The boundary of the deformation space of the fundamental group of some hyperbolic 3-manifolds fibering over the circle

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Abstract By using Thurston's bending construction we obtain a sequence of faithful discrete representations \( \rho_n \) of the fundamental group \( \pi_1 \) of a closed hyperbolic 3-manifold fibering over the circle into the isometry group \( \text{Iso} \mathbb{H}^4 \) of the hyperbolic space \( \mathbb{H}^4 \). The algebraic limit of \( \rho_n \) contains a finitely generated subgroup \( F \) whose 3-dimensional quotient \( \Omega(F) = F/\pi_1 \) is a quotient group, where \( \Omega(F) \) is the discontinuity domain of \( F \) acting on the sphere at infinity \( S^3 = \partial \mathbb{H}^4 \). Moreover \( F \) is isomorphic to the fundamental group of a closed surface and contains infinitely many conjugacy classes of maximal parabolic subgroups.

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1 Introduction and statement of results

By a Kleinian (discontinuous) group \( G \) we mean a subgroup of the group \( \text{Conf}(\mathbb{S}^n) = SO^+ (1; n+1) \) of conformal transformations of \( \mathbb{R}^n = \mathbb{S}^n = \mathbb{R}^n \) which acts discontinuously on a non-empty set \( \Omega(G) = \mathbb{S}^n \) called its domain of discontinuity. It may be connected or not; we will say that \( G \) is a function group if there is a connected component \( \Omega_G \subseteq \Omega(G) \) that is invariant under the action of the whole group: \( G \Omega_G = \Omega_G \). The quotient spaces \( M_G = \Omega_G / G \) and \( M(G) = \Omega(G) / G \) are n-manifolds in the case in which \( G \) is torsion-free. The complement \( \Omega(G) = \mathbb{S}^n \cup \Omega(G) \subseteq \mathbb{H}^{n+1} \) is the limit set of \( G \).

A finitely generated Kleinian group \( G \) is called geometrically finite if for some \( \epsilon > 0 \) there exists an \( \epsilon \) neighbourhood of \( H_G = G \) in \( \mathbb{H}^{n+1} \) which is of finite hyperbolic volume. Here \( H_G \subseteq \mathbb{H}^{n+1} \) is the convex hull of \( \Omega(G) \).

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Let us consider for \( n = 3 \) a hyperbolic 3-manifold \( M = \mathbb{H}^3 = \Gamma \backslash \text{PSL}_2 \mathbb{C} \) fibered over the circle \( S^1 \) with fiber a closed surface \( S \). The notation is \( M = \sim S^1 \). A representation \( \rho : \Gamma \to \text{Conf}(S^3) \) is called admissible if the following conditions are satisfied.

1. \( \rho : \Gamma \to \text{Conf}(S^3) \) is faithful and \( \rho(\Gamma) = \Gamma_0 \) is Kleinian.
2. \( \rho \) preserves the type of each element, i.e., \( \rho(\gamma) \) is loxodromic for all \( \gamma \in \Gamma \).
3. \( \rho \) is induced by a homeomorphism \( f : \Omega(\Gamma) \to \Omega(\Gamma_0) \), namely \( f \gamma f^{-1} = (\gamma), \gamma \in \Gamma \).

The set of all admissible representations modulo conjugation in \( \text{Conf}(S^3) \) is called the deformation space \( \text{Def}(\Gamma) \) of the group \( \Gamma \).

The set \( \text{Def}(\Gamma) \) inherits the topology of convergence on generators of \( \Gamma \) on compact subsets in \( S^3 \) because \( \text{Def}(\Gamma) \cap \text{Conf}(S^3) = k \in \mathbb{N} \) (is conjugation in \( \text{Conf}(S^3) \)). As \( \text{Def}(\Gamma) \) is a bounded domain [13] two questions have arisen. The first is to describe the cases when \( \text{Def}(\Gamma) \) is non-trivial and the second is to study the boundary \( @\text{Def}(\Gamma) \), as was done for the classical Teichmüller space [2], [10]. The answer to the first question is still unknown even in the case when \( M \) is Haken. We will consider the case when \( M \) contains many totally geodesic surfaces. Each of them produces a curve in \( \text{Def}(\Gamma) \) by Thurston's "bending" construction [19]. Our main interest is in groups which appear on the boundary \( @\text{Def}(\Gamma) \). These are higher dimensional analogs of \( \text{B} \) (groups which arise as the limits of sequences of quasifuchsian groups in classical Teichmüller space.

