Holomorphic disks and genus bounds

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Abstract

We prove that, like the Seiberg-Witten monopole homology, the Heegaard Floer homology for a three-manifold determines its Thurston norm. As a consequence, we show that knot Floer homology detects the genus of a knot. This leads to new proofs of certain results previously obtained using Seiberg-Witten monopole Floer homology (in collaboration with Kronheimer and Mrowka). It also leads to a purely Morse-theoretic interpretation of the genus of a knot. The method of proof shows that the canonical element of Heegaard Floer homology associated to a weakly symplectically fillable contact structure is non-trivial. In particular, for certain three-manifolds, Heegaard Floer homology gives obstructions to the existence of taut foliations.

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1 Introduction

The purpose of this paper is to verify that the Heegaard Floer homology of \cite{27} determines the Thurston semi-norm of its underlying three-manifold. This further underlines the relationship between Heegaard Floer homology and Seiberg-Witten monopole Floer homology of \cite{16}, for which an analogous result has been established by Kronheimer and Mrowka, cf. \cite{18}.

Recall that Heegaard Floer homology $\widehat{HF}(Y)$ is a finitely generated, $\mathbb{Z}=\mathbb{Z}^\infty$ graded $\mathbb{Z}[\pi_1(Y)]$-module associated to a closed, oriented three-manifold $Y$. This group in turn admits a natural splitting indexed by Spin$^c$ structures $s$ over $Y$,

$$\widehat{HF}(Y) = \bigoplus_{s} \widehat{HF}(Y; s):$$

(We adopt here notation from \cite{27}; the hat signifies here the simplest variant of Heegaard Floer homology, while the underline signifies that we are using the construction with \textit{twisted coefficients}, cf. Section 8 of \cite{26}.)

The Thurston semi-norm \cite{39} on the two-dimensional homology of $Y$ is the function

$$\|\cdot\|: H_2(Y; \mathbb{Z}) \to \mathbb{Z}^0$$

de ned as follows. The complexity of a compact, oriented two-manifold $\chi$ is the sum over all the connected components $i$ with positive genus $g(i)$ of the quantity $2g(i) - 2$. The Thurston semi-norm of a homology class $2H_2(Y; \mathbb{Z})$ is the minimum complexity of any embedded representative of $\chi$.

(Thurston extends this function by linearity to a semi-norm $: H_2(Y; \mathbb{Q}) \to \mathbb{Q}$.)

Our result now is the following:

**Theorem 1.1** The Spin$^c$ structures $s$ over $Y$ for which the Heegaard Floer homology $\widehat{HF}(Y; s)$ is non-trivial determine the Thurston semi-norm on $Y$, in the sense that:

$$\|\cdot\| = \max_{s} \min_{i,j} jh\mathbf{c}_1(s); ij$$

for any $2H_2(Y; \mathbb{Z})$.

The above theorem has a consequence for the \textit{knot Floer homology} of \cite{31}, \cite{35}. For simplicity, we state this for the case of knots in $S^3$.

Recall that knot Floer homology is a bigraded Abelian group associated to an oriented knot \( K \subset S^3 \),

\[
\widetilde{HF}^*(K) = \bigoplus_{d \in \mathbb{Z}, s \in \mathbb{Z}} \mathbb{Z}.
\]

These groups are a refinement of the Alexander polynomial of \( K \), in the sense that

\[
\sum_s \mathbb{Z} \mathbb{Z} \widetilde{HF}^*(K; s) T^s = \kappa(T);
\]

where here \( T \) is a formal variable, \( \kappa(T) \) denotes the symmetrized Alexander polynomial of \( K \), and

\[
\mathbb{Z} \mathbb{Z} \widetilde{HF}^*(K; s) = \sum_{d \in \mathbb{Z}} (-1)^d \text{rk} \widetilde{HF}^d(K; s);
\]

(cf. Equation 1 of [31]). One consequence of the proof of Theorem 1.1 is the following quantitative sense in which \( \widetilde{HF}^* \) distinguishes the unknot:

**Theorem 1.2** Let \( K \subset S^3 \) be a knot, then the Seifert genus of \( K \) is the largest integer \( s \) for which the group \( \widetilde{HF}^*(K; s) \neq 0 \).

This result in turn leads to an alternate proof of a theorem proved jointly by Kronheimer, Mrowka, and us [19], first conjectured by Gordon [13] (the cases where \( p = 0 \) and \( p = 1 \) follow from theorems of Gabai [9] and Gordon and Luecke [14] respectively):

**Corollary 1.3** [19] Let \( K \subset S^3 \) be a knot with the property that for some integer \( p \), \( S^3_p(K) \) is diffeomorphic to \( S^3_p(U) \) (where here \( U \) is the unknot) under an orientation-preserving diffeomorphism, then \( K \) is the unknot.

The first ingredient in the proof of Theorem 1.1 is a theorem of Gabai [8] which expresses the minimal genus problem in terms of taut foliations. This result, together with a theorem of Eliashberg and Thurston [5] gives a reformulation in terms of certain symplectically semi-fillable contact structures. The final breakthrough which makes this paper possible is an embedding theorem of Eliashberg [3], see also [6] and [25], which shows that a symplectic semi-filling of a three-manifold can be embedded in a closed, symplectic four-manifold. From this, we then appeal to a theorem [34], which implies the non-vanishing of the Heegaard Floer homology of a three-manifold which separates a closed, symplectic four-manifold. This result, in turn, rests on the topological quantum field-theoretic properties of Heegaard Floer homology, together with the
suitable handle-decomposition of an arbitrary symplectic four-manifold induced from the Lefschetz pencils provided by Donaldson [2]. (The non-vanishing result from [34] is analogous to a non-vanishing theorem for the Seiberg-Witten invariants of symplectic manifolds proved by Taubes, cf. [36] and [37].)

1.1 Contact structures

In another direction, the strategy of proof for Theorem 1.1 shows that, just like its gauge-theoretic counterpart, the Seiberg-Witten monopole Floer homology, Heegaard Floer homology provides obstructions to the existence of weakly symplectically fillable contact structures on a given three-manifold, compare [17].

For simplicity, we restrict attention now to the case where $Y$ is a rational homology three-sphere, and hence $HF(Y) = \hat{HF}(Y)$. In [30], we constructed an invariant $c(\ ) \hat{H}F(Y)$, which we showed to be non-trivial for Stein fillable contact structures. In Section 4, we generalize this to the case of symplectically semi-fillable contact structures (see Theorem 4.2 for a precise statement). It is very interesting to see if this non-vanishing result can be generalized to the case of tight contact structures. (Of course, in the case where $b_1(Y) > 0$, a reasonable formulation of this question requires the use of twisted coefficients, cf. Section 4 below.)

In Section 4 we also prove a non-vanishing theorem using the “reduced Heegaard Floer homology” $HF^+_{\text{red}}(Y)$ (for the image of $c(\ )$ under a natural map $HF(Y) \to HF^+(Y)$), in the case where $b_1^+(W) > 0$ or $W$ is a weak symplectic semi-fillable contact structure with more than one boundary component. According to a result of Eliashberg and Thurston [5], a taut foliation $\mathcal{F}$ on $Y$ induces such a structure.

