Construction of 2-local finite groups of a type studied by Solomon and Benson

Ran Levi
Bob Oliver

Department of Mathematical Sciences, University of Aberdeen
Meston Building 339, Aberdeen AB24 3UE, UK
and
LAGA { UMR 7539 of the CNRS, Institut Galilée
Av J.-B Clement, 93430 Villetaneuse, France

Email: ran@maths.abdn.ac.uk and bob@math.univ-paris13.fr

Abstract

A p-local finite group is an algebraic structure with a classifying space which has many of the properties of p-completed classifying spaces of finite groups. In this paper, we construct a family of 2-local finite groups, which are exotic in the following sense: they are based on certain fusion systems over the Sylow 2-subgroup of Spin_7(q) (q an odd prime power) shown by Solomon not to occur as the 2-fusion in any actual finite group. Thus, the resulting classifying spaces are not homotopy equivalent to the 2-completed classifying space of any finite group. As predicted by Benson, these classifying spaces are also very closely related to the Dwyer-Wilkerson space BDI(4).

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As one step in the classification of finite simple groups, Ron Solomon [22] considered the problem of classifying all finite simple groups whose Sylow 2-subgroups are isomorphic to those of the Conway group $\text{Co}_3$. The end result of his paper was that $\text{Co}_3$ is the only such group. In the process of proving this, he needed to consider groups $G$ in which all involutions are conjugate, and such that for any involution $x \in 2G$, there are subgroups $K \triangleleft H \triangleleft C_G(x)$ such that $K$ and $C_G(x) = H$ have odd order and $H = K = \text{Spin}_7(q)$ for some odd prime power $q$. Solomon showed that such a group $G$ does not exist. The proof of this statement was also interesting, in the sense that the 2-local structure of the group in question appeared to be internally consistent, and it was only by analyzing its interaction with the $p$-local structure (where $p$ is the prime of which $q$ is a power) that he found a contradiction.

In a later paper [3], Dave Benson, inspired by Solomon's work, constructed certain spaces which can be thought of as the 2-completed classifying spaces which the groups studied by Solomon would have if they existed. He started with the spaces $\text{BDI}(4)$ constructed by Dwyer and Wilkerson having the property that

$$H(\text{BDI}(4); \mathbb{F}_2) = \mathbb{F}_2[x_1; x_2; x_3; x_4]^{	ext{GL}_4(2)}$$

(the rank four Dickson algebra at the prime 2). Benson then considered, for each odd prime power $q$, the homotopy fixed point set of the $\mathbb{Z}$-action on $\text{BDI}(4)$ generated by an "Adams operation" $q$ constructed by Dwyer and Wilkerson. This homotopy fixed point set is denoted here $\text{BDI}_4(q)$.

In this paper, we construct a family of 2-local finite groups, in the sense of [6], which have the 2-local structure considered by Solomon, and whose classifying spaces are homotopy equivalent to Benson's spaces $\text{BDI}_4(q)$. The results of [6] combined with those here allow us to make much more precise the statement that these spaces have many of the properties which the 2-completed classifying spaces of the groups studied by Solomon would have had if they existed. To explain what this means, we first recall some definitions.

A fusion system over a finite group $S$ is a category whose objects are the subgroups of $S$, and whose morphisms are monomorphisms of groups which include all those induced by conjugation by elements of $S$. A fusion system is saturated if it satisfies certain axioms formulated by Puig [19], and also listed in [6, Definition 1.2] as well as at the beginning of Section 1 in this paper. In particular, for any finite group $G$ and any $2 \leq \text{Syl}_p(G)$, the category $F_S(G)$ whose objects are the subgroups of $S$ and whose morphisms are those monomorphisms between subgroups induced by conjugation in $G$ is a saturated fusion system over $S$.
If $F$ is a saturated fusion system over $S$, then a subgroup $P \leq S$ is called *centric* if $C_S(P) = Z(P)$ for all $P$ isomorphic to $P$ in the category $F$. A centric linking system associated to $F$ consists of a category $L$ whose objects are the $F$-centric subgroups of $S$, together with a functor $L \rightarrow F$ which is the inclusion on objects, is surjective on all morphism sets and which satisfies certain additional axioms (see [6, Definition 1.7]). These axioms suffice to ensure that the $p$-completed nerve $jL_p^\wedge$ has all of the properties needed to regard it as a "classifying space" of the fusion system $F$. A $p$-local finite group consists of a triple $(S; F; L)$, where $S$ is a finite $p$-group, $F$ is a saturated fusion system over $S$, and $L$ is a linking system associated to $F$. The classifying space of a $p$-local finite group $(S; F; L)$ is the $p$-completed nerve $jL_p^\wedge$ (which is $p$-complete since $jL$ is always $p$-good [6, Proposition 1.12]). For example, if $G$ is a finite group and $S = \text{Sol}_p(G)$, then there is an explicitly defined centric linking system $L_S^\wedge(G)$ associated to $F_S(G)$, and the classifying space of the triple $(S; F_S(G); L_S^\wedge(G))$ is the space $jL_S^\wedge(G)j_p^\wedge \cdot BG_p^\wedge$.

Exotic examples of $p$-local finite groups for odd primes $p$ (i.e., examples which do not represent actual groups) have already been constructed in [6], but using ad hoc methods which seemed to work only at odd primes.

In this paper, we first construct a fusion system $F_{\text{Sol}}(q)$ (for any odd prime power $q$) over a $2$-Sylow subgroup $S$ of $\text{Spin}_7(q)$, with the properties that all elements of order 2 in $S$ are conjugate (i.e., the subgroups they generated are all isomorphic in the category), and the "centralizer fusion system" (see the beginning of Section 1) of each such element is isomorphic to the fusion system of $\text{Spin}_7(q)$. We then show that $F_{\text{Sol}}(q)$ is saturated, and has a unique associated linking system $L_S^\wedge(q)$. We thus obtain a $2$-local finite group $(S; F_{\text{Sol}}(q); L_S^\wedge(q))$ where by Solomon's theorem [22] (as explained in more detail in Proposition 3.4), $F_{\text{Sol}}(q)$ is not the fusion system of any finite group.

Let $B\text{Sol}(q) \overset{\text{def}}{=} jL_S^\wedge(q)j_p^\wedge$ denote the classifying space of $(S; F_{\text{Sol}}(q); L_S^\wedge(q))$. Thus, $B\text{Sol}(q)$ does not have the homotopy type of $BG_p^\wedge$ for any finite group $G$, but does have many of the nice properties of the $2$-completed classifying space of a finite group (as described in [6]).

Relating $B\text{Sol}(q)$ to $BDI_4(q)$ requires taking the "union" of the categories $L_S^{\text{Sol}}(q^n)$ for all $n \geq 1$. This however is complicated by the fact that an inclusion of $\ell$-ends $F_p^n \rightarrow F_p^m$ (i.e., $m \geq n$) does not induce an inclusion of centric linking systems. Hence we have to replace the centric linking systems $L_S^{\text{Sol}}(q^n)$ by the full subcategories $L_S^{\text{Sol}}(q^n)$ whose objects are those $2$-subgroups which are centric in $F_S^{\text{Sol}}(q^n) = \bigcap_{n \geq 1} F_S^{\text{Sol}}(q^n)$, and show that the inclusion induces a homotopy equivalence $B\text{Sol}(q^n) \overset{\text{def}}{=} jL_S^{\text{Sol}}(q^n)j_p^\wedge \cdot B\text{Sol}(q^n)$. Inclusions of $\ell$-ends

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do induce inclusions of these categories, so we can then define $L^c_{Sol}(q^1) \overset{\text{def}}{=} \bigcup_{n=1}^{\infty} L^c_{Sol}(q^n)$, and spaces

$$B \text{ Sol}(q^1) \overset{\text{def}}{=} \bigcup_{n=1}^{\infty} B \text{ Sol}(q^n) \overset{\text{def}}{=} B^{D_4}(q^n) \overset{\text{def}}{=} B \text{ DSI}(4)$$

The category $L^c_{Sol}(q^1)$ has an 'Adams map' $q$ induced by the Frobenius automorphism $x \rightarrow x^q$ of $F_q$. We then show that $B \text{ Sol}(q^1)$ is $B \text{ DSI}(4)$, the space of Dwyer and Wilkerson mentioned above, and also that $B \text{ Sol}(q)$ is equivalent to the homotopy fixed point set of the $\mathbb{Z}$-action on $B \text{ Sol}(q^1)$ generated by $B \text{ DSI}(4)$. The space $B \text{ Sol}(q)$ is thus equivalent to Benson's spaces $B \text{ DSI}(4)$ for any odd prime power $q$.