One of the most fundamental questions is to describe the topological type of the orbifold \( M(\Gamma) = \Omega(\Gamma) = \mathbb{H}^3 = \Gamma \) (a manifold in the case when \( \Gamma \) is torsion-free), in particular, when \( \Gamma \) is a function group it is important to know when the fundamental group \( \pi_1(M_G = \Omega_\Gamma = \mathbb{H}^3) \) turns out to be finitely generated, or even more generally when it has finite homotopy type.

In dimension 2 the famous theorem of Ahlfors [1] says that a finitely generated non-elementary Kleinian group \( G \) \( \text{Conf}(\mathbb{H}^2) \) has a factor-space \( \Omega(G) = G \) consisting of a finite number of Riemann surfaces \( S_1; \ldots; S_n \) each having a finite hyperbolic area.

We discovered in [7] that the weakest topological version of Ahlfors' theorem does not hold starting already with dimension 3. Namely we constructed a finitely generated function group \( F \) \( \text{Conf}(S^3) \) such that the group \( \pi_1(\Omega_F = \mathbb{H}^3) \) is not finitely generated. Afterwards it was pointed out in [15] that this group is in fact not finitely presented.

It has also been shown that there exists a finitely generated Kleinian group with in finitely many conjugacy classes of parabolics [6].
In [14] we constructed a finitely generated group $F_1$ such that $1(\Omega_{F_1} = F_1)$ is not finitely generated and having in finitely many non-conjugate elliptic elements; moreover $F_1$ appears as an in finitely presented subgroup of a geometrically finite Kleinian group in $\mathbf{H}^4$ without parabolic elements. On the other hand, it was shown in [4] that a finitely generated but in finitely presented group can also appear as a subgroup of a cocompact group in $\text{SO}(1; 4)$.

**Theorem 1** Let $\Gamma = _1(M)$ be the fundamental group of a hyperbolic 3-manifold $M$ being over the circle with fibre a closed surface $S$. Suppose that $\Gamma$ is commensurable with the reflection group $R$ determined by the faces of a right-angular polyhedron $D \subset \mathbf{H}^3$. Then there exists a finite-index subgroup $L \subset \Gamma$ and a path $t : [0; 1] \rightarrow \text{Def}(\Gamma)$ such that $t$ converges to a faithful representation $\rho : _1(\Gamma) \rightarrow \text{Def}(\Gamma)$ (as $t \rightarrow 1$) and the following hold:

1. $1(F_L)$ contains in finitely many conjugacy classes of maximal parabolic subgroups,
2. $1(\Omega_{F_L}) = 1(F_L)$ is in finitely generated,

where $F_L = L \setminus 1$ is isomorphic to the fundamental group of a closed hyperbolic surface which finitely covers $S$ and $1(F_L)$ acts discontinuously on an invariant component $\Omega_{F_L}$ of $S^3$.

**Remark** Groups satisfying all the conditions of Theorem 1 do exist. An example of Thurston, of the reflection group in the faces of the right-angular dodecahedron, which is commensurable with a group of a closed surface bundle, is given in [18].

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## 2 Outline of the proof

Before giving a formal proof of the Theorem let us describe it informally.

Our construction is inspired essentially by papers [6], [8] and [14]. In the first two a free Kleinian group of finite rank satisfying the conclusion (2) was produced, whereas now we give an example of a closed surface group with this property. Our present construction is essentially easier than that of [14]. Also, we produce a curve in the deformation space whose limit point is the group in question.
Step 1 We start with an uniform lattice $\Gamma \subset \text{PSL}_2 \mathbb{C}$ commensurable with the reflection group $R$ whose limit set is the Euclidean $2$-sphere $\partial B_1 \subset S^3$. There exists a Fuchsian subgroup $H_2$ of $\Gamma$ leaving invariant a vertical plane whose intersection with $B_1$ is a round circle, its limit set $(H_2)$ (see figure 1). The group $H_2$ also leaves invariant a geodesic plane $w_2 \subset B_1$. Consider the action of the group $\Gamma$ in the outside ball $B_1 = S^3 \setminus \overline{B}_1$. For some finite-index subgroup $\Gamma_1$ of $\Gamma$ we construct a new group $G_1$ obtained by Maskit's Combination theorem from $\Gamma_1$ and $G_1$ combined along the common subgroup $H_2 = \text{Stab}(w_2)$, where $w_2$ is the reflection in $w_2$. The new group $G_1$ is still isomorphic to some subgroup $G$ of $\Gamma$ of finite index essentially because the same construction can be done inside $B_1$ by reflecting the picture along the geodesic plane $w_2$. Thus $G_1$ belongs to the deformation space $\text{Def}(G_1)$. One can obtain a fundamental domain $\mathcal{R}(G_1) \subset B_1$ of $G_1$ which is situated in a small neighbourhood of the spheres $\partial B_1$.