One consequence of this is an obstruction to the existence of such a filling (or taut foliation) for a certain class of three-manifolds $Y$. An L-space [29] is a rational homology three-sphere with the property that $HF(Y)$ is a free Z{module whose rank coincides with the number of elements in $H_1(Y;\mathbb{Z})$. Examples include all lens spaces, and indeed all Seifert fibered spaces with positive scalar curvature. More interesting examples are constructed as follows:

- If $K \subset S^3$ is a knot for which $S^3_r(K)$ is an L-space for some $r > 0$, then so is $S^3_r(K)$ for all rational $r > p$. A number of L-spaces are constructed in [29].

It is interesting to note the following theorem of Nemethi: a three-manifold $Y$ is an L-space which is obtained as a plumbing of spheres if and only if it is the link of a rational surface singularity [24]. L-spaces in the context of Seiberg-Witten monopole Floer homology are constructed in Section 3 of [19].

though the constructions there apply equally well in the context of Heegaard Floer homology).

The following theorem should be compared with [20], [25] and [19] (see also [21]):

**Theorem 1.4** An L-space $Y$ has no symplectic semi-fillings with disconnected boundary; and all its symplectic fillings have $b_2^+(W) = 0$. In particular, $Y$ admits no taut foliation.

### 1.2 Morse theory and minimal genus

Theorem 1.1 admits a reformulation which relates the minimal genus problem directly in terms of Morse theory on the underlying three-manifold. For simplicity, we state this in the case where $M$ is the complement of a knot $K \subset S^3$.

Fix a knot $K \subset S^3$. A perfect Morse function is said to be compatible with $K$ if $K$ is realized as a union of two of the flows which connect the index three and zero critical points (for some choice of generic Riemannian metric on $S^3$). Thus, the knot $K$ is specified by a Heegaard diagram for $S^3$, equipped with two distinguished points $w$ and $z$ where the knot $K$ meets the Heegaard surface. In this case, a simultaneous trajectory is a collection of gradient flowlines for the Morse function which connect all the remaining (index two and one) critical points of $f$. From the point of view of Heegaard diagrams, a simultaneous trajectory is an intersection point in the $g$-fold symmetric product of $\text{Sym}^g(\ )$, where $g$ is the genus of $\ )$ of two $g$-dimensional tori $\mathbb{T} = \{1 \cdots g\}$ and $\mathbb{T}' = \{1 \cdots g\}'$, where here $f_i g_{i=1}^g$ resp. $f_i g_{i=1}^g$ denote the attaching circles of the two handlebodies.

Let $X = X(f; )$ denote the set of simultaneous trajectories. Any two simultaneous trajectories differ by a one-cycle in the knot complement $M$ and hence, if we make an identification $H_1(M; \mathbb{Z}) = \mathbb{Z}$, we obtain a difference map

$$s : X \to \mathbb{Z} :$$

There is a unique map $s : X \to \mathbb{Z}$ with the properties that $s(x) - s(y) = (x; y)$ for all $x; y \in X$, and also $\# f x \ s(x) = i g \ (\mod 2)$ for all $i \in \mathbb{Z}$.

Although we will not need this here, it is worth pointing out that simultaneous trajectories can be viewed as a generalization of some very familiar objects from knot theory. To this end, note that a knot projection, together with a distinguished edge, induces in a natural way a compatible Heegaard diagram. The
simultaneous trajectories for this Heegaard diagram can be identified with the "Kauffman states" for the knot projection; see [15] for an account of Kauffman states, and [33] for their relationship with simultaneous trajectories.

The following is a corollary of Theorem 1.1.

**Corollary 1.5** The Seifert genus of a knot $K$ is the minimum over all compatible Heegaard diagrams for $K$ of the maximum of $s(x)$ over all the simultaneous trajectories.

It is very interesting to compare the above purely Morse-theoretic characterization of the Seifert genus with Kronheimer and Mrowka's purely differential-geometric characterization of the Thurston semi-norm on homology in terms of scalar curvature, arising from the Seiberg-Witten equations, cf. [18]. It would also be interesting to find a more elementary proof of the above result.

1.3 Remark

This paper completely avoids the machinery of gauge theory and the Seiberg-Witten equations. However, much of the general strategy adopted here is based on the proofs of analogous results in monopole Floer homology which were obtained by Kronheimer and Mrowka, cf. [18]. It is also worth pointing out that although the construction of Heegaard Floer homology is completely different from the construction of Seiberg-Witten monopole Floer homology, the invariants are conjectured to be isomorphic. (This conjecture should be viewed in the light of the celebrated theorem of Taubes relating the Seiberg-Witten invariants of closed symplectic manifolds with their Gromov-Witten invariants, cf. [38].)

1.4 Organization

We include some preliminaries on contact geometry in Section 2, and a quick review of Heegaard Floer homology in Section 3. In Section 4, we prove the non-vanishing results for symplectically semi-fillable contact structures (including Theorem 1.4). In Section 5 we turn to the proofs of Theorems 1.1 and 1.2 and Corollaries 1.3 and 1.5.
1.5 Acknowledgements

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2 Contact geometric preliminaries

The three-manifolds we consider in this paper will always be oriented and connected (unless specified otherwise). A contact structure is a nowhere integrable two-plane distribution in $TY$. The contact structures we consider in this paper will always be cooriented, and hence (since our three-manifolds are also oriented) the two-plane distributions are also oriented. Indeed, they can be described as the kernel of some smooth one-form with the property that its exterior derivative is a volume form for $Y$ (with respect to its given orientation). The form induces the orientation on $Y$.

A contact structure over $Y$ naturally gives rise to a Spin$^c$ structure, its canonical Spin$^c$ structure, written $\xi$, cf. [17]. Indeed, Spin$^c$ structures in dimension three can be viewed as equivalence classes of nowhere vanishing vector fields over $Y$, where two vector fields are considered equivalent if they are homotopic in the complement of a ball in $Y$, cf. [40], [12]. Dually, an oriented two-plane distribution gives rise to an equivalence class of nowhere vanishing vector fields (which are transverse to the distribution, and form a positive basis for $TY$). Now, the canonical Spin$^c$ structure of a contact structure is the Spin$^c$ structure associated to its two-plane distribution. The first Chern class of the canonical Spin$^c$ structure $\xi$ is the first Chern class of $\xi$, thought of now as a complex line bundle over $Y$.

Four-manifolds considered in this paper are also oriented. A symplectic four-manifold $(W; \omega)$ is a smooth four-manifold equipped with a smooth two-form $\omega$ satisfying $\omega^2 = 0$ and also the non-degeneracy condition that $\omega^\wedge$ is a volume form for $W$ (compatible with its given orientation).
Let $(W;\omega)$ be a compact, symplectic four-manifold $W$ with boundary $Y$. A four-manifold $W$ is said to have convex boundary if there is a contact structure over $Y$ with the property that the restriction of $\omega$ to the two-planes of is everywhere positive, cf. [4]. Indeed, if we fix the contact structure $Y$ over $Y$, we say that $W$ is a convex weak symplectic filling of $(Y;\omega)$. If $W$ is a convex weak symplectic filling of a possibly disconnected three-manifold $Y^0$ with contact structure $\xi^0$, and if $Y \cap Y^0$ is a connected subset with induced contact structure, then we say that $W$ is a convex, weak semi-filling of $(Y;\omega)$. Of course, if a symplectic four-manifold $W$ has boundary $Y$, equipped with a contact structure for which the restriction of $\omega$ is everywhere negative, we say that $W$ has concave boundary, and that $W$ is a concave weak symplectic filling of $Y$. (We use the term "weak" here to be consistent with the accepted terminology from contact geometry. We will, however, never use the notion of strong symplectic fillings in this paper.)

