4" \) is an edge of \( l \).) Let \( l_0 \) be an isosceles triangle with vertices \( v_0; v_1; v_4 \) and edges \("0,1;"1,4;"0,4\) such that \( l(3,4) = l(0,4) = l(2,4) \) and \( \angle v_4 l_0 = \angle C_4 t_0 \). Since \( l(0,4); l(1,4); l(1,2); l(2,4) \) we then observe that \( l(0,1) \neq l(0,1) \). As \( \angle c_3 t_2 \neq \angle c_4 t_0 \) we have that \( l(3,4) \neq l(3,4) \).

Therefore \( l(0,1) \neq l(0,1) \). We have equality if and only if \( \angle c_2 t_2 = \angle c_4 t_0 \) and \( l(2,3) = l(2,4) = l(0,4) = l(2,4) \). From above \( \angle c_2 t_2 = \angle c_4 t_0 \) if and only if \( l(1,2) = l(1,2); l(3,0) = l(0,1) \) and \( l(2,3) = l(2,3) \). So we have that \( l(0,1) = l(3,4) \) and \( l(1,2) = l(2,3) = l(3,0) = l(0,1) \).

That is: \( t_1; T_1 \) are isometric equilateral triangles and \( t_0; T_0; t_2; t_3 \) (respectively \( T_2; T_3 \)) are isometric isosceles triangles. By considering angle sums at \( c_4; c_5 : \angle c_4 t_2 = \angle c_4 t_3 = \angle c_5 t_2 = \angle c_5 t_3 \). So: \( t_1; T_1 \) are isometric equilateral triangles and \( t_0; T_0; t_2; t_3 ; T_2; T_3 \) are isometric isosceles triangles. By the angle sum at \( c_3 : \angle c_3 t_2 = \angle c_3 t_3 = \angle c_3 t_2 = \angle c_3 t_3 \) = \( 4 \) and so \( \angle c_3 t_0 = \angle c_3 t_0 = \angle c_3 T_0 = \angle c_3 T_0 = \angle 4 \). Again, by considering angle sums at \( c_0; c_1 \) all the angles are \( =4 \), all of the edges are of equal length. So \( O \) is the octahedral orbifold.

\[ \square \]

Figure 7: Arguments for \( \angle c_3 t_2 \) and for \( \angle c_3 t_2 \)

References


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