Step 2 There is another geodesic plane $w_1 \subset B_1$ disjoint from $w_2$ whose stabilizer in $\Gamma_1$ is $H_1$ (see figure 2). Denote by $B_2$ the ball $(B_1)$. Take a sphere $B_1$ passing through the circle $w_3 \setminus B_2$ of the limit set of the group $H_1$ and tangent to the isometric spheres of some element $g_2 \in \Gamma_1$, where $H_1$ is a subgroup of $\Gamma_1$ stabilizing $w_1$. We now construct a family of Euclidean spheres $t \in [0;1]$ and corresponding groups $G_t$ obtained as before from $G_1$ and $G_1$ by using the combination method along common closed surface subgroups. We prove then that there is a path $t: t \in [0;1] \rightarrow \text{Def}(L^0)$ such that $L^0 = L_0$, $t = G_t$ where $L^0$ is some finite-index subgroup of $R$. One can equally say that $G_t$ is obtained by using Thurston's bending deformation. The main point is now to prove that the limit
group $G_1 = \lim_{t \to 1} (L^0)$ is discontinuous and has a fundamental domain obtained from the part of $R\left(G_1\right)$ by doubling along the sphere. The group $G_1$ is also isomorphic to $L^0$ and so contains a fundamental group $N$ of a closed surface bundle over the circle, which is isomorphic to the group $L = \Gamma \setminus L^0$. Let $F$ be the fundamental group of the fiber given by $\Gamma \setminus L^0$. Since two isometric spheres of the element $g_1$ in $\Gamma_1$ are tangent to each other, we get a new accidental parabolic element $g = g_1 g_2 g_1^{-1} g_2$ in the group $G_1$. By a choice of $g_1$ made from the very beginning we assure that $g \not\in F$, so we have a pseudo-Anosov action of some element $t \not\in N \setminus F$ such that the orbit $t^n g t^{-n} (n \in \mathbb{Z})$ gives us infinitely many conjugacy classes of maximal parabolic subgroups of $F$. Now Scott's compact core theorem implies that $\Omega_F = F$ is not finitely generated.

End of outline

3 Preliminaries

We will consider the Poincare model of hyperbolic space $H^3$ in the unit ball $B_1$ equipped with the hyperbolic metric. By a right-angled polyhedron $D \subset H^3$ we mean a polyhedron all of whose dihedral angles are $\pi/2$.
Consider the tessellation of $H^3$ by images of $D$ under the reflection group $R$ from Theorem 1. Denote by $W = H^3$ the collection of geodesic planes $w$ such that there exists $r \in R$, for which $r(w) \cap \triangle$ is a face of $D$.

It is easy to see that if $1$ and $2$ are two faces of $D$ with $1 \cap 2 = \emptyset$, then also the geodesic planes $\sim_1$ and $\sim_2$ have no point in common. One can easily show that the distance between $1$ and $2$, as well as that of $\sim_1$ and $\sim_2$, is realized by a common perpendicular for which $\sim \cap \triangle = \emptyset$.

Let $\Gamma_0 = R \setminus \Gamma$ which is a subgroup of a finite index in both groups $R$ and $\Gamma$. By passing to a subgroup of a finite index and preserving notation, we may assume that $\Gamma_0$ is a normal subgroup in $R$; $|R : \Gamma_0| < 1$. For a plane $w \in W$ we write $H_w = \text{Stab}(w; \Gamma_0) = f g 2 \Gamma_0$; $gw = wg$. It is not hard to see that $H_w$ is a Fuchsian group of the first kind commensurable with the reflection group determined by the edges of some face of the polyhedron $r(D_1); r \in R$.

Let us now take two disjoint planes $w_1$ and $w_2$ from $W$ containing opposite faces of $D$ and let $\bot$ be their common perpendicular; up to conjugation in $\text{Isom} H^3$ we can assume that $\bot$ is a Euclidean diameter of $B_1$. Denote $B_1 = S^3 \text{nd}(B_1)$ as well (where $\text{cl}(\cdot)$ is the closure of a set). We have the following:

**Lemma 1.** For every horosphere $\sim_3$ in $B_1$ centered at the point $2 \bot \triangle$ (see figure 1) there exists $\sim_0 > 0$ such that for every "close sphere $\sim_1$ in $B_1$ to $\sim_3$ (" $\sim_0")$ orthogonal to the plane $2$ there exists a geodesic plane $w$ and an element $g_1 2 [H_w; H_w]$ (commutator subgroup) such that:

$$l_{g_1} \sim_1 \in \triangle; \quad \text{and} \quad g_1(l_{g_1} \sim_1) = l_{g_1} l_{g_1} \sim_1;$$

where $l_{g_1} l_{g_1} = l_{g_1}^{-1}$ are isometric spheres of $g_1$.