If a contact structure $(Y;\xi)$ admits a weak convex symplectic filling, it is called weakly fillable. Note that every contact structure $(Y;\xi)$ can be realized as the concave boundary of some symplectic four-manifold (cf. [7], [10], and [3]). This is one justification for dropping the modifier "convex" from the terminology "weakly fillable". If a contact structure $(Y;\xi)$ admits a weak symplectic semi-filling, then it is called weakly semi-fillable. According to a recent result of Eliashberg (cf. [3], restated in Theorem 4.1 below) any weakly semi-fillable contact structure is weakly fillable, as well.

A symplectic structure $(W;\omega)$ endows $W$ with a canonical Spin$^c$ structure, denoted $\mathfrak{k}(\omega)$, cf. [36]. This can be thought of as the canonical Spin$^c$ structure associated to any almost-complex structure $J$ over $W$ compatible with $\omega$, compare [36]. In particular, the first Chern class of the canonical Spin$^c$ structure $\mathfrak{k}(\omega)$ is the first Chern class of its complexified tangent bundle. If $(W;\omega)$ has convex boundary $(Y;\xi)$, then the restriction of the canonical Spin$^c$ structure over $W$ to $Y$ is the canonical Spin$^c$ structure of the contact structure.

### 2.1 Foliations and contact structures

Recall that a taut foliation is a foliation $F$ which comes with a two-form $\alpha$ which is positive on the leaves of $F$ (note that like our contact structures, all the foliations we consider here are cooriented and hence oriented). An irreducible three-manifold is a three-manifold $Y$ with $\chi(Y) = 0$. A fundamental result of Gabai states that if $Y$ is irreducible and $\chi(Y) = 0$, $Y$ is an embedded surface which minimizes complexity in its homology class, and with has no spherical or
toroidal components, then there is a smooth, taut foliation $F$ which contains $\mathcal{O}$ as a union of compact leaves. In particular, this shows that if $Y$ is an irreducible three-manifold with non-trivial Thurston semi-norm, and $\mathcal{Y}$ is an embedded surface which minimizes complexity in its homology class, then there is a smooth, taut foliation $F$ with the property that $h_2(F);[ j] = \pm$ ( ).

(Here, we let $F$ be a taut foliation whose closed leaves include all the components of $Y$ with genus greater than one.)

The link between taut foliations and semi-fillable contact structures is provided by an observation of Eliashberg and Thurston, cf. [5], according to which if $Y$ admits a smooth, taut foliation $F$, then $W = [-1; 1]$ $Y$ can be given the structure of a convex symplectic manifold, where here the two-plane fields over $f \g Y$ are homotopic to the two-plane field of tangencies to $F$.

3 Heegaard Floer homology

Heegaard Floer homology is a collection of $\mathbb{Z} = \mathbb{Z}$-graded homology theories associated to three-manifolds, which are functorial under smooth four-dimensional cobordisms (cf. [27] for their constructions, and [28] for the verification of their functorial properties).

There are four variants, $\mathbb{H}F(Y)$, $\mathbb{H}F^-(Y)$, $\mathbb{H}F^1(Y)$, and $\mathbb{H}F^+(Y)$. $\mathbb{H}F^-(Y)$ is the homology of a complex over the polynomial ring $\mathbb{Z}[U]$, $\mathbb{H}F^1(Y)$ is the associated "localization" (i.e. it is the homology of the complex associated to tensoring with the ring of Laurent polynomials over $U$), $\mathbb{H}F^+(Y)$ is associated to the cokernel of the localization map, and finally $\mathbb{H}F(Y)$ is the homology of the complex associated to setting $U = 0$. Indeed, all these groups admit splittings indexed by Spin$^c$ structures over $Y$. The various groups are related by long exact sequences

$$\vdots \longrightarrow \mathbb{H}F(Y; t) \overset{i}{\longrightarrow} \mathbb{H}F^+(Y; t) \overset{U}{\longrightarrow} \mathbb{H}F^+(Y; t) \longrightarrow \vdots \quad \text{(1)}$$

where here $t \in \text{Spin}^c(Y)$. The reduced Heegaard Floer homology $\mathbb{H}F^{-}_\text{red}(Y; t)$ is the cokernel of the map $i$. Sometimes we distinguish this from $\mathbb{H}F^{-}_\text{reg}(Y; t)$, which is the kernel of the map $j$, though these two $\mathbb{Z}[U]$ modules are identified in the long exact sequence above.

For $Y = S^3$, we have that $\mathbb{H}F(S^3) = \mathbb{Z}$. We can now lift the $\mathbb{Z} = \mathbb{Z}$ grading to an absolute $\mathbb{Z}$-grading on all the groups, using the following conventions. The

group $\mathfrak{HF}(S^3) = \mathbb{Z}$ is supported in dimension zero, the maps $i, j$, and $d$ from Equation (1) preserve degree, and $U$ decreases degree by two. Indeed, for $S^3$, we have an identification of $\mathbb{Z}[U]$ modules:

$$
\begin{array}{cccc}
0 & \rightarrow & \mathfrak{HF}^{-}(S^3) & \rightarrow & \mathfrak{HF}^{1}(S^3) & \rightarrow & \mathfrak{HF}^{+}(S^3) & \rightarrow & 0 \\
& = & \gamma & = & \gamma & = & \gamma & = & \gamma \\
0 & \rightarrow & U \mathbb{Z}[U] & \rightarrow & \mathbb{Z}[U; U^{-1}] & \rightarrow & \mathbb{Z}[U; U^{-1}]=U \mathbb{Z}[U] & \rightarrow & 0
\end{array}
$$

where here the element $1 \in \mathbb{Z}[U; U^{-1}]$ lies in grading zero and $U$ decreases grading by two. (See [32] for a definition of absolute gradings in more general settings.)

To state functoriality, we must first discuss maps associated to cobordisms. Let $W_1$ be a smooth, oriented four-manifold with $\partial W_1 = -Y_1 \cup Y_2$, where here $Y_1$ and $Y_2$ are connected. (Here, of course, $-Y_1$ denotes the three-manifold underlying $Y_1$, endowed with the opposite orientation.) In this case, we sometimes write $W_1: Y_1 \rightarrow Y_2$; or, turning this around, we can view the same four-manifold as giving a cobordism $W_1: -Y_2 \rightarrow -Y_1$. There is an associated map

$$
\mathfrak{B}_{W_1}: \mathfrak{HF}(Y_1) \rightarrow \mathfrak{HF}(Y_2);
$$

well-defined up to an overall multiplication by $1$, which can be decomposed along Spin$^c$ structures over $W_1$:

$$
\mathfrak{B}_{W_1; s}: \mathfrak{HF}(Y_1; t_1) \rightarrow \mathfrak{HF}(Y_2; t_2);
$$

where here $t_i = s j_{Y_i}$, i.e. so that

$$
\mathfrak{B}_{W_1} = \prod_{s \in \text{Spin}^c(W_1)} \mathfrak{B}_{W_1; s}.
$$