The paper is organized as follows. Two propositions used for constructing saturated fusion systems, one very general and one more specialized, are proven in Section 1. These are then applied in Section 2 to construct the fusion systems $F_{Sol}(q)$, and to prove that they are saturated. In Section 3 we prove the existence and uniqueness of a centric linking systems associated to $F_{Sol}(q)$ and study their automorphisms. Also in Section 3 is the proof that $F_{Sol}(q)$ is not the fusion system of any finite group. The connections with the space $B \text{ DSI}(4)$ of Dwyer and Wilkerson is shown in Section 4. Some background material on the spinor groups $\text{Spin}(V; b)$ over fields of characteristic $6 \neq 2$ is collected in an appendix.

We would like to thank Dave Benson, Ron Solomon, and Carles Broto for their help while working on this paper.

1 Constructing saturated fusion systems

In this section, we first prove a general result which is useful for constructing saturated fusion systems. This is then followed by a more technical result, which is designed to handle the specific construction in Section 2.

We first recall some definitions from [6]. A fusion system over a $p$-group $S$ is a category $F$ whose objects are the subgroups of $F$, such that

$$\text{Hom}_F(P; Q) \quad \text{Mor}_F(P; Q) \quad \text{Inj}(P; Q)$$

for all $P; Q \in S$, and such that each morphism in $F$ factors as the composite of an $F$-isomorphism followed by an inclusion. We write $\text{Hom}_F(P; Q) = \text{Mor}_F(P; Q)$ to emphasize that the morphisms are all homomorphisms of groups.
We say that two subgroups \( P; Q \) are \( F \) conjugate if they are isomorphic in \( F \). A subgroup \( P \) is fully centralized (fully normalized) in \( F \) if \( jC(S(P)) = jC(S(P)^0) \) and each \( jN(S(P)) = jN(S(P)^0) \) for all \( P \in S \) which is \( F \) conjugate to \( P \). A saturated fusion system is a fusion system \( F \) over \( S \) which satisfies the following two additional conditions:

(I) For each fully normalized subgroup \( P \) there is a set \( \text{Aut}_S(P) \) and each \( g \in \text{Aut}_S(P) \) such that

\[
\text{N}_P = g \text{N}_S(P) \quad \text{and} \quad g^{-1} \text{Aut}_S(P) \quad \text{for all} \quad P \in S.
\]

(II) For each \( P \) and each \( \langle \text{Hom}_F(P; S) \rangle \) such that \( \langle P \rangle \) is fully centralized in \( F \), if we set

\[
\text{N}_P = g \text{N}_S(P) \quad \text{and} \quad g^{-1} \text{Aut}_S(P) \quad \text{for all} \quad P \in S.
\]

For example, if \( G \) is a finite group and \( S \) is \( \text{Syl}_p(G) \) then the category \( F_S(G) \) whose objects are the subgroups of \( S \) and whose morphisms are the homomorphisms induced by conjugation in \( G \) is a saturated fusion system over \( S \). A subgroup \( P \in S \) is fully centralized in \( F_S(G) \) if and only if \( C_S(P) \) is \( \text{Syl}_p(C_G(P)) \), and \( P \) is fully normalized in \( F_S(G) \) if and only if \( N_S(P) \) is \( \text{Syl}_p(N_G(P)) \).

For any fusion system \( F \) over a \( p \)-group \( S \) and any subgroup \( P \), the centralizer fusion system \( C_F(P) \) over \( C_S(P) \) is defined by setting

\[
\text{Hom}_{C_F(P)}(Q; Q^0) = \langle \text{Hom}_F(P; Q; Q^0) \rangle \quad \text{for all} \quad Q; Q^0 \quad \text{such that} \quad C_S(P) \quad \text{is} \quad \text{fully centralized in} \quad F.
\]

**Proposition 1.1** Let \( F \) be any fusion system over a \( p \)-group \( S \). Then \( F \) is saturated if and only if there is a set \( \mathcal{X} \) of elements of order \( p \) in \( S \) such that the following conditions hold:

(a) Each \( x \in \mathcal{X} \) is \( F \) conjugate to some element of \( \mathcal{X} \).

(b) If \( x \) and \( y \) are \( F \) conjugate and \( y \in \mathcal{X} \), then there is some morphism \( \text{Hom}_F(C_S(x); C_S(y)) \) such that \( (x) = y \).

(c) For each \( x \in \mathcal{X} \), \( C_F(x) \) is a saturated fusion system over \( C_S(x) \).

**Proof** Throughout the proof, conditions (I) and (II) always refer to the conditions in the definition of a saturated fusion system, as stated above or in [6, Definition 1.2].
We first check that $h$ is fully centralized. Then condition (a) holds by definition, (b) follows from condition (ii), and (c) holds by [6, Proposition A.6] or [19].

Assume conversely that $x$ is chosen such that conditions (a-c) hold for $F$. Denote

$$U = (P; x)^P S; jx = p; x 2 Z(P)^T; \text{ some } T 2 \text{ Syl}_p(\text{Aut}_F(P)), T^* \text{ Aut}_S(P);$$

where $Z(P)^T$ is the subgroup of elements of $Z(P)$ fixed by the action of $T$. Let $U_0 = U$ be the set of pairs $(P; x)$ such that $x 2 \mathcal{X}$. For each $1 \neq P \subseteq S$, there is some $x$ such that $(P; x) 2 U$ (since every action of a $P$-group on $Z(P)$ has nontrivial fixed set); but $x$ need not be unique.

We first check that

$$(P; x) 2 U_0; P \text{ fully centralized in } C_F(x) \implies P \text{ fully centralized in } F. \quad (1)$$

Assume otherwise: that $(P; x) 2 U_0$ and $P$ is fully centralized in $C_F(x)$, but $P$ is not fully centralized in $F$. Let $P^0 S$ and $2 \text{ Isoc}_F(P; P^0)$ be such that $jC_S(P)j < jC_S(P^0)j$. Set $x^0 = '(x) \in Z(P^0)$. By (b), there exists $2 \text{ Hom}_F(C_S(x); C_S(x))$ such that $(x^0) = x$. Set $P^0 = (P^0)$. Then $2 \text{ Isoc}_F(x'(P; P^0))$, and in particular $P^0$ is $C_F(x)$-conjugate to $P$. Also, since $C_S(P^0) \subseteq C_S(x^0)$, sends $C_S(x)$ injectively into $C_S(P^0)$, and $jC_S(P)j < jC_S(P^0)j$, $jC_S(P^0)j$. Since $C_S(P) = C_{C_F(x)}(P)$ and $C_S(P^0) = C_{C_F(x)}(P^0)$, this contradicts the original assumption that $P$ is fully centralized in $C_F(x)$.