**Proof.** Up to further conjugation in $\text{Isom} B_1$ preserving $\bot$, we may assume that $\sim_3$ is the vertical plane tangent to $\triangle$ at $2 \bot \triangle$. Take $w = w_1$ and let $g_1 2 [H_{w_1}; H_{w_1}]$ be any primitive element corresponding to a simple dividing loop on the surface $w_1 = H_{w_1}$.

Suppose first that $l_{g_1} \sim_3 = \emptyset$. In this case we proceed as follows. Put $w = w_1 \in W$, where $w_i$ denotes the reflection in plane $w_i$ (i = 1, 2). Then $l_{g_1}$ is a hyperbolic element whose invariant axis is $\sim$. Consider the sequence of planes $\sim(w_{g_1})$. We claim that, for some $n$, $\sim(l_{g_1}) \sim_3 \in \triangle$. In fact this follows directly from the fact that the this point of the hyperbolic element is a conical limit point of $\Gamma_0$, and so the approximating sequence $\sim(n l_{g_1})$ should intersect a fixed horosphere (or equivalently by sending to the infinity and passing to the half-space model one can see that becomes now a dilation $z \sim z (\sim > 0)$ which implies that the translations of the image of $l_{g_1}$ by
We can suppose that \( w \) is normal in \( \Gamma_0 \). Since \( \Gamma_0 \) is normal in \( \Gamma \) it now follows that \( n g_1^{-1} n^{-1} 2 [H \backslash n(w_1); H \backslash n(w_1)] \). \( \Gamma_0 \) and \( n(I_g) = I_n g_1^{-1} \). The latter is true since \( g_1 \) preserves each Euclidean plane passing through \( B_1 \). And, hence \( (n g_1^{-1} n^{−1}) j n(I_g) \) is an Euclidean isometry. So up to replacing \( w_1 \) by \( n(w_1) \) and \( g_1 \) by \( n g_1^{-1} \) if needed, we may assume that \( I_{g_1} \). The same conclusion is then obviously true for a plane \( 1 \) \( B_1 \) sufficiently close to \( 3 \).

For \( '1 = I_{g_1} \backslash 1 \) we now claim that \( g_1('1) = '2 = H^0 g_1 \backslash 1. \) Indeed, \( g_1 = 1 g_1 \) where \( 2 \) is orthogonal to \( 1 \) and contains \( ' \) (Figure 1). Evidently

\[
g_1('1) = 1 (I_{g_1} \backslash 1) = 1 (I_{g_1} \backslash 1) = H^0 g_1 \backslash 1 \quad (2)
\]

since \( 1(1) = 1. \) The lemma is proved. \( \Box \)

So we can suppose that \( w_1 = W \) is chosen satisfying all the conclusions of Lemma 1. Let \( w_2 = W \) be a geodesic plane disjoint from \( w_1 \) and let \( ' \) be their common perpendicular passing through the origin of \( B_1 \). Now consider the Euclidean plane orthogonal to \( ' \) (Figure 2) such that

\[
\backslash \backslash \backslash \backslash \backslash _{B_1} = \backslash \backslash \backslash \backslash \backslash _W = \backslash \backslash \backslash \backslash \backslash _W = \backslash \backslash \backslash \backslash \backslash _W = \backslash \backslash \backslash \backslash \backslash _W
\]

It is not hard to see that \( \text{Stab} (1; \Gamma) = \text{Stab} (w_2; \Gamma) = H_{w_2} \). Reflecting our picture in the plane \( w_2 \) we get

\[
B_2 = \langle B_1 \rangle; \quad w_2 = \langle w_2 \rangle \quad \text{and}
H_{w_3} = H_{w_1} \quad (\text{Figure 2})
\]

By Lemma 1 we can now find a Euclidean sphere centered on \( ' \) which goes through the circle \( \backslash \backslash \backslash \backslash _{B_2} \) and is tangent to \( I_{g_1} \) (Figure 2). Moreover, by Lemma 1, \( 1 = 1 \) is tangent also to \( I_{g_1}^0 \).

Denote \( 1 = 1. \) The lemma is proved. \( \Box \)

**Lemma 2** There exists a subgroup \( \Gamma_1 \) of \( \Gamma_0 \) of finite index such that the following conditions hold:

(a) The boundary of the isometric fundamental domain \( P(\Gamma_1) = B_1 \) lies in a regular \( ' \) (neighbourhood of \( \backslash \backslash \backslash \backslash _B \) \( B_1 = S^2 \backslash \text{nd}(B_1); ' > 0 \).