There are similarly induced maps $\mathfrak{F}_{W_1; s}^+$ on $\mathfrak{HF}^+$ which are equivariant under the action of $\mathbb{Z}[U]$. For $\mathfrak{HF}^1$ and $\mathfrak{HF}^-$, there are again induced maps $\mathfrak{F}_{W_1; s}^1$ and $\mathfrak{F}_{W_1; s}^-$ for each fixed Spin$^c$ structure $s \in \text{Spin}^c(W_1)$ (but now, we can no longer sum maps over all Spin$^c$ structures, since in nitely many might be non-trivial). Indeed, these maps are compatible with the natural maps from Diagram (1); for example, all the squares in the following diagram commute:

$$
\begin{array}{cccc}
0 & \rightarrow & \mathfrak{HF}^{-}(Y_1; t_1) & \rightarrow & \mathfrak{HF}^{1}(Y_1; t_1) & \rightarrow & \mathfrak{HF}^{+}(Y_1; t_1) & \rightarrow & 0 \\
& = & \gamma & = & \gamma & = & \gamma & = & \gamma \\
0 & \rightarrow & U \mathbb{Z}[U] & \rightarrow & \mathbb{Z}[U; U^{-1}] & \rightarrow & \mathbb{Z}[U; U^{-1}]=U \mathbb{Z}[U] & \rightarrow & 0
\end{array}
$$

Functoriality of Floer homology is to be interpreted in the following sense. Let $W_1: Y_1 \to Y_2$ and $W_2: Y_2 \to Y_3$. We can form then the composite cobordism

$$W_1 \# Y_2 W_2: Y_1 \to Y_3.$$  

We claim that for each $s_1 \in \text{Spin}^c(W_i)$ with $s_1|_{Y_2} = s_2|_{Y_2}$, we have that

$$X F_{W:s} = F_{W_2:s_2} \circ F_{W_1:s_1};$$  

with analogous formulas for $HF^-$, $HF^1$, and $HF^+$ as well (this is the composition law, Theorem 3.4 of [28]).

Of these theories, $HF^1$ is the weakest at distinguishing manifolds. For example, if $W: Y_1 \to Y_2$ is a cobordism with $b_2^s(W) > 0$, then for any Spin$^c$ structure $s \in \text{Spin}^c(W)$ the induced map

$$F_{W:s}: HF^1(Y_1; s_{|Y_1}) \to HF^1(Y_2; s_{|Y_2})$$  

vanishes (cf. Lemma 8.2 of [28]).

Floer homology can be used to construct an invariant for smooth four-manifolds $X$ with $b_2^s(X) > 1$ (here, $b_2^s(X)$ denotes the dimension of the maximal subspace of $H^2(X; \mathbb{R})$ on which the cup-product pairing is positive-definite) endowed with a Spin$^c$ structure $s \in 2 \text{Spin}^c(X)$

$$x_{s}: \mathbb{Z}[U] \to \mathbb{Z};$$  

which is well-defined up to an overall sign. This invariant is analogous to the Seiberg-Witten invariant, cf. [41]. This map is a homogeneous element in $\text{Hom}(\mathbb{Z}[U]; \mathbb{Z})$ with degree given by

$$\frac{c_1(s)^2 - 2 \cdot (X) - 3 \cdot (X)}{4};$$

For a fixed four-manifold $X$, the invariant $x_{s}$ is non-trivial for only finitely many $s \in 2 \text{Spin}^c(X)$. (Note that the four-manifold invariant $x_{s}$ constructed in [28] is slightly more general, as it incorporates the action of $H_1(X; \mathbb{Z})$, but we do not need this extra structure for our present applications.)

The invariant is constructed as follows. Let $X$ be a four-manifold, and $N$ a separating hypersurface in $X$ with $0 = H^1(N; \mathbb{Z}) \to H^2(X; \mathbb{Z})$, so that $X = X_1 \cup_N X_2$, with $b_2^s(X_i) > 0$ for $i = 1, 2$. (Here, $H^2(Y; \mathbb{Z}) \to H^2(X; \mathbb{Z})$ is the connecting homomorphism in the Mayer-Vietoris sequence for the decomposition of $X$ into $X_1$ and $X_2$.) Such a separating three-manifold
is called an admissible cut in the terminology of [28]. Given such a cut, delete balls $B_1$ and $B_2$ from $X_1$ and $X_2$ respectively, and consider the diagram:

$$
\begin{array}{c}
\text{HF}^-(S^3) \xrightarrow{\gamma} \text{HF}^1(S^3) \\
\text{HF}^1(N; t) \xrightarrow{\gamma} \text{HF}^+(N; t) \\
\text{HF}^1(S^3) \xrightarrow{\gamma} \text{HF}^+(S^3)
\end{array}
$$

where here $t = s_jN$ and $s_i = s_jX_i$. Since the two maps indicated with 0 vanish (as $b_2(X_i - B_i) > 0$), there is a well-defined map

$$F_{X-B_1-B_2}^{\text{mix}}: \text{HF}^-(S^3) \to \text{HF}^+(S^3);$$

which factors through $\text{HF}^{+\text{red}}(N; t)$.

The invariant $X; s$ corresponds to $F_{X-B_1-B_2}^{\text{mix}}$ under the natural identification

$$\text{Hom}(\mathbb{Z}[U]/\mathbb{Z}[U; U^{-1}], \mathbb{Z}) = \text{Hom}(\mathbb{Z}[U], \mathbb{Z})$$

According to Theorem 9.1 of [28], $X; s$ is a smooth four-manifold invariant.

The following property of the invariant is immediate from its definition: if $X = X_1 [N] X_2$ where $N$ is a rational homology three-sphere with $HF^{+\text{red}}(N) = 0$, and the four-manifolds $X_i$ have the property that $b_2(X_i) > 0$, then for each $s$ 2 Spin$^c(X)$,

$$X; s = 0;$$

The second property which we rely on heavily in this paper is the following analogue of a theorem of Taubes [36] and [37] for the Seiberg-Witten invariants for four-manifolds: if $(X; 1)$ is a smooth, closed, symplectic four-manifold with $b_2^+(X) > 1$, then if $s(1)$ 2 Spin$^c(X)$ denotes its canonical Spin$^c$ structure, then we have that

$$X; s = 1;$$

while if $s$ 2 Spin$^c(X)$ is any Spin$^c$ structure for which $X; s = 0$, then we have that

$$\text{hc}_i(s) = \text{hc}_i(1);$$

with equality if $s = s(1)$. This result is Theorem 1.1 of [34], and its proof relies on a combination of techniques from Heegaard Floer homology (specifically, the surgery long exact sequence from [26]) and Donaldson's Lefschetz pencils for symplectic manifolds, [2].
3.1 Three-manifolds with \( b_1(Y) > 0 \)

There is a version of Floer homology with "twisted coefficients" which is relevant in the case where \( b_1(Y) > 0 \). Fundamental to this construction is a chain complex \( \mathcal{CF}(Y) \) (and also corresponding complexes \( \mathcal{CF}^-, \mathcal{CF}^1 \), and \( \mathcal{CF}^+ \)) with coefficients in \( \mathbb{Z}[H^1(Y; \mathbb{Z})] \) which is a lift of the complex \( \mathcal{CF}(Y) \) (whose homology calculates \( \mathcal{HF}(Y) \)), in the following sense. Let \( \mathbb{Z} \) be the module over \( \mathbb{Z}[H^1(Y; \mathbb{Z})] \), where the elements of \( H^1(Y; \mathbb{Z}) \) act trivially. Then, there is an identification \( \mathcal{CF}(Y) = \mathcal{CF}(Y) \otimes_{\mathbb{Z}[H^1(Y; \mathbb{Z})]} \mathbb{Z} \). Thus, there is a change of coefficient spectral sequences which relates the homology of \( \mathcal{CF}(Y) \), written \( \mathcal{HF}(Y) \), with \( \mathcal{HF}(Y) \).