By definition, for each $(P; x) 2 U$, $N_S(P) C_S(x)$ and hence $\text{Aut}_{C_S(x)}(P) = \text{Aut}_{S}(P)$. By assumption, there is $T 2 \text{ Syl}_p(\text{Aut}_F(P))$ such that $(x) = x$ for all $2 T$; i.e., such that $T^* \text{ Aut}_{C_F(x)}(P)$. In particular, it follows that

$$8(P; x) 2 U: \text{Aut}_S(P) 2 \text{ Syl}_p(\text{Aut}_F(P)) ( C_{C_F(x)}(P) 2 \text{ Syl}_p(\text{Aut}_{C_F(x)}(P));$$

We are now ready to prove condition (I) for $F$; namely, to show for each $P \subseteq S$ fully normalized in $F$ that $P$ is fully centralized and $\text{Aut}_S(P) 2 \text{ Syl}_p(\text{Aut}_F(P))$. By definition, $jN_S(P)j jN_S(P^0)j$ for all $P^0 F$-conjugate to $P$. Choose $x 2 Z(P)$ such that $(P; x) 2 U$; and let $T 2 \text{ Syl}_p(\text{Aut}_F(P))$ be such that $T^* \text{ Aut}_S(P)$ and $x 2 Z(P)^T$. By (a) and (b), there is an element $y 2 \mathcal{X}$ and a homomorphism $2 \text{ Hom}_C(C_S(x); C_S(y))$ such that $(x) = y$. Set $P^0 = P$, and set $T^0 = T^{-1} 2 \text{ Syl}_p(\text{Aut}_F(T^0))$. Since $T^* \text{ Aut}_S(P)$ by definition of $U$, and $(N_S(P)) = N_S(P^0)$ by the maximality assumption, we see that $T^0 \text{ Aut}_S(P^0)$. Also, $y 2 Z(P^0)^T (T^0y = y$ since $Tx = x)$, and this shows that $(P^0; y) 2 U_0$. The maximality of $jN_S(P^0)j = jN_{C_S(y)}(P^0)j$ implies that $P^0$ is fully normalized in $C_F(y)$. Hence by condition (I) for the saturated
fusion system $C_F(y)$, together with (1) and (2), $P$ fully centralized in $F$ and $\text{Aut}_F(P) \cong \text{Syl}_p(\text{Aut}_F(P))$.

It remains to prove condition (II) for $F$. Fix $1 \neq P = S$ and $\mu \text{ Hom}_F(P; S)$ such that $P \overset{\text{def}}{=} \text{Aut}_F(P)$.

We must show that:

$$N \cdot = g \cdot 2 \cdot N_S(P) \cdot c_g^{-1} \cdot 2 \cdot \text{Aut}_S(P)$$

We can now define $N_S(P)$ which is fully centralized in $F$, and set:

$$x \cdot 2 \cdot Z(N \cdot) \text{ and hence } N \cdot C_S(x); \text{ and } N_S(P) \cdot C_S(x): \quad (3)$$

Fix $y$ and $x$ which is $F$ to a homomorphism $-2 \cdot \text{Hom}_F(P; S)$, and choose some $x^0 \cdot Z(P)$ of order $p$ which is central under the action of $\text{Aut}_F(P)$, and set $x = c^{-1} \cdot 2 \cdot Z(P)$. For all $g \cdot 2 \cdot N \cdot , c_g^{-1} \cdot 2 \cdot \text{Aut}_S(P)$, $x$ is $0$, and hence $c_g(x) = x$. Thus:

$$x \cdot 2 \cdot Z(N \cdot) \text{ and hence } N \cdot C_S(x); \text{ and } N_S(P) \cdot C_S(x): \quad (3)$$

Set $= \cdot 0 \cdot 2 \cdot \text{Iso}_F(Q; P)$.

By construction, $(y) = y$, and thus $2 \cdot \text{Iso}_F(y)(Q; P)$. Since $P^0$ is fully centralized in $F$, (4) implies that $Q^0$ is fully centralized in $F$. Hence condition (II), when applied to the saturated fusion system $C_F(y)$, shows that $\cdot$ extends to a homomorphism $-2 \cdot \text{Hom}_F(y)(N; C_S(y))$, where:

$$N \cdot = g \cdot 2 \cdot N_{C_S(y)}(Q) \cdot c_g^{-1} \cdot 2 \cdot \text{Aut}_{C_S(y)}(Q^0):$$

Also, for all $g \cdot 2 \cdot N \cdot , C_S(x)$ (see (3)),

$$c^{-1} = c_g^h \cdot c_g^{-1} = (c_g^h)^{-1} = c_g^h = c_g^h \cdot 2 \cdot \text{Aut}_{C_S(y)}(Q^0)$$

for some $h \cdot 2 \cdot N_S(P)$ such that $c_g^{-1} = c_g$. This shows that $N \cdot \cdot N$; and also (since $C_S(Q^0) = \cdot C_S(P^0)$, by (4)) that:

$$(N \cdot) = \cdot C_S(x)(Q^0):$$

We can now define:

$$\cdot \overset{\text{def}}{=} (c_g^h)^{-1} = (\cdot c_g^h) = 2 \cdot \text{Hom}_F(N \cdot; S);$$

and $\cdot j_p$ = 1.
Proposition 1.1 will also be applied in a separate paper of Carles Broto and Jesper Møller [7] to give a construction of some "exotic" $p$-local finite groups at certain odd primes.

Our goal now is to construct certain saturated fusion systems, by starting with the fusion system of $\text{Spin}_7(q)$ for some odd prime power $q$, and then adding to that the automorphisms of some subgroup of $\text{Spin}_7(q)$. This is a special case of the general problem of studying fusion systems generated by fusion subsystems, and then showing that they are saturated. We first x some notation. If $F_1$ and $F_2$ are two fusion systems over the same group $S$, then $\mathcal{H}(F_1; F_2)$ denotes the fusion system over $S$ generated by $F_1$ and $F_2$: the smallest fusion system over $S$ which contains both $F_1$ and $F_2$. More generally, if $F$ is a fusion system over $S$, and $F_0$ is a fusion system over a subgroup $S_0$ of $S$, then $\mathcal{H}(F; F_0)$ denotes the fusion system over $S$ generated by the morphisms in $F$ between subgroups of $S$, together with morphisms in $F_0$ between subgroups of $S_0$ only. In other words, a morphism in $\mathcal{H}(F; F_0)$ is a composite

\[ P_0 \rightarrow P_1 \rightarrow P_2 \rightarrow \cdots \rightarrow P_{k-1} \rightarrow P_k; \]

where for each $i$, either \( i \text{ Hom}_F(P_{i-1}; P_i) \), or \( i \text{ Hom}_{F_0}(P_{i-1}; P_i) \) (and $P_{i-1}; P_i$ $S_0$).

As usual, when $G$ is a finite group and $S \triangleleft \text{Syl}_p(G)$, then $F_S(G)$ denotes the fusion system of $G$ over $S$. If $\Gamma \triangleleft \text{Aut}(G)$ is a group of automorphisms which contains $\text{Inn}(G)$, then $F_S(\Gamma)$ will denote the fusion system over $S$ whose morphisms consist of all restrictions of automorphisms in $\Gamma$ to monomorphisms between subgroups of $S$.

The next proposition provides some fairly specialized conditions which imply that the fusion system generated by the fusion system of a group $G$ together with certain automorphisms of a subgroup of $G$ is saturated.

**Proposition 1.2** Fix a finite group $G$, a prime $p$ dividing $|G|$, and a Sylow $p$-subgroup $S \triangleleft \text{Syl}_p(G)$. Fix a normal subgroup $Z \unlhd G$ of order $p$, an elementary abelian subgroup $U \unlhd S$ of rank two containing $Z$ such that $C_S(U) \triangleleft \text{Syl}_p(C_G(U))$, and a subgroup $\Gamma \triangleleft \text{Aut}(C_G(U))$ containing $\text{Inn}(C_G(U))$ such that $\gamma(U) = U$ for all $\gamma \in \Gamma$. Set

\[ S_0 = C_S(U) \quad \text{and} \quad F \overset{\text{def}}{=} \mathcal{H}(F_S(G); F_{S_0}(\Gamma)); \]

and assume the following hold.

(a) All subgroups of order $p$ in $S$ different from $Z$ are $G$-conjugate.
(b) $\Gamma$ permutes transitively the subgroups of order $p$ in $U$.

(c) $f' \ 2 \ \Gamma \ j' (Z) = Zg = Aut_{N_G(U)}(C_G(U))$.

(d) For each $E \in S$ which is elementary abelian of rank three, contains $U$, and is fully centralized in $F_S(G)$,

$$f \ 2 \ \text{Aut}_F(C_S(E)) j' (Z) = Zg = \text{Aut}_G(C_S(E)),$$

(e) For all $E, E^0 \in S$ which are elementary abelian of rank three and contain $U$, if $E$ and $E^0$ are $\Gamma$-conjugate, then they are $G$-conjugate.