(b) \( \langle I_γ = \rangle ; \quad γ 2 Γ_1 \} g_1 \; g_1^{-1} \).

(c) For subgroups \( H_1 = Γ_1 \backslash H_{w_1}; H_2 = Γ_1 \backslash H_{w_2} \) there exists another fundamental domain \( R(Γ_1) \backslash \backslash _B \) of \( Γ_1 \) such that

\[
R(Γ_1) \backslash \langle [0] \rangle = P(\Gamma) \backslash \langle [0] \rangle;
\]

where \( P(\Gamma) \) is an isometric fundamental domain for the group \( \Gamma = \langle H_1; H_2 i \rangle \).

(d) \( g_1 2 Γ_1 \backslash [H_1; H_1]. \)
Proof This Lemma can be obtained by repeating the arguments of [14, Main Lemma]. We just sketch these considerations. First, we choose a subgroup $G_0$ of a finite index satisfying conditions (a) and (b) such that $g_1 \in G_0$ by using the property of separability of infinite cyclic subgroups in $G_0$.

To obtain (c) we will intend $\Gamma_1$ by using Scott’s LERF property of the group $\Gamma_0$ with respect to its geometrically finite subgroups (see [16], [17]). To this end we proceed as follows: the group $H$ is geometrically finite as a result of Klein{Maskit free combination from $H_1$ and $H_2$, which are both geometrically finite subgroups of $\Gamma_0$. The LERF property now says that for the element $g_1, g_1$ there exists a subgroup of $\Gamma_0$ of finite index which contains $H$ and does not contain $g_1$. Call this subgroup $\Gamma_1$. Evidently, $g_1 \in [H_1; H_1] \Gamma_1$ by construction. For the complete proof, see [14, Main Lemma].

Let us introduce the following notation: $\Omega_1^\tau = B_1 \cap \gamma_2 \Gamma_1$. $\gamma(-)$ is the component of $S^3 \setminus$ for which $w_2 = -$. Let $\Gamma_1^0 = \text{Stab}(\Omega; \Gamma_1)$.

The complete proof of the following assertion can be also found in [14, Lemma 3].

Lemma 3 The group $G_1 = H_1^0$; $\Gamma_1^0$ i is discontinuous and

1. $G_1 = \Gamma_1^0 H_2$ ( $\Gamma_1^0$ ).

2. $G_1$ is isomorphic to a subgroup $G_1 \quad R$ of infinite index.

Sketch of proof (1) This follows from the fact that the plane $\Omega_1^\tau$ is strongly invariant under $H_2$ in $\Gamma_1^0$ by [14, Lemma 3.c], which means $H_2 = \quad$ and $\gamma_2 \Gamma_1^0 H_2$. One can now get assertion (1) from Maskit’s First Combination theorem [11].

(2) Consider the reflection $w_2$ in the geodesic plane $w_2 \quad B_1$. We claim that the group $G_1 = H_1^0$; $w_2 \Gamma_1^0$ i is isomorphic to $G_1$. Indeed, $w_2$ is also strongly invariant under $H_2$ in $\Gamma_1^0$ and we again observe that $G_1 = \Gamma_1^0 H_2 (w_2 \Gamma_1^0 w_2) = G_1$ because $w_2 j w_2 = \quad j = \text{id}$.

Now $w_2 \quad R$. Therefore, $G_1 \quad R$ and $G_1$ has a compact fundamental domain $R(G_1) = R(\Gamma_1^0) \setminus w_2 (R(\Gamma_1^0))$. The covering $H^3 (G_1 \quad \Gamma_0)$ is finite since $jR : \Gamma_0 | 1$ and, hence, the manifold $M (G_1 \quad \Gamma_0) = H^3 (G_1 \quad \Gamma_0)$ is compact. Thus, the covering $M (G_1 \quad \Gamma_0)$ is finite as well and so $jG_0 : G_1 \quad \Gamma_0 < 1$.

Corollary 4 There exists a path $t : [0; 1]$ ! Def $(G_1)$ such that $0 = G_1$ and $1 = G_1$. 