Indeed, given any module \( M \) over \( \mathbb{Z}[H^1(Y; \mathbb{Z})] \), we can form the group \( \mathcal{HF}(Y; M) = \mathcal{HF}(Y) \otimes_{\mathbb{Z}[H^1(Y; \mathbb{Z})]} M \); which gives Floer homology with coefficients twisted by \( M \). The analogous construction in the other versions of Floer homology gives groups \( \mathcal{HF}^-(Y; M) \), \( \mathcal{HF}^0(Y; M) \), and \( \mathcal{HF}^+(Y; M) \). All of these are related by exact sequences analogous to those in Diagram (1). In particular, we can form a reduced group \( \mathcal{HF}^+_{\text{red}}(Y; M) \), which is the cokernel of the localization map \( \mathcal{HF}^+(Y; M) \to \mathcal{HF}^+(Y; M) \).

In particular, if we fix a two-dimensional cohomology class \( ![\gamma] \in H^2(Y; \mathbb{R}) \), we can view \( \mathbb{Z}[\mathbb{R}] \) as a module over \( \mathbb{Z}[H^1(Y; \mathbb{Z})] \) via the ring homomorphism

\[
[y] \mapsto T^r \gamma \]

(where here \( T^r \) denotes the group-ring element associated to the real number \( r \)). This gives us a notion of twisted coefficients which we denote by \( \mathcal{HF}(Y; ![\gamma]) \).

This can be thought of explicitly as follows. Choose a Morse function on \( Y \) compatible with a Heegaard decomposition \( (X; ; \gamma; ;z) \), and \( x \) also a two-cocycle \( ![\gamma] \) over \( Y \) which represents \( ![\gamma] \). We obtain a map from Whitney disks \( u \) in \( \text{Sym}^2() \) (for \( T \) and \( T \)) to two-chains in \( Y \): \( u \) induces a two-chain in \( Y \) with boundaries along the \( X \) and \( Y \). These boundaries are then coned off by following gradient trajectories for the \( X \) and \( Y \) circles. Since \( ![\gamma] \) is a cocycle, the evaluation of \( ![\gamma] \) on \( u \) depends only on the homotopy class of \( u \). We denote this evaluation by \( ![\gamma](u) \). (This determines an additive assignment in the terminology of Section 8 of [26].) The differential on \( \mathcal{HF}^+(Y; ![\gamma]) \) is given by

\[
\partial_s [x; i] = \sum_{y^2 \in \mathbb{T} \setminus \mathbb{T}} \sum_{f \in 2^z(x; y)} \frac{M(f)}{R} \cdot T^{|\gamma|} \cdot ![\gamma; i - n_2(x)] \]

where here we adopt notation from [26]: \( \pi_2(x; y) \) denotes the space of homotopy classes of Whitney disks in \( \text{Sym}^g(\cdot) \) for \( T \) and \( T \) connecting \( x \) and \( y \), \( (\cdot) \) denotes the formal dimension of its space \( M(\cdot) \) of holomorphic representatives, and \( n_z(\cdot) \) denotes the intersection number of \( \cdot \) with the subvariety \( f_z \text{Sym}^g(\cdot) \).

Now, if \( W : Y_1 \rightarrow Y_2 \), and \( M_1 \) is a module over \( H^1(Y_1; \mathbb{Z}) \), there is an induced map

\[
F^+_W : \text{HF}^+(Y_1; M_1) \rightarrow \text{HF}^+(Y_2; M_1 \otimes_{H^1(Y_1; \mathbb{Z})} H^2(W; Y_1[1] Y_2));
\]

well-defined up to the action by some unit in \( \mathbb{Z}[H^2(Y_1[1] Y_2; \mathbb{Z})] \), defined as in Subsection 3.1 [28]. (Indeed, in that discussion, the construction is separated according to \( \text{Spin}^c \) structures over \( W \), which we drop at the moment for notational simplicity.) In the case of \( \{\text{twisted coefficients}, this gives rise to a map\]

\[
F^+_W, ! : \text{HF}^+(Y_1; ! \mathcal{I} Y_1) \rightarrow \text{HF}^+(Y_2; ! \mathcal{I} Y_2)
\]

(again, well-defined up to multiplication by \( T^c \) for some \( c \in \mathbb{R} \)) which can be concretely described as follows.

Suppose for simplicity that \( W \) is represented as a two-handle addition, so that there is a corresponding \( \text{Heegaard triple}^\ast \) \( (\cdot; \cdot; \cdot; \cdot) \). The corresponding four-manifold \( X; \gamma \) represents \( W \) minus a one-complex. Fix now a two-cocycle \( ! \) representing \( [1] 2 H^2(W; \mathbb{R}) \). Again, a Whitney triangle \( u \) in \( \text{Sym}^g(\cdot) \) for \( T \), \( T \), and \( T_U \) (with vertices at \( x \), \( y \), and \( w \)) determines a two-chain in \( X; \gamma \), whose evaluation on \( ! \) depends on \( u \) only through its induced homotopy class \( \pi_2(x; y; w) \), denoted by \( \mathcal{I} ! \cdot \). Now,

\[
F^+_W, ! [x; i] = \#(M(\cdot)) \mathcal{T} ! [y; i - n_z(\cdot)]; (3)
\]

where \( 2 T \setminus T_U \) represents a canonical generator for the Floer homology \( \text{HF} = H(U^{-1} \text{CF}^\ast) \) of the three-manifold determined by \( (\cdot; \cdot; \cdot; z) \), which is a connected sum \( \#(S^2 - S^2) \). This can be extended to arbitrary (smooth, connected) cobordisms from \( Y_1 \) to \( Y_2 \) as in [28].

(In the present discussion, since we have suppressed \( \text{Spin}^c \) structures from the notation, a subtlety arises. The expression analogous to Equation (3), only using \( \text{HF}^\ast \), is not well-defined since, in principle, there might be in nitely many different homotopy classes which induce non-trivial maps \( \cdot \) i.e. we are trying to sum the maps on \( \text{HF}^\ast \) induced by infinitely many different \( \text{Spin}^c \) structures. However, if the cobordism \( W \) has \( b_2^2(W) > 0 \), then there are
only finitely many Spin$^c$ structures which induce non-zero maps, according to Theorem 3.3 of [28].

Note that when $W$ is a cobordism between two integral homology three-spheres, the above construction is related to the construction in the untwisted case by the formula

$$F^+_W([!]) = \text{Th}_3(s)([!]) [W] \prod_{s_2 \text{Spin}^c(W)} F^+_W,$$

for some constant $c \in \mathbb{R}$.

## 4 Invariants of weakly fillable contact structures

We briefly review the construction here of the Heegaard Floer homology element associated to a contact structure over the three-manifold $Y$, $c(\cdot) \in \mathbb{Z}$.

After sketching the construction, we describe a refinement which lives in Floer homology with twisted coefficients.