Then $F$ is a saturated fusion system over $S$. Also, for any $P \in S$ such that $Z = P$,

$$f' \ 2 \ \text{Hom}_G(P; S) j' (Z) = Zg = \text{Hom}_G(P; S). \quad (1)$$

Proposition 1.2 follows from the following three lemmas. Throughout the proofs of these lemmas, references to points (a)-(e) mean to those points in the hypotheses of the proposition, unless otherwise stated.

**Lemma 1.3** Under the hypotheses of Proposition 1.2, for any $P \in S$ and any central subgroup $Z^0 \leq Z(P)$ of order $p$,

$$Z \notin Z^0 \ U \Rightarrow 9' \ 2 \ \text{Hom}_G(P; S_0) \text{ such that } j' (Z^0) = Z \quad (1)$$

and

$$Z^0 \leq U \Rightarrow 9 \ 2 \ \text{Hom}_G(P; S_0) \text{ such that } j' (Z^0) U. \quad (2)$$

**Proof** Note first that $Z \leq Z(S)$, since it is a normal subgroup of order $p$ in a $p$-group.

Assume $Z \notin Z^0 \ U$. Then $U = ZZ^0$, and

$$P \in C_S(Z^0) = C_S(ZZ^0) = C_S(U) = S_0$$

since $Z = Z(P)$ by assumption. By (b), there is $2 \ \Gamma$ such that $j' (Z^0) = Z$. Since $S_0 \ 2 \ \text{Syl}_p(C_G(U))$, there is $h \ 2 \ C_G(U)$ such that $\hbar (P; h^{-1}) S_0$; and since

$$c_h 2 \ \text{Aut}_{N_G(U)}(C_G(U)) \ \Gamma$$

by (c), $\defeq c_h 2 \ \text{Hom}_G(P; S_0)$ and sends $Z^0$ to $Z$.

If $Z^0 \leq U$, then by (a), there is $g \ 2 \ G$ such that $gZ^0g^{-1} \ U \setminus Z$. Since $Z$ is central in $S$, $gZ^0g^{-1}$ is central in $gP \ g^{-1}$, and $U$ is generated by $Z$ and $gZ^0g^{-1}$, it follows that $gP \ g^{-1} \ C_G(U)$. Since $S_0 \ 2 \ \text{Syl}_p(C_G(U))$, there is $h \ 2 \ C_G(U)$ such that $h(gP \ g^{-1}h^{-1}) S_0$; and we can take $c_{h^0} 2 \ \text{Hom}_G(P; S_0)$. \qed
Lemma 1.4 Assume the hypotheses of Proposition 1.2, and let
\[ F = \text{Mor}_G(\mathbb{Z}; \mathbb{Z}) \]
be the fusion system generated by \( G \) and \( \Gamma \). Then for all \( P; P^0 \rightarrow S \) which contain \( Z \),
\[ f' : 2 \text{Hom}_I(P; P^0), j'(Z) = Zg = \text{Hom}_G(P; P^0) : \]

Proof Upon replacing \( P^0 \) by \( ' (P) \rightarrow P \), we can assume that \( ' \) is an isomorphism, and thus that it factors as a composite of isomorphisms
\[ P = P_0 \rightarrow P_1 \rightarrow P_2 \rightarrow \cdots \rightarrow P_k \rightarrow P = P^0, \]
where for each \( i \), \( i : 2 \text{Hom}_G(P_{i-1}; P_i) \) or \( i : 2 \text{Hom}_G(P_{i-1}; P_i) \) be the subgroups of order \( p \) such that \( Z_0 = Z_k = Z \) and \( Z_i = i(Z_{i-1}) \).

To simplify the discussion, we say that a morphism in \( F \) is of type \( (G) \) if it is given by conjugation by an element of \( G \), and of type \( (\Gamma) \) if it is the restriction of an automorphism of \( G \). More generally, we say that a morphism is of type \( (G; \Gamma) \) if it is the composite of a morphism of type \( (G) \) followed by one of type \( (\Gamma) \), etc. We regard Id_{\Gamma}, for all \( P \rightarrow S \), to be of both types, even if \( P \not< S_0 \).

For each \( i \), using Lemma 1.3, choose some \( i : 2 \text{Hom}_G(P_i; U; S) \) such that \( i(Z_i) = Z \). More precisely, using points (1) and (2) in Lemma 1.3, we can choose \( i \) to be of type \( (\Gamma) \) if \( Z_i \not< U \) (the inclusion if \( Z_i = Z \)), and to be of type \( (G; \Gamma) \) if \( Z_i U \not< U \). Set \( P^0_i = i(P_i) \). To keep track of the effect of morphisms on the subgroups \( Z_i \), we write them as morphisms between pairs, as shown below. Thus, \( ' \) factors as a composite of isomorphisms
\[ (P_0; Z) \rightarrow (P_1; Z_1) \rightarrow \cdots \rightarrow (P_k; Z_k) : \]

If \( ' \) is of type \( (G) \), then this composite (after replacing adjacent morphisms of the same type by their composite) is of type \( (G; \Gamma) \). If \( ' \) is of type \( (\Gamma) \), then the composite is again of type \( (\Gamma; G; \Gamma) \) if either \( Z_{i-1} \not< U \) or \( Z_i \not< U \), and is of type \( (\Gamma; G; \Gamma; G; \Gamma) \) if neither \( Z_{i-1} \not< \) nor \( Z_i \) is contained in \( U \). So we are reduced to assuming that \( ' \) is of one of these two forms.

Case 1 Assume first that \( ' \) is of type \( (\Gamma; G; \Gamma) \); i.e., a composite of isomorphisms of the form
\[ (P_0; Z) \rightarrow (P_1; Z_1) \rightarrow (P_2; Z_2) \rightarrow (P_3; Z) : \]
Then $Z_1 = Z$ if and only if $Z_2 = Z$ because $'_2$ is of type $(G)$. If $Z_1 = Z_2 = Z$, then $'_1$ and $'_3$ are of type $(G)$ by (c), and the result follows.

If $Z_1 \not\subset Z \not\subset Z_2$, then $U = ZZ_2 = ZZ_2$, and thus $'_2(U) = U$. Neither $'_1$ nor $'_3$ can be the identity, so $P_i \cong C_5(U)$ for all $i$ by definition of $\text{Hom}_T(-; -)$, and hence $'_2$ is of type $(\Gamma)$ by (c). It follows that $Z$ is of type $(G)$ by (c) again.

**Case 2** Assume now that $'$ is of type $(\Gamma; G; \Gamma; G; \Gamma)$; more precisely, that it is a composite of the form

$$
\begin{align*}
(P_0; Z) \xrightarrow{\iota_4} (P_1; Z_1) \xrightarrow{\iota_3} (P_2; Z_2) \xrightarrow{\iota_2} (P_3; Z_3) \xrightarrow{\iota_1} (P_4; Z_4) \xrightarrow{\iota_0} (P_5; Z);
\end{align*}
$$

where $Z_2; Z_3 \not\subset U$. Then $Z_1; Z_4 \not\subset U$ and are distinct from $Z$, and the groups $P_0; P_1; P_4; P_5$ all contain $U$ since $'_1$ and $'_5$ (being of type $(\Gamma)$) leave $U$ invariant. In particular, $P_2$ and $P_3$ contain $Z$, since $P_1$ and $P_4$ do and $'_2; '_4$ are of type $(G)$. We can also assume that $U = P_2; P_3$, since otherwise $P_2 \setminus U = Z$ or $P_3 \setminus U = Z$, $'_3(Z) = Z$, and hence $'_3$ is of type $(G)$ by (c) again. Finally, we assume that $P_2; P_3 \cong C_5(U)$, since otherwise $'_3 = \text{Id}$.

Let $E_i \cong P_i$ be the rank three elementary abelian subgroups defined by the requirements that $E_2 = UZ_2$, $E_3 = UZ_3$, and $'_i(E_{i-1}) = E_i$. In particular, $E_i = Z(P_i)$ for $i = 2; 3$ (since $Z \subset Z(P_i)$, and $U \subset Z(P_i)$ by the above remarks); and hence $E_i \cong Z(P_i)$ for all $i$. Also, $U = ZZ_4$ since $'_4(Z) = Z$, and thus $U = '_5(U) \supset E_5$. Via similar considerations for $E_0$ and $E_1$, we see that $U \cap E_i$ for all $i$.