Deformation space of hyperbolic 3-manifolds

Let us consider now the family of spheres \( w_1 = w_2 = (H_2); \quad w_2; \quad t : \quad t \cdot [0; 1] \), we construct the family of groups \( G_t = f_t^0 \); \( f_t^0 \), by the arguments of Lemma 3. Consider now the action of \( t^0 \) in \( B_1 \), where \( p_1 : B_1 \to B_1 \) is the covering map. The surfaces \( p_1( t^0 ) \) are all embedded and parallel due to condition (b). If now \( \Omega^0 = \Omega^1 \) is the component of \( G_t \) containing 1, then the manifold \( M_{G_t} = \Omega^1 \cdot G_t \) is \( \Omega^1 \) along the boundary \( p_1( t^0 ) \). Thus, for all \( t \cdot [0; 1] \), \( M_t \) are all homeomorphic and there exists a continuous family of homeomorphisms \( f_t : \Omega^1 \to \Omega^1(G_t) \) such that \( G_t = f_t(G_t), G_1 = f_1(G_t) \).

By construction the domain \( R(G_t) = R(G_t) \to \Omega^1(G_t) \) is fundamental for the action of \( G_t \) in \( \Omega^1(G_t) \).

Claim 5 \( R(G_t) \) is fundamental for the action of \( G_t \) in \( \Omega^1(G_t) \).

Proof Recall that \( t(t) \) means the right (left) component of \( S^3 \) in \( (I_t \cdot 2 \cdot 2) \). Then \( + \to - \to R(G_t) \) is fundamental for the action of \( G_t \) in \( \Omega^1(G_t) \).

Also, \( \Omega^1(G_t) = \Omega^1(G_t) \to R(G_t) \) is fundamental for the action of \( G_t \) in \( \Omega^1(G_t) \).

Let us consider now the family of spheres \( t \cdot w = w_3 \cdot w_1 \cdot w_3; \quad t = 1 \cdot 0 \cdot 1 \cdot 0; \quad t \cdot [0; 1] \), where \( t \cdot \text{ext}(B_1) \setminus \text{ext}(B_2) \to \text{ext}(B_1) \cdot \text{ext}(B_2) \to \text{ext}(B_2) \) (recall \( \text{ext}(t) \) is the exterior of a set in \( R^3 \) ), \( p_t = 1 \cdot 0 \cdot 1 \cdot 0 \). Denote by \( t \cdot \text{ext}(t) \) the corresponding reflections. As before take the domain \( \Omega = \Omega^1 \cdot n(G_1(0)) \) and the group \( G_t^0 = \text{Stab}(\Omega; G_1) \), where \( o = \text{ext}(0) \) is the unbounded component of \( R^3 \) in \( 0 \).

Denote \( G_t = I_t^0 \); \( I_t^0 \); \( I_t^0 \). Evidently, \( G_t = \lim_{t \to 1} G_t^0 \).

Lemma 6 The groups \( G_t \) are discontinuous, \( t \cdot [0; 1] \).

Proof First, let us prove the lemma for \( t = 1 \). By Claim 5 we have now that \( R(G_t) \) is fundamental for the action of \( G_t \) in \( \Omega^1(G_t) \).

Moreover we claim also that

\[
\begin{align*}
\text{g} : \quad & t = 1, \quad g \cdot 2 \cdot G_1 \cdot 1 \cdot H_3, \quad H_3 = t, \quad \text{where} \quad H_3 = H_1.
\end{align*}
\]

To prove (3) we only need to show that \( g(1 \cdot (H_3)) \) is a point of approximation (see [14, Claim 1]).
All conditions of Maskit’s First Combination theorem are now satisfied for the groups $G_0$ and $G_0^i$ ($t \neq 1$) [11] and we obtain also

$$G_1 = G_1^0 \langle \gamma_1 \rangle$$

(4)

where the $G_1$ are all discontinuous, $t \in [0,1)$.

Let us now consider the group $G_1$ and the domain $R(G_1) = R(G_1) \setminus (R(G_1))$. Our goal now is to show that $R(G_1)$ is a fundamental domain for the action of $G_1$ in $\Omega G_1$ ($1 \leq \Omega G_1$). If now $h_1; g_1; \ldots; g_i$ is a set of generators of $G_0$ then $S = h_1; g_1; \ldots; g_i; g_1; g_1^0; \ldots; g_i^0$ are generators of $G_1$, where $g_i^0 = h_i$ and $g_1 = g_1_1$. Observe that the element $g_1$ is included in $S$ because some of its isometric spheres belong to the boundary $\partial R(G_1)$.

We want to apply the Poincare Polyhedron theorem [12]. Indeed, an arbitrary cycle of edges in $\partial R(G_1)$ consists either of edges situated in $\partial R(G_1) \setminus \text{int}(S)$, and $\partial (R(G_1)) \setminus \text{int}(S)$, or is an edge cycle $'1 = l_{g_1} \setminus l_{g_2}; '2 = l_{g_1} \setminus l_{g_2}$, where $l_{g_1}; l_{g_2}$ are the isometric spheres of $g_1$ and $g_1(k = 1; 2)$. The sum of angles in any cycle of the first type is $2\pi$ because $R(G_1)$ is a fundamental domain [12].