The contact invariant is constructed with the help of some work of Giroux. Specifically, in [11], Giroux shows that contact structures over $Y$ are in one-to-one correspondence with equivalence classes of open book decompositions of $Y$, under an equivalence relation given by a suitable notion of stabilization. Indeed, after stabilizing, one can realize the open book with connected binding, and with genus $g > 1$ (both are convenient technical devices). In particular, performing surgery on the binding, we obtain a cobordism (obtained by a single two-handle addition) $W_0: Y \rightarrow Y_0$, where here the three-manifold $Y_0$ fibers over the circle. We call this cobordism a Giroux two-handle subordinate to the contact structure over $Y$. This cobordism is used to construct $c(\cdot)$, but to describe how, we must discuss the Heegaard Floer homology for three-manifolds which fiber over the circle.

Let $Z$ be a (closed, oriented) three-manifold endowed with the structure of a fiber bundle $\pi: Z \rightarrow S^1$. This structure endows $Z$ with a canonical Spin$^c$ structure $t(\cdot) \in \text{Spin}^c(Z)$ (induced by the two-plane distribution of tangents to the fiber). According to [34], if the genus $g$ of the fiber is greater than one, then

$$\text{HF}^+(Z; t(\cdot)) = \mathbb{Z};$$

In particular, there is a homogeneous generator $c_0(\cdot)$ for $\mathbb{Z}$ which maps to the generator $c_0^+(\cdot)$ of $\text{HF}^+(Z; t(\cdot))$. This generator is, of course, uniquely determined up to sign.
With these remarks in place, we can give the definition of the invariant $c(\cdot)$ associated to a contact structure over $Y$. If $Y$ is given a contact structure, fix a compatible open book decomposition (with connected binding, and fiber genus $g > 1$), and consider the corresponding Giroux two-handle $W_0: -Y_0 \to -Y$ (which we have "turned around" here), and let

$$\hat{H}_{W_0}: \mathcal{H}F(-Y_0) \to \mathcal{H}F(-Y)$$

be the induced map. Then, define $c(\cdot) \to \mathcal{H}F(-Y) \setminus \mathcal{H}F(Y)$ to be the image $\hat{H}_{W_0}(c_0(\cdot))$. It is shown in [30] that this element is uniquely associated (up to sign) to the contact structure, i.e. it is independent of the choice of compatible open book. In fact, the element $c(\cdot)$ is supported in the summand $\mathcal{H}F(Y; \tau(\cdot)) \to \mathcal{H}F(Y)$, where here $\tau(\cdot)$ is the canonical $Spin^c$ structure associated to the contact structure, in the sense described in Section 2. (In particular, the canonical $Spin^c$ structure of the fibration structure on $-Y_0$ is $Spin^c$ cobordant to the canonical $Spin^c$ structure of the contact structure over $-Y$ via the Giroux two-handle.)

With the help of Giroux's characterization of Stein fillable contact structures, it is shown in [30] that $c(\cdot)$ is non-trivial for a Stein structure. This non-vanishing result can be strengthened considerably with the help of the following result of Eliashberg [3].

**Theorem 4.1** (Eliashberg [3]) Let $(Y; \cdot)$ be a contact three-manifold, which is the convex boundary of some symplectic four-manifold $(W; !)$. Then, any Giroux two-handle $W_0: Y \to -Y_0$ can be completed to give a compact symplectic manifold $(V; !)$ with concave boundary $\partial V = (Y; \cdot)$, so that $!$ extends smoothly over $X = W \setminus V$.

Although Eliashberg's is the construction we need, concave fillings have been constructed previously in a number of different contexts, see for example [22], [1], [7], [10], [25]. Indeed, since the first posting of the present article, Etnyre pointed out to us an alternate proof of Eliashberg's theorem [6], see also [25].

In the construction, $V$ is given as the union of the Giroux two-handle with a surface bundle $V_0$ over a surface-with-boundary which extends the fiber bundle structure over $Y_0$. Moreover, the fibers of $V_0$ are symplectic. By forming a symplectic sum if necessary, one can arrange for $\mathcal{B}(V) = \infty$ to be arbitrarily large.

To state the stronger non-vanishing theorem, we use a refinement of the contact element using twisted coefficients. We can repeat the construction of $c(\cdot)$ with
coefficients in any module $M$ over $\mathbb{Z}[H^1(Y;\mathbb{Z})]$ (compare Remark 4.5 of [30]), to get an element
\[ c(\; ; M) \in \mathbb{H}^2_{HF}(Y ; M) = \mathbb{Z}[H^1(Y;\mathbb{Z})]. \]

As the notation suggests, this is an element $c(\; ; M) \in \mathbb{H}^2_{HF}(Y ; M)$, which is well-defined up to overall multiplication by a unit in the group-ring $\mathbb{Z}[H^1(Y;\mathbb{Z})]$. Let $c^+ (\; ; M)$ denote the image of $c(\; ; M)$ under the natural map $\mathbb{H}^2_{HF}(Y ; M) \to \mathbb{H}^2_{HF}(Y ; M)$, and let $c^+_{red}(\; ; M)$ denote its image under the projection $\mathbb{H}^2_{HF}(Y ; M) \to \mathbb{H}^2_{HF}(Y ; M)$.

In our applications, we will typically take the module $M$ to be $\mathbb{Z}[\mathbb{R}]$, with the action specified by some two-form $\alpha$ over $Y$, so that we get $c(\; ; \alpha) \in \mathbb{H}^2_{HF}(Y ; \alpha)$. The following theorem should be compared with a theorem of Kronheimer and Mrowka [17], see also Section 6 of [19]:

**Theorem 4.2** Let $(W;\alpha)$ be a weak filling of a contact structure $(Y;\alpha)$. Then, the associated contact invariant $c(\; ; \alpha)$ is non-trivial. Indeed, it is non-torsion and primitive (as is its image in $\mathbb{H}^2_{HF}(Y;\alpha)$). Indeed, if $(W;\alpha)$ is a weak-semi-filling of $(Y;\alpha)$ with disconnected boundary or $(W;\alpha)$ is a weak filling of $Y$ with $b_2^+(W) > 0$, then the reduced invariant $c^+_{red}(\; ; \alpha)$ is non-trivial (and indeed non-torsion and primitive).