Set $H = C_G(U)$ for short. Let $E_3$ be the set of all elementary abelian subgroups $E \subset S$ of rank three which contain $U$, and with the property that $C_S(E) \cong \text{Syl}_p(C_H(E))$. Since $C_5(E) \cong C_5(U) = S_0 \cong \text{Syl}_p(H)$, the last condition implies that $E$ is fully centralized in the fusion system $F_{S_0}(H)$. If $E \subset S$ is any rank three elementary abelian subgroup which contains $U$, then there is some $a \in H$ such that $E^a = aEa^{-1}$ is of type $(\Gamma)$; hence $F_{S_0}(H)$ is saturated and $U \not\subset H$. Then $C_5 \cong \text{Iso}(E; E^9) \setminus \text{Iso}(E; E^9)$ by (c). So upon composing with such isomorphisms, we can assume that $E_i \subset E_3$ for all $i$, and also that $'_i(C_5(E_{i-1})) = C_5(E_i)$ for each $i$.

In this way, $'$ can be assumed to extend to an $F$-isomorphism $'$ from $C_5(E_0)$ to $C_5(E_5)$ which sends $Z$ to itself. By (e), the rank three subgroups $E_i$ are all $G$-conjugate to each other. Choose $g \in G$ such that $gEg^{-1} = E_0$. Then $gC_5(E_5)g^{-1}$ and $C_5(E_0)$ are both $\text{Syl}_p$-subgroups of $C_G(E_0)$, so there is $h \in C_G(E_0)$ such that $(hg)C_5(E_5)(hg)^{-1} = C_5(E_0)$. By (d), $g_{hg} \cong 2 \text{Aut}_F(C_5(E_0))$ is of type $(G)$; and thus $'_2 \cong \text{Iso}(P_0; P_5)$.

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To finish the proof of Proposition 1.2, it remains only to show:

**Lemma 1.5** Under the hypotheses of Proposition 1.2, the fusion system $F$ generated by $F_S(G)$ and $F_{S_0}(\Gamma)$ is saturated.

**Proof** We apply Proposition 1.1, by letting $\mathcal{X}$ be the set of generators of $Z$. Condition (a) of the proposition (every $x \in S$ of order $p$ is $F$-conjugate to an element of $\mathcal{X}$) holds by Lemma 1.3. Condition (c) holds since $C_F(Z)$ is the fusion system of the group $C_G(Z)$ by Lemma 1.4, and hence is saturated by [6, Proposition 1.3].

It remains to prove condition (b) of Proposition 1.1. We must show that if $y; z \in S$ are $F$-conjugate and $hZ = Z$, then there is $2 \text{Hom}_F(C_S(y); C_S(z))$ such that $(y) = z$. If $y \in U$, then by Lemma 1.3(2), there is $2 \text{Hom}_F(C_S(y); S_0)$ such that $(y) \in U$. If $y \not\in U \setminus Z$, then by Lemma 1.3(1), there is $2 \text{Hom}_F(C_S(y); S_0)$ such that $(y) \in Z$. We are thus reduced to the case where $y; z \in Z$ (and are $F$-conjugate).

In this case, then by Lemma 1.4, there is $g \in G$ such that $z = g y g^{-1}$. Since $Z \triangleleft G$, $[G:C_G(Z)]$ is prime to $p$, so $S$ and $gSg^{-1}$ are both Sylow $p$-subgroups of $C_G(Z)$, and hence are $C_G(Z)$-conjugate. We can thus choose $g$ such that $z = g y g^{-1}$ and $gSg^{-1} = S$. Since $C_S(y) = C_S(z) = S (Z \triangleleft Z(S)$ since it is a normal subgroup of order $p$), this shows that $c_G 2 \text{Iso}_G(C_S(y); C_S(z))$, and finishes the proof of (b) in Proposition 1.1. 

## 2 A fusion system of a type considered by Solomon

The main result of this section and the next is the following theorem:

**Theorem 2.1** Let $q$ be an odd prime power, and $x \in S \triangleleft Syl_2(\text{Spin}_7(q))$. Let $Z(\text{Spin}_7(q))$ be the central element of order 2. Then there is a saturated fusion system $F = F_{\text{Sol}}(q)$ which satisfies the following conditions:

(a) $C_F(z) = F_S(\text{Spin}_7(q))$ as fusion systems over $S$.

(b) All involutions of $S$ are $F$-conjugate.

Furthermore, there is a unique centric linking system $L = L^c_{\text{Sol}}(q)$ associated to $F$. 

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Theorem 2.1 will be proven in Propositions 2.11 and 3.3. Later, at the end of Section 3, we explain why Solomon’s theorem [22] implies that these fusion systems are not the fusion systems of any finite groups, and hence that the spaces $B\Sol(q)$ are not homotopy equivalent to the 2-completed classifying spaces of any finite groups.

Background results needed for computations in Spin$(V;b)$ have been collected in Appendix A. We focus attention here on $SO_7(q)$ and $Spin_7(q)$. In fact, since we want to compare the constructions over $\mathbb{F}_q$ with those over its field extensions, most of the constructions will first be made in the groups $SO_7(\mathbb{F}_q)$ and $Spin_7(\mathbb{F}_q)$.

We now fix, for the rest of the section, an odd prime power $q$. It will be convenient to write $Spin_7(q^1) = Spin_7(\mathbb{F}_q)$, etc. In order to make certain computations more explicit, we set

$$V_1 = M_2(\mathbb{F}_q), \quad M_2^0(\mathbb{F}_q) = (\mathbb{F}_q)^7 \quad \text{and} \quad b(A;B) = \det(A) + \det(B).$$

(where $M_2^0(-)$ is the group of $(2\times 2)$ matrices of trace zero), and for each $n \geq 1$ set $V_n = M_2(\mathbb{F}_q^n), \quad M_2^0(\mathbb{F}_q^n) \quad V_1$. Then $b$ is a nonsingular quadratic form on $V_1$ and on $V_n$. Identify $SO_7(q^1) = SO(V_1; b)$ and $SO_7(q^n) = SO(V_n; b)$, and similarly for $Spin_7(q^1)$ and $Spin_7(q^n)$. For all $2 Spin(M_2(\mathbb{F}_q); \det)$ and $2 Spin(M_2^0(\mathbb{F}_q); \det)$, we write for their image in $Spin_7(q^1)$ under the natural homomorphism

$$4:3: Spin_4(q^1) \quad Spin_3(q^1) \quad Spin_7(q^1).$$

There are isomorphisms

$$e_4: SL_2(q^1) \quad SL_2(q^1) \rightarrow Spin_4(q^1) \quad \text{and} \quad e_3: SL_2(q^1) \rightarrow Spin_3(q^1)$$

which are defined explicitly in Proposition A.5, and which restrict to isomorphisms

$$SL_2(q^1) = Spin_4(q^1)$$

$$SL_2(q^n) = Spin_3(q^n)$$

for each $n$. Let

$$z = e_4(-I;-I) \quad 1 = 1 \quad e_3(-I) 2 Z(Spin_7(q))$$

denote the central element of order two, and set

$$z_1 = e_4(-I;1) \quad 12 Spin_7(q):$$

Here, $12 Spin_k(q)$ ($k = 3; 4$) denotes the identity element. Define $U = hz; z_1 i$. 

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Definition 2.2 Define 

\[ ! : SL_2(q^1)^3 \longrightarrow Spin_7(q^1) \]

by setting

\[ ! (A_1; A_2; A_3) = e_4(A_1; A_2) e_3(A_3) \]

for \( A_1; A_2; A_3 \in SL_2(q^1) \). Set

\[ H(q^1) = ! (SL_2(q^1)^3) \quad \text{and} \quad [A_1; A_2; A_3] = ! (A_1; A_2; A_3) : \]

Since \( e_3 \) and \( e_4 \) are isomorphisms, \( \text{Ker}(!) = \text{Ker}(4;3) \), and thus

\[ \text{Ker}(!) = h(−I; −I; −I) i : \]

In particular, \( H(q^1) = (SL_2(q^1)^3) \neq (I;I;I)g \). Also,

\[ z = [I;I;−I] \quad \text{and} \quad z_1 = [−I;I;I] ; \]

and thus

\[ U = [ I; I; I ] \]

(with all combinations of signs).