We now claim that the element $g = g_1^{-1}g_1$ is parabolic with a fixed point $d = l_{g_1} \setminus l_{g_2}$. Indeed, $g_1^{-1}g_1 = l_{g_1}^{-2}$ because $g_1 = l_{g_1}^{-1}l_{g_1}$ and $2$ is orthogonal to (figure 2). Now it is easy to check that $g(d) = d$, $g(l_{g_1}) = \text{int}(l_{g_1})$ and $g(\text{int}(l_{g_1})) = \text{ext}(g(l_{g_1}))$, therefore the elements $g$ and $g^0 = g_1g_1^{-1}$ are parabolics.

All conditions of the Maskit-Poincare theorem are valid at the edges ’1’ also and, hence, $G_1$ is discontinuous. Lemma 6 is proved. □

**Lemma 7** The group $G_3$ is isomorphic to a subgroup $L^0 \subset R$ of a finite index.

**Proof** We repeat our construction of $G_0$ by modelling it in $\mathbb{H}^3$ so as to get the required isomorphism.

Recall that we started from the group $\Gamma_1^0 \subset \text{Isom}(\mathbb{H}^3)$ and showed that $G_1 = H_1^0; \quad \Gamma_1^0 \cap G_1 = H_1^0; \quad w_1 \Gamma_1^0 w_1$ (see Lemma 4). Next we constructed $G_0$ by using reflection in $0 = 0$ such that $0 \setminus w_3 = (H_3); \quad 0 \setminus B_1 = \vdots; \quad w_3 = w_1$.

Let $w_2 = w_2(w_1) \in \mathbb{H}^3$; then $2 = W$. Again let us take the subgroup $G_1$ of $G_1$ which is $G_1 = \text{Stab}(\mathbb{H}^3 \cap G_1)$, where $\cap$ is a subspace $\mathbb{H}^3 \cap$ not containing $w_2$. 

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By construction the fundamental domain \( R(G_1) = R(\Gamma_0) \setminus w_2(R(\Gamma_0)) \) of the group \( G_1 \) satisfies \( R(G_1) \setminus = \text{P}(H_3^0 = \text{Stab}(\Gamma_1)) \). Again by Maskit’s First Combination Theorem we have a group \( L^0: \)

\[
L^0 = G_1 \text{ Stab}(\Gamma_1)
\]

(5)

We constructed an isomorphism \( \Gamma_0 \) of \( G_1 \) in Lemma 4 such that \( \Gamma_1 = \text{Stab}(\Gamma_0) \). It follows now from (4) and (5) that the map \( \Gamma_1 \) can be extended to an isomorphism \( L^0 ! \to G_0 \).

Index \( \text{[ } R : L^0 \text{]} \) is finite because \( L^0 \) has a compact fundamental domain. The Lemma is proved.

Recall that we identify \([ ]2 \text{ Def}(L^0) \) with \( (L^0) \).

**Lemma 8** There exists a path \( t : [0; 1] \to \text{cl}(\text{Def}(L^0)) \) such that \( 0 = L^0, 1 = G_2 \) @\( \text{Def}(L^0), \quad t([0; 1]) = \text{Def}(L^0) \).

**Proof** We have constructed a path \( t : [0; 1] \to \text{Def}(G_1) \) in Corollary 4 such that \( 0 = G_1, 1 = G_2 \) and \( t \) is a family of admissible representations. Let further \( t \Gamma_1 = \emptyset \). Obviously, the representations \( t \Gamma_0 \) are also admissible and \( t \Gamma_0 \) is a family of admissible representations \( t : L^0 ! \to \text{Def}(L^0) \).

We can easily extend our family \( t \) to a family of admissible representations \( t : L^0 ! \to \text{Def}(L^0) \) by the formula \( t = t \Gamma_0 \), where \( t \) are the spheres constructed in Corollary 4.

Observe that \( t \) and now take a new continuous family of spheres \( t \) for which \( t \setminus w_2 = (H_{\delta}) = w_3 \setminus B_2 \) and \( 1 = w_3; 2 = 0 \) where \( w_3 \) is the sphere containing \( w_2 (t \in [0; 1]) \).

Again we have a path \( t \Lambda (L^0) = hG_0^i; t \Gamma_0 ^i \). Composing the path \( t \) with \( t \) and with the path corresponding to spheres \( t \) connecting \( 0 \) with \( 1 \) we get required path \( t \). The Lemma is proved.