**Proof** Let $(W;\alpha)$ be a symplectic filling of $(Y;\alpha)$ with convex boundary.

Consider Eliashberg's cobordism bounding $Y$, $V = W_0 \cup Y_0 V_0$, where here $W_0 : Y \to (-Y_0)$, $Y_0$ is the Giroux two-handle and $V_0$ is a surface bundle over a surface-with-boundary. Now, the union
\[ X = V_0 \cup (-Y) \cup W \]
is a closed, symplectic four-manifold. (As the notation suggests, we have "turned around" $W_0$, to think of it as a cobordism from $-Y_0$ to $-Y$; similarly for $V_0$.) Arrange for $b_2^+(V_0) > 1$, and decompose $V_0$ further by introducing an admissible cut by $N$. Now, $N$ decompose $X$ into two pieces $X = X_1 \cup N X_2$, where $b_2^+(X_1) > 0$, and we can suppose now that $X_2$ contains the Giroux cobordism, i.e.
\[ X_2 = (V_0 - X_1) \cup (-Y) \cup W: \quad (4) \]

Now, by the definition of $c$, for any given $s \in \text{Spin}^c(X)$, there is an element $2 \mathbb{H}^2_{HF}(N;\alpha)$ with the property that
\[ x_{s} = F_{X_2 \cup B_2}(\; ; ) \]

(By definition of , the element here is any element of \( HF^+(N; s|N) \) whose image under the connecting homomorphism in the second exact sequence in Equation (1) coincides with the image of a generator of \( HF^-(S^2) \) under the map \( F_{X_1-B_1}: HF^-(S^2) \to HF^-(N; s|N) \).) Applying the product formula for the decomposition of Equation (4), we get that
\[
\begin{align*}
X_{:; t(|)!} &= F^{+}_{W-B_{2_2}} F^{+}_{W_0} F^{+}_{V_0-X_{1}} (:) \\
2H^{1}(Y_{;:})
\end{align*}
\]
In terms of \( \{ \text{twisted coefficients} \} \), we have that
\[
\begin{align*}
X_{:; t(|)!} &= T^{H} [ c_{1}(t(|)! + )];X ] = F^{+}_{W-B_{2}};[! ] F^{+}_{W_0};[! ] F^{+}_{V_0-X_{1}};[! ] (:) \\
2H^{1}(Y_{;:})
\end{align*}
\]
(Here, \( 2HF^{+}(N; s|N; ![!]) \) is the analogue of the class considered earlier.) But \( HF^{+}(Y_{0}; t) = \mathbb{Z}[\mathbb{R}] \) is generated by \( c^{+}_{0}(:) \) (where here \( Y_{0} \to S^{1} \) is the projection obtained from restricting the bundle structure over \( Y_{0} \), and \( t \) is the restriction of \( t(|) \) to \( Y_{0} \)), so there is some element \( p(T) 2 \mathbb{Z}[\mathbb{R}] \) with the property that \( F^{+}_{Y_{0}-nd(F)}(:) = p(T) \) \( c^{+}(:) \). Thus,
\[
\begin{align*}
X_{:; t(|)!} &= T^{H} [ c_{1}(t(|)! + )];X ] = p(T) F^{+}_{W-B_{2}} (c^{+}(; ![!])) \\
2H^{1}(Y_{;:})
\end{align*}
\]
The left-hand-side here gives a polynomial in \( T \) (well defined up to an overall sign and multiple of \( T \)) whose lowest-order term is one, according to Theorem 1.1 of [34] (recalled in Section 3). It follows at once that \( F^{+}_{W-B_{2}} (c^{+}(; ![!])) \) is non-trivial. Indeed, it also follows that \( F^{+}_{W-B_{2}} (c^{+}(; ![!])) \) is a primitive homology class (since the leading coefficient is 1), and no multiple of it zero. This implies the same for \( c^{+}(; ![!]) \).

Now, when \( b^{+}_{2}(W) > 0 \), we use \( Y \) as a cut for \( X \) to show that the induced element \( c^{+}_{0}(; ![!]) \) is non-trivial (primitive and torsion). In the case where \( Y \) is semi-labile with disconnected boundary, we can close \( X \) by the remaining boundary components as in Theorem 4.1 to construct a new symplectic filling \( W^{0} \) of \( Y \) with one boundary component and \( b^{+}_{2}(W^{0}) > 0 \), reducing to the previous case.

**Proof of Theorem 1.4**. A three-manifold \( Y \) is an \( L \) space if it is a rational homology three-sphere and \( HF(Y) \) is a free \( \mathbb{Z} \{ \text{module of rank } jH_{1}(Y; \mathbb{Z}) \} \). Note that for an \( L \)-space, \( HF^{+}_{\text{red}}(Y) \otimes \mathbb{Q} = 0 \). This is an easy application of the long exact sequence (1), together with the fact that the intersection of the kernel of \( U: HF^{+}(Y) \to HF^{1}(Y) \) with the image of \( HF^{1}(Y) \) inside \( HF^{+}(Y) \) has rank \( jH_{1}(Y; \mathbb{Z}) \), since \( HF^{1}(Y) = \mathbb{Z}[U; U^{-1}] \) (cf. Theorem 10.1.

of \([26]\)), the map from \(HF^{1}(Y)\) to \(HF^{+}(Y)\) is an isomorphism in all sufficiently large degrees (i.e. \(U^{-n}\) for \(n\) sufficiently large), and it is trivial in all sufficiently small degrees.

For a three-manifold \(Y\) with \(b_2(Y) = 0\), \(HF^{+}(Y;[!]) = HF^{+}(Y) \otimes_{\mathbb{Z}} \mathbb{R}\), since \([!]\) \(2H^2(Y;\mathbb{Q})\) is exact. Thus, the reduced group in which \(c^*_\text{red}(\cdot;[!])\) lives consists only of torsion classes, and the result now follows from Theorem 4.2.

Sometimes, it is easier to use \(\mathbb{Z} = p\mathbb{Z}\) coefficients (especially when \(p = 2\)). To this end, we say that \(Y\) a rational homology three-sphere is a \(\mathbb{Z} = p\mathbb{Z}\) \{L\} space for some prime \(p\) if \(\mathcal{HF}(Y;\mathbb{Z} = p\mathbb{Z})\) has rank \(jH_1(Y;\mathbb{Z})\) over \(\mathbb{Z} = p\mathbb{Z}\) (of course, an \(L\) space is automatically a \(\mathbb{Z} = p\mathbb{Z}\) \{L\} space for all \(p\)). Since \(c^*([!])\) is primitive, the above argument shows that a \(\mathbb{Z} = p\mathbb{Z}\) \{L\} space (for any prime \(p\)) cannot support a taut foliation.

The need to use twisted coefficients in the statement of Theorem 4.2 is illustrated by the three-manifold \(Y\) obtained as zero-surgery on the trefoil. The reduced Heegaard Floer homology with untwisted coefficients is trivial (cf. Equation 26 of \([32]\)), but this three-manifold admits a taut foliation. (In particular the reduced Heegaard Floer homology of this manifold with twisted coefficients is non-trivial, cf. Lemma 8.6 of \([32]\).)

5 The Thurston norm

We turn our attention to the proof of Theorem 1.1.

**Proof of Theorem 1.1** It is shown in Section 1.6 of \([26]\) that if \(\mathcal{HF}(Y; s) \neq 0\), then

\[
jHc_1(s); i j \quad (\cdot) = \quad (5)
\]

(The result is stated there for \(HF^{+}\) with untwisted coefficients, but the argument there applies to the case of \(\mathcal{HF}\).) It remains to prove that if \(Y\) is an embedded surface which minimizes complexity in its homology class, then there is a Spin\(^c\) structure \(s\) with \(\mathcal{HF}(Y; s) \neq 0\) and

\[
hc_1(s); [\cdot] = -4(\cdot) = \quad (6)
\]

The Künneth principle for connected sums (cf. Theorem 1.5 of \([26]\)) states that

\[
\mathcal{HF}(Y_1 \# Y_2; s_1 \# s_2) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathcal{HF}(Y_1; s_1) \otimes_{\mathbb{Z}} \mathcal{HF}(Y_2; s_2) \otimes_{\mathbb{Z}} \mathbb{Q}.
\]

In particular, if \(\mathcal{HF}(Y_1; s_1) \otimes_{\mathbb{Z}} \mathbb{Q}\) and \(\mathcal{HF}(Y_2; s_2) \otimes_{\mathbb{Z}} \mathbb{Q}\) are non-trivial, then so is \(\mathcal{HF}(Y_1 \# Y_2; s_1 \# s_2) \otimes_{\mathbb{Z}} \mathbb{Q}\). Since every closed three-manifold admits a

connected sum decomposition where the summands are all either irreducible or copies of $S^2 \# S^1$ [23], it suffices to verify that $\mathcal{H}_F(Y; s) \otimes \mathbb{Q}$ is non-trivial for the elementary summands of $Y$. (It is straightforward to see that $Y_1 \# Y_2(1 + 2) = Y_1(1) + Y_2(2)$ in $Y_1 \# Y_2$, where here $i^2 H_2(Y_i)$, under the natural identification $H_2(Y_1 \# Y_2) = H_2(Y_1) \oplus H_2(Y_2)$.)