For each \( n < 1 \), the natural homomorphism

\[ Spin_7(q^n) \longrightarrow SO_7(q^n) \]

has kernel and cokernel both of order 2. The image of this homomorphism is the commutator subgroup \( \Omega_7(q^n) \triangleleft SO_7(q^n) \), which is partly described by Lemma A.4(a). In contrast, since all elements of \( F_q \) are squares, the natural homomorphism from \( Spin_7(q^1) \) to \( SO_7(q^1) \) is surjective.

Lemma 2.3 There is an element \( 2 N_{Spin_7(q)}(U) \) of order 2 such that

\[ [A_1; A_2; A_3]^{-1} = [A_2; A_1; A_3] \] (1)

for all \( A_1; A_2; A_3 \in SL_2(q^1) \).

Proof Let \( 2 SO_7(q) \) be the involution defined by setting

\[ −(X; Y) = (−(X); −Y) \]

for \( (X; Y) \in V_1 = M_2(F_q) \cap M_2(F_q) \), where

\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} : \]

Let \( 2 Spin_7(q^1) \) be a lifting of \( − \). The \((-1)\)eigenspace of \( − \) on \( V_1 \) has orthogonal basis

\[ (I;0); 0; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; 0; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; 0; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ; \]

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and in particular has discriminant 1 with respect to this basis. Hence by Lemma A.4(a), $-2 \Omega_7(q)$, and so $2 \text{Spin}_7(q)$. Since in addition, the $(-1)$ eigenspace of $-1$ is 4-dimensional, Lemma A.4(b) applies to show that $-2 = 1$.

By definition of the isomorphisms $e_3$ and $e_4$, for all $A_i \ 2 \text{SL}_2(q^i)$ $(i = 1; 2; 3)$ and all $(X; Y) \ 2 \ V_1$, 

$$[A_1; A_2; A_3][X; Y] = (A_1 X A_2^{-1}; A_3 Y A_3^{-1});$$

Here, $\text{Spin}_7(q)$ acts on $V_1$ via its projection to $\text{SO}_7(q)$. Also, for all $X; Y \ 2 \ M_2(\mathbb{F}_q)$,

$$(X) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1}$$

and in particular $(X Y) = (Y)(X)$; and $(X) = X^{-1}$ if $\det(X) = 1$. Hence for all $A_1; A_2; A_3 \ 2 \text{SL}_2(q^i)$ and all $(X; Y) \ 2 \ V_1$,

$$[A_1; A_2; A_3]^{-1}(X; Y) = (-A_1 (X) A_2^{-1}; -A_3 Y A_3^{-1})$$

$$= (A_2 X A_1^{-1}; A_3 Y A_3^{-1}) = [A_2; A_1; A_3][X; Y];$$

This shows that (1) holds modulo $\mathbb{Z}$ is $\mathbb{Z}(\text{Spin}_7(q^i)))$. We thus have two automorphisms of $H(q^i) = (\text{SL}_2(q^i)^3)$ of $\text{Spin}_7(q^i)$, and the permutation automorphism $\text{Spin}_7(q^i)$, which are liftings of the same automorphism of $H(q^i)$, respectively, since each automorphism of $H(q^i)$ is a lifting of an automorphism of $H(q^i)$, and thus (1) holds. Also, since $U$ is the subgroup of all elements $[1; 1; 1]$ with all combinations of signs, formula (1) shows that $2 \text{N}_{\text{Spin}_7(q)}(U)$.}

**Definition 2.4** For each $n$, set

$$H(q^n) = H(q^n) \backslash \text{Spin}_7(q^n) \quad \text{and} \quad H_0(q^n) = ! (\text{SL}_2(q^n)^3) \quad H(q^n):$$

Define

$$\Gamma_n = \text{Inn}(H(q^n)) \rtimes b_3 \quad \text{Aut}(H(q^n));$$

where $b_3$ denotes the group of permutation automorphisms

$$b_3 = [A_1; A_2; A_3][A_1; A_2; A_3] \quad 2 \ 3 \quad \text{Aut}(H(q^n));$$

For each $n$, let $q^n$ be the automorphism of $\text{Spin}_7(q^n)$ induced by the $n$-th isomorphism $(q^n \ Y \ q^{n'})$. By Lemma A.3, $\text{Spin}_7(q^n)$ is the $\text{xed subgroup of } q^n$. Hence each element of $H(q^n)$ is of the form $[A_1; A_2; A_3]$, where either $A_i \ 2 \text{SL}_2(q^n)$ for each $i$ (and the element lies in $H_0(q^n)$), or $q^n(A_i) = -A_i$ for each $i$. This shows that $H_0(q^n)$ has index 2 in $H(q^n)$.

The goal is now to choose compatible Sylow subgroups $S(q^n) \ 2 \text{Syl}_2(\text{Spin}_7(q^n))$ (all $n \ 1$) contained in $N(H(q^n))$, and let $F_{\text{Sol}}(q^n)$ be the fusion system over $S(q^n)$ generated by conjugation in $\text{Spin}_7(q^n)$ and by restrictions of $\Gamma_n$.

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Proposition 2.5  The following hold for each \( n \geq 1 \).

(a) \( H(q^n) = C_{\text{Spin}_7(q^n)}(U) \).

(b) \( N_{\text{Spin}_7(q^n)}(U) = N_{\text{Spin}_7(q^n)}(H(q^n)) = H(q^n) \text{ h i} \), and contains a Sylow \( 2 \) subgroup of \( \text{Spin}_7(q^n) \).

Proof  Let \( z_1 \) be the image of \( z_1 \) in \( \text{Spin}_7(q) \). Set \( V_+ = M_2(F_q) \) and \( V_- = M_2(F_q) \): the eigenspaces of \( z_1 \) acting on \( V \). By Lemma A.4(c),
\[
C_{\text{Spin}_7(q^n)}(U) = C_{\text{Spin}_7(q^n)}(z_1)
\]
is the group of all elements \( 2 \text{ Spin}_7(q^n) \) whose image \( 2 \text{ SO}_7(q^n) \) has the form
\[
- = - + \quad \text{where} \quad 2 \text{ SO}(V);
\]
In other words,
\[
C_{\text{Spin}_7(q^n)}(U) = 4:3 \text{ Spin}_4(q^n), \text{ Spin}_3(q^n) = 1 (\text{SL}_2(q^n)^3) = H(q^n);
\]
Furthermore, since
\[
z_1^{-1} = [-I; I; I]^{-1} = [I; -I; I] = zz_1
\]
by Lemma 2.3, and since any element of \( N_{\text{Spin}_7(q^n)}(U) \) centralizes \( z \), conjugation by \( \text{centralizes \( z \), conjugation by } \text{ generates Out}_{\text{Spin}_7(q^n)}(U) \). Hence
\[
N_{\text{Spin}_7(q^n)}(U) = H(q^n) \text{ h i}.
\]
Point (a), and the first part of point (b), now follow upon taking intersections with \( \text{Spin}_7(q^n) \).

If \( N_{\text{Spin}_7(q^n)}(U) \) did not contain a Sylow \( 2 \) subgroup of \( \text{Spin}_7(q^n) \), then since every noncentral involution of \( \text{Spin}_7(q^n) \) is conjugate to \( z_1 \) (Proposition A.8), the Sylow \( 2 \) subgroups of \( \text{Spin}_7(q^n) \) would have no normal subgroup isomorphic to \( C_2^2 \). By a theorem of Hall (cf [15, Theorem 5.4.10]), this would imply that they are cyclic, dihedral, quaternion, or semidihedral. This is clearly not the case, so \( N_{\text{Spin}_7(q^n)}(U) \) must contain a Sylow \( 2 \) subgroup of \( \text{Spin}_7(q^n) \), and this finishes the proof of point (b).