## 4 Proof of Theorem 1

(1) Denote by \( F = G_1 \) a Bieber group of our initial manifold \( M \), and let also \( F_0 = \Gamma_0 \setminus F \).

By J. Jorgensen’s theorem [5] the limit \( \lim t \Gamma_1 = \lim t \Gamma_0 \) is an isomorphism \( t : L^0 ! \to G_1 \). Let us consider the subgroup \( L = L^0 \setminus \Gamma_0 \); \( j \Gamma_0 : L \gamma < 1 \). Put also \( F_L = L \setminus F_0 \) for its normal subgroup. We have also the curve \( t(L) \to \text{Def}(L) \). Let \( N = t(L); F = t(F_L) \). Let us show that \( g = g^{-1} \gamma \) for \( g \in F \). To this

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end let us recall that the element $g_1$ was chosen from the very beginning being in $[H_{w_1};H_{w_1}]$ (Lemma 1). Recalling also that $^{-1}(g_1) = g_2$, by construction we get $g_2 = g_1$; $w_2(w_1); g_1 2 [H_{w_1};H_{w_1}] [F_0;F_0]$ (see Lemma 1). The group $\Gamma_0$ was chosen to be normal in the reflection group $R$, and since $[\Gamma_0;\Gamma_0] = F$, it is straightforward to see that

\[
\text{r} [F_0;F_0] \text{r}^{-1} F_0; \text{r} 2 R:
\]

Hence, $g_2 2 F_0$, and for the element $g_0 = (g_2)^{-1}$ we immediately obtain $g_0 2 F_0 = F_0 \setminus L^0$. It follows that $^{1}(g_0) = g = g_2^{-1} g_1 2 F_0 \setminus \Gamma_1 = F$ as was promised.

We have that $N$ is isomorphic to the semi-direct product of $F$ and the infinite cyclic group $Z$, so taking the element $t^2 N n F$ projecting to the generator of $N = F$, we observe that the elements

\[
g_1 = t^n g t^{-n} 2 F; \quad g 2 F; \quad n 2 Z
\]

are all parabolics. Since $N$ contains no abelian subgroups of rank bigger than 1 and $t^2 F F$ (n 2 Z) one can easily see that the elements (6) are also non-conjugate in $F$. We have proved (1) of the Theorem.

(2) By the construction, the fundamental polyhedron $R(G_1)$ of the group $G_1$ contains only one conjugacy class of parabolic elements $g$ of rank 1. There is a strongly invariant cusp neighborhood $B_g = [0;1] \setminus (0;1)$ which comes from the construction of $R(G_1)$. So each parabolic $g_i$ of type (6) gives rise to a submanifold

\[
B_{g_i} \cap T_1 = T_n \cap [0;1); \quad T_n = S^1 \setminus S^1
\]

in the manifold $M(F) = \Omega_{\mathbb{Z}} F$. Therefore $M(F)$ contains in nitely many parabolic ends (7) bounded by tori $T_n$. They all are non-parallel in $M(F)$ and therefore by Scott's \textquoteleft core\textquoteright theorem the group $\overline{1}(M(F))$ is not nitely generated [16].

Remark By using the argument of [14] one can prove:

**Theorem 2** There is a (non-faithful) representation $\overline{1+}$ which is "close to $\overline{1}$ for some small $\epsilon > 0"$ such that the group $\overline{1+}(F_L)$ is in nitely generated, has in nitely many non-conjugate elliptic elements. Moreover, $\overline{1+}(F_L)$ is a normal in nitely presented subgroup of a geometrically nitely group $\overline{1+}(L)$ without parabolics.

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To prove the theorem one can continue to deform the group for $1 < t < 1 + " (these representations will no longer be faithful) in order to get an elliptic element $g_t$ whose isometric spheres form an angle $(t)$ instead of being tangent. To do this in our Lemma 2, instead of the sphere tangent to the isometric spheres of $g_1$, one needs to consider a nearby sphere $1 + "$ forming angle $(t)$ with them. If $(" = \frac{n}{20}$ and $n > 0$ is large enough the group $1 + " (F_L)$ is Kleinian, has in nitely many non-conjugate elliptic elements of the order $n$ (obtained as above as an orbit of $g_{1 + "}$ by a pseudo-Anosov automorphism of the $1 + " (F_L)$). The construction gives us that $1 + " (F_L)$ is a normal and nitely generated but in nitely presented subgroup of the geometrically nitely group $1 + " (L)$ without parabolic elements. In particular $1 + " (L)$ is a Gromov hyperbolic group (see [14, Lemmas 5{7]).

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