We first observe that if $Y$ has trivial Thurston semi-norm (for example, when $b_1(Y) = 0$ or $Y = S^2 \# S^1$), then there is an element $s \in \text{Spin}^c(Y)$ for which $\mathcal{H}_F(Y; s) \not\cong 0$. Indeed, it is shown in Theorem 10.1 of [26] that $\mathcal{H}_F^3(Y; s) = \mathbb{Z}[U; U^{-1}]$ for any $s$ with $c_1(s) = 0$. Also, for such $\text{Spin}^c$ structures, the map from $\mathcal{H}_F^1(Y; s)$ to $\mathcal{H}_F^+(Y; s)$ is non-trivial. The non-triviality of $\mathcal{H}_F(Y; s)$ follows at once (using the analogue of Exact Sequence (1) for the case of twisted coefficients).

In the case where $Y$ is an irreducible three-manifold with non-trivial Thurston norm, and $F$ is a surface which minimizes complexity in its homology class, Gabai [8] constructs a smooth taut foliation $F$ for which

$$h_c(F; s) = -4.$$ 

According to a theorem of Eliashberg and Thurston, then $[−1; 1]$ $Y$ can be equipped with a convex symplectic form, which extends $F$, thought of as a foliation over $0 \in Y$. In particular, their result gives a weakly symplectically semi-llable contact structure with $h_c(F; s) = -4$. It follows now from Theorem 4.2 that $c_1(s) = 0$ and $2 \mathcal{H}_F(Y; s) \otimes \mathbb{Q} \not\cong 0$.

One approach to Theorem 1.2 would directly relate knot Floer homology with the twisted Floer homology of the zero-surgery. We opt, however, to give an alternate proof which uses the relation between the knot Floer homology and the Floer homology of the zero-surgery in the untwisted case, and adapts the proof rather than the statement of Theorem 1.1. The relevant relationship between these groups can be found in Corollary 4.5 of [31], according to which if $d > 1$ is the smallest integer for which $\mathcal{H}_F(K; d) \not\cong 0$, then

$$\mathcal{H}_F(K; d) = H^+(S^3_0(K); d - 1);$$ 

where here we have identified $\text{Spin}^c(S^3_0(K)) = \mathbb{Z}$ by the map $s \mapsto h_c(s); [s] = 2$, where $[s] \mathcal{H}_F(S^3_0(K); s) = \mathbb{Z}$ is some generator. (Note that the choice of generator is not particularly important, as $H^+(S^3_0(K); i) = H^+(S^3_0(K); -i)$, according to the conjugation invariance of Heegaard Floer homology, Theorem 2.4 of [26].)

This result will be used in conjunction with the \"adjunction inequality\" for knot Floer homology, Theorem 5 of [31], which shows that $\mathcal{H}_F(K; i) = 0$ for
all \( j > g(K) \); and indeed, the proof of that result proceeds by constructing a compatible doubly-pointed Heegaard diagram (from a genus-minimizing Seifert surface for \( K \)) which has no simultaneous trajectories \( x \) with \( s(x) > g(K) \).

**Proof of Theorem 1.2** Let \( K \) be a knot with genus \( g \). Assume for the moment that \( g > 1 \). Let \( Y \) be the three-manifold obtained as zero-framed surgery on \( S^3 \) along \( K \), and let \( [\cdot] \in H_2(Y;\mathbb{Z}) \) denote a generator. In this case, Gabai [9] constructs a taut foliation \( F \) over \( Y \) with \( h_c(F) = 2 - 2g \).

Eliashberg's theorem [3] now provides a symplectic four-manifold \( X = X_1 \cup_Y X_2 \), where here \( b_2^+(X_1) > 0 \). According to the product formula Equation (2), the sum

\[
\sum_{X; t(\cdot)+2H^1(Y)}
\]

is calculated by a homomorphism which factors through the Floer homology \( HF^+(Y; t(\cdot)) \). On the other hand, \( c_2(t(\cdot)) \) gives a cohomology class whose evaluation on a generator for \( H_2(Y;\mathbb{Z}) \) is non-trivial when \( g > 1 \) (for a suitable evaluation of \( c_2(s(\cdot)) \), it follows that the various terms in the sum are homogeneous of different degrees. But by Theorem 1.1 of [34], it follows that the term corresponding to \( t(\cdot) \) (and hence the sum) is non-trivial. It follows now that \( HF^+(Y; t(\cdot)) = HF^+(S^3(K); g - 1) \) (for suitably chosen generator) is non-trivial and hence, in view of Equation (7), Theorem 1.2 follows for knots with genus at least two.

Suppose that \( g = 1 \). In this case, we have a Künneth principle for the knot Floer homology (cf. Equation 5 of [31]), according to which (since \( HF(K; s) = 0 \) for all \( s > 1 \)),

\[
HF(K \# K; 2) \otimes_{\mathbb{Z}}\mathbb{Q} = HF(K; 1) \otimes_{\mathbb{Q}} HF(K; 1)
\]

But \( K \# K \) is a knot with genus 2, and hence \( HF(K \# K; 2) \) is non-trivial; and hence, so is \( HF(K; 1) \).

**Proof of Corollary 1.3** According to the integral surgeries long exact sequence for Heegaard Floer homology (in its graded form), if \( S^3(K) = L(p, 1) \), the Alexander polynomial of \( K \) is trivial (indeed \( HF^+(S^3(K)) = HF^+(S^2 \# S^1) \)), cf. Theorem 1.8 of [32]. In [29], it is shown that if \( S^3_p(K) \) is a lens space for some integer \( p \), then the knot Floer homology \( HF^+(K; 1) \) is determined by the Alexander polynomial \( \Delta_K(T) \) (cf. Theorem 1.2 of [29]) which in the present case is trivial. Thus, in view of Theorem 1.2, the knot \( K \) is trivial. \(\square\)
Proof of Corollary 1.5 In the proof of Theorem 5 of [31], we demonstrate that if a knot has genus $g$, then there is a compatible Heegaard diagram with no simultaneous trajectories $x$ for which $s(x) > g$. In the opposite direction, note that $HF_K(K; d)$ is generated by simultaneous trajectories with $s(x) = d$. According to Theorem 1.2, $HF_K(K; g) \neq 0$, and hence any compatible Heegaard diagram must contain some simultaneous trajectories $x$ with $s(x) = g$.

References


[29] P S Ozsvath, Z Szabo, On knot Floer homology and lens space surgeries, arXiv v: nat h. GT/ 0303017