Alternatively, point (b) follows from the standard formulas for the orders of these groups (cf [24, pages 19, 140]), which show that
\[
\frac{j_{\text{Spin}_7(q^n)}}{j_{H(q^n) \text{ h i}}} = \frac{q^{2n}(q^{2n} - 1)(q^{2n} - 1)(q^{2n} - 1)}{2 [q^n(q^{2n} - 1)]^3} = q^{6n}(q^{2n} + q^{2n} + 1) \frac{q^{2n} + 1}{2}
\]
is odd.

\[ \square \]
Following the notation of Definition A.7, we say that an elementary abelian 2-subgroup of $\text{Spin}_7(q^n)$ which is contained in $H(q^n)$ is $i = N_{\text{Spin}_7(q^n)}(U)$.

**Definition 2.6** Fix elements $A; B \in \text{SL}_2(q)$ such that $hA; Bi = Q_8$ (a quaternion group of order 8), and set $\mathcal{A} = [A; A; A]$ and $\mathcal{B} = [B; B; B]$. Let $C(q^3)$, $C_{\text{SL}_2(q^3)}(A)$ be the subgroup of elements of 2-power order in the centralizer (which is abelian), and set $Q(q^3) = hC(q^3); Bi$. Define

$$S_0(q^3) = ! (Q(q^3)^3) \quad H_0(q^3)$$

and

$$S(q^3) = S_0(q^3) \text{ h i } H(q^3) \quad \text{Spin}_7(q^3):$$

Here, $2\text{Spin}_7(q)$ is the element of Lemma 2.3. Finally, for each $n \geq 1$, define

$$C(q^n) = C(q^3) \setminus \text{SL}_2(q^3); \quad Q(q^n) = Q(q^3) \setminus \text{SL}_2(q^3);$$

$$S_0(q^n) = S_0(q^3) \setminus \text{Spin}_7(q^3); \quad \text{and } S(q^n) = S(q^3) \setminus \text{Spin}_7(q^3):$$

Since the two eigenvalues of $A$ are distinct, its centralizer in $\text{SL}_2(q^3)$ is conjugate to the subgroup of diagonal matrices, which is abelian. Thus $C(q^3)$ is conjugate to the subgroup of diagonal matrices of 2-power order. This shows that each finite subgroup of $C(q^3)$ is cyclic, and that each finite subgroup of $Q(q^3)$ is cyclic or quaternion.

**Lemma 2.7** For all $n$, $S(q^n)$ contains a Sylow 2-subgroup of $\text{Spin}_7(q^n)$.

**Proof** By [23, 6.23], $A$ is contained in a cyclic subgroup of order $q^n - 1$ or $q^n + 1$ (depending on which of them is divisible by 4). Also, the normalizer of this cyclic subgroup is a quaternion group of order 2$(q^n - 1)$, and the formula $j_{\text{SL}_2(q^n)} = q^n(q^{2n} - 1)$ shows that this quaternion group has odd index. Thus by construction, $Q(q^n)$ is a Sylow 2-subgroup of $\text{SL}_2(q^n)$. Hence $! (Q(q^n)^3)$ is a Sylow 2-subgroup of $H_0(q^n)$, so $(Q(q^n)^3) \setminus \text{Spin}_7(q^n)$ is a Sylow 2-subgroup of $H(q^n)$. It follows that $S(q^n)$ contains a Sylow 2-subgroup of $H(q^n)$, and hence also of $\text{Spin}_7(q^n)$ by Proposition 2.5(b).

Following the notation of Definition A.7, we say that an elementary abelian 2-subgroup of $\text{Spin}_7(q^n)$ has type I if its eigenspaces all have square discriminant, and has type II otherwise. Let $E_r$ be the set of elementary abelian subgroups of rank $r$ in $\text{Spin}_7(q^n)$ which contain $z$, and let $E^1_r$ and $E^2_r$ be the sets of those of type I or II, respectively. In Proposition A.8, we show that there are two conjugacy classes of subgroups in $E^1_1$ and one conjugacy class of...
subgroups in $E_4$. In Proposition A.9, an invariant $x_C(E)$ of $E$ is defined, for all $E$, $E_4$ (and where $C$ is one of the conjugacy classes in $E_4$) as a tool for determining the conjugacy class of a subgroup. More precisely, $E$ has type I if and only if $x_C(E) = 1$, and $E$, $E_4$ if and only if $x_C(E) = 1$. The next lemma provides some more detailed information about the rank four subgroups and these invariants.

Recall that we define $\hat{A} = [A; A; A]$ and $\hat{B} = [B; B; B]$.

**Lemma 2.8** Fix $n = 1$, set $E = \langle z; z_1, \hat{A}, \hat{B}, S(q') \rangle$, and let $C$ be the Spin$_7(q')$-conjugacy class of $E$. Let $E^U$ be the set of all elementary abelian subgroups $E \subseteq S(q')$ of rank 4 which contain $U = \langle z, z_1 \rangle$. Fix a generator $X \in C(q')$ (the 2-power torsion in $C_{S_5}(q')$), and choose $Y \in C(q^{2n})$ such that $Y^2 = X$. Then the following hold.

(a) $E$ has type I.
(b) $E^U = E_{ijk}; E^0_{ijk}; X^j; X^k; X^{ij}; X^{ik}; X^{jk}; X^{ijk}; X^i; X^j; X^k; X$ (a finite set), where

$$E_{ijk} = \langle z, z_1; \hat{A}, \hat{B}, X^i; X^j; X^k \rangle.$$

and

$$E^0_{ijk} = \langle z, z_1; \hat{A}, \hat{B}, X^i; X^j; X^k; X \rangle.$$

(c) $x_C(E_{ijk}) = \langle (1) \rangle; (1); (1) \rangle \rangle$ and $x_C(E^0_{ijk}) = \langle (1) \rangle; (1); (1) \rangle \rangle \hat{A}$.

(d) All of the subgroups $E^0_{ijk}$ have type II. The subgroup $E^0_{ijk}$ has type I if and only if $i \equiv j \equiv k \pmod{2}$, and lies in $C$ (is conjugate to $E$) if and only if $i \equiv j \equiv k \pmod{2}$. The subgroups $E_{000}$, $E_{001}$, and $E_{100}$ thus represent the three conjugacy classes of rank four elementary abelian subgroups of Spin$_7(q')$ (and $E = E_{000}$).

(e) For any $\Gamma_n \subseteq \text{Aut}(H(q'))$ (see Definition 2.4), if $E_{000}$, $E_{001}$, $2 E_{000}$ are such that $\langle E_0 \rangle = E_{000}$, then $\langle x_C(E_0) \rangle = x_C(E_0)$.

**Proof (a)** The set

$$(1; 0); (A; 0); (B; 0); (AB; 0); (0; A); (0; B); (0; AB)$$

is a basis of eigenvectors for the action of $E$ on $V_n = M_2(\mathbb{F}_{q^2}) = M_2(\mathbb{F}_{q^2})$.

(Since the matrices $A$, $B$, and $AB$ all have order 4 and determinant one, each has as eigenvalues the two distinct fourth roots of unity, and hence they all have trace zero.) Since all of these have determinant one, $E$ has type I by definition.
(b) Consider the subgroups

\[ R_0 = \left\{ \sigma \in C(q^2)^3 \mid S(q^2) = [X^i; X^j; X^k] [X^i Y; X^j Y; X^k Y] i; j; k \neq 2 \right\} \]

and

\[ R_1 = C_{S(q^2)}(H^i; A_i) = R_0 \Omega_i: \]

Clearly, each subgroup \( E \) of \( E\) is contained in

\[ C_{S(q^2)}(U) = S_0(q^2) = R_0 [B^i; B^j; B^k]: \]

All involutions in this subgroup are contained in \( R_1 = R_0 [B; B; B], \) and thus \( E \) is contained in \( R_1 \). Hence \( E \) has rank 3, which implies that \( E \) is an involutory \( A \) in the 2-torsion subgroup. Since all elements of order two in the coset \( R_0 \Omega_i \) have the form

\[ [X^i B; X^j B; X^k B] \text{ or } [X^i Y B; X^j Y B; X^k Y B] \]

for some \( i; j; k \), this shows that \( E \) must be one of the groups \( E_{ijk} \) or \( E_{ij} \).

(Note in particular that \( E = E_{000} \).)

(c) By Proposition A.9(a), the element \( x_{C}(E) \) is characterized uniquely by the property that \( x_{C}(E) = g^{-1} q^i(g) \) for some \( g \in \text{Spin}(q^2) \) such that \( gEg^{-1} \) is a 2-element group. We now apply this explicitly to the subgroups \( E_{ijk} \) and \( E_{ij} \).

For each \( i \), \( Y^{-1}(X^i B)Y^i = Y^{-1}X^i B = B \). Hence for each \( i; j; k \),

\[ [Y^i, Y^j, Y^k]^{-1} E_{ijk} [Y^i, Y^j, Y^k] = E \]

and

\[ q^i ([Y^i, Y^j, Y^k]) = [Y^i, Y^j, Y^k] [-i)^i; (-i)^j; (-i)^k]: \]

Hence

\[ x_{C}(E_{ijk}) = [-i, -j, -k]: \]

Similarly, if we choose \( Z \) in \( C_{SL_2(q^2)}(A) \) such that \( Z^2 = A \), then for each \( i \),

\[ (Y^i Z)^{-1}(X^i Y B)(Y^i Z) = B: \]

Hence for each \( i; j; k \),

\[ [Y^i Z, Y^j Z, Y^k Z]^{-1} E_{ijk} [Y^i Z, Y^j Z, Y^k Z] = E. \]

Since \( q^i(Z) = Z A, \)

\[ q^i ([Y^i Z, Y^j Z, Y^k Z]) = [Y^i Z, Y^j Z, Y^k Z] [-i)^i A; (-i)^j A; (-i)^k A]: \]

and hence

\[ x_{C}(E_{ijk}^0) = [-i) A; (-i)^j A; (-i)^k A]: \]

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This now follows immediately from point (c) and Proposition A.9(b,c).

By Definition 2.4, \( \Gamma_n \) is generated by \( \text{Inn}(H(q^n)) \) and the permutations of the three factors in \( H(q^n) = (\text{SL}_2(q^n))^3 \). If \( q \) is a permutation automorphism, then it permutes the elements of \( E_n^0 \), and preserves the elements \( x_C(-) \) by the formulas in (c). If \( 2 \ \text{Inn}(H(q^n)) \) and \( (E^n) = E^n_0 \) for \( E^n_0 E^n_0 E^n_0 \), then \( (x_C(E^n)) = x_C(E^n_0) \) by definition of \( x_C(-) \); and so the same property holds for all elements of \( \Gamma_n \).

Following the notation introduced in Section 1, \( \text{Hom}_{\text{Spin}_7}(P;Q) \) for \( P;Q \) denotes the set of homomorphisms from \( P \) to \( Q \) induced by conjugation by some element of \( \text{Spin}_7(q^n) \). Also, if \( P;Q \) \( S(q^n) \) \( H(q^n) \), \( \text{Hom}_{\Gamma_n}(P;Q) \) denotes the set of homomorphisms induced by restriction of an element of \( \Gamma_n \). Let \( F_n = F_{\text{ad}}(q^n) \) be the fusion system over \( S(q^n) \) generated by \( \text{Spin}_7(q^n) \) and \( \Gamma_n \). In other words, for each \( P;Q \) \( S(q^n) \), \( \text{Hom}_{\Gamma_n}(P;Q) \) is the set of all composites

\[
P = P_0 \rightarrow P_1 \rightarrow P_2 \rightarrow \cdots \rightarrow P_{k-1} \rightarrow P_k = Q;
\]

where \( P_i \) \( S(q^n) \) for all \( i \), and each \( P_i \) lies in \( \text{Hom}_{\text{Spin}_7}(P_{i-1};P_i) \) or \( \text{Hom}_{\Gamma_n}(P_{i-1};P_i) \). This clearly defines a fusion system over \( S(q^n) \).

**Proposition 2.9** Fix \( n \). Let \( E \) \( S(q^n) \) be an elementary abelian subgroup of rank 3 which contains \( U \), and such that

\[
C_{S(q^n)}(E) \leq Syl_2(C_{\text{Spin}_7}(q^n)(E));
\]

Then

\[
f' \ 2 \ \text{Aut}_{F_n}(C_{S(q^n)}(E))j' (z) = zg = \text{Aut}_{\text{Spin}_7(q^n)}(C_{S(q^n)}(E)); \quad (1)
\]

**Proof** Set

\[
\text{Spin} = \text{Spin}_7(q^n); \quad S = S(q^n); \quad \Gamma = \Gamma_n; \quad \text{and} \quad F = F_n
\]

for short. Consider the subgroups

\[
R_0 = R_0(q^n) \overset{\text{def}}{=} (C(q^n)^3) \setminus S \quad \text{and} \quad R_1 = R_1(q^n) \overset{\text{def}}{=} C_{S}(H;\bar{a}_i) = H R_0; \bar{a}_i;
\]

Here, \( R_0 \) is generated by elements of the form \( [X_1;X_2;X_3] \), where either \( X_1 \otimes C(q^n) \), or \( X_1 = X_2 = X_3 = X \) \( 2 C(q^n) \) and \( \bar{a}^2(X) = -X \). Also, \( C(q^n) \) \( Syl_2(C_{\text{SL}_2(q^n)}(A)) \) is cyclic of order \( 2^k \) 4, where \( 2^k \) is the largest power which divides \( q^n \) 1; and \( C(q^n) \) is cyclic of order \( 2^{k+1} \). So

\[
R_0 = (C_{2k})^3 \quad \text{and} \quad R_1 = R_0 \rtimes H \bar{a}_i;
\]

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where $\mathcal{B} = \langle B; B; B \rangle$ has order 2 and acts on $R_0$ via $(g^T g^{-1})$. Note that 
\[ hU; \mathcal{A} = \langle 1; 1 \rangle; \langle A; A; A \rangle = C_2^3 \]
is the 2-torsion subgroup of $R_0$.

We claim that $R_0$ is the only subgroup of $S$ isomorphic to $(C_{2^k})^3$. \hspace{1cm} (2)

To see this, let $R^0 \subseteq S$ be any subgroup isomorphic to $(C_{2^k})^3$, and let $E^0 = C_2^3$ be its 2-torsion subgroup. Recall that for any 2-group $P$, the Frattini subgroup $\text{Fr}(P)$ is the subgroup generated by commutators and squares in $P$. Thus \[ E^0 \text{ Fr}(R^0) \subseteq \text{Fr}(S) \]
(note that $\langle B; B; I \rangle = (\langle B; I ; I \rangle)^2$). Any elementary abelian subgroup of rank 4 in $\text{Fr}(S)$ would have to contain $hU; \mathcal{A}$ (the 2-torsion in $R_0 = C_{2^k}$), and this is impossible since no element of the coset $R_0 \langle B; B; I \rangle$ commutes with $\mathcal{A}$. Thus, $\text{rk}(\text{Fr}(S)) = 3$. Hence $U \subseteq E^0$, since otherwise $hU; E^0$ would be an elementary abelian subgroup of $\text{Fr}(S)$ of rank 4. This in turn implies that $R^0 \cong C_3(U)$, and hence that $E^0 \text{ Fr}(C_3(U)) = R_0$. Thus $E^0 = hU; \mathcal{A}$ (the 2-torsion in $R_0$ again). Hence $R^0 \cong C_3(hU; \mathcal{A}) = hR_0; \mathcal{B}i$, and it follows that $R^0 = R_0$. This finishes the proof of (2).

Choose generators $x_1; x_2; x_3 \in R_0$ as follows. Fix $X \subseteq C_{\mathbb{Z}_2(q^m)}(A)$ of order $2^k$, and $Y \subseteq C_{\mathbb{Z}_2(q^m)}(A)$ of order $2^{k+1}$ such that $Y^2 = X$. Set $x_1 = [X; I; X]$, $x_2 = [X; I; I]$, and $x_3 = [Y; Y; Y]$. Thus, $x_1^{2^{k-1}} = z$, $x_2^{2^{k-1}} = z_1$, and $x_3^{2^{k-1}} = \mathcal{A}$.

Now let $E \subseteq \text{S}(q^m)$ be an elementary abelian subgroup of rank 3 which contains $U$, and such that $C_3(E) \subseteq \text{Syl}_2(C_{\text{Spin}}(E))$. In particular, $E \cap R_1 = C_3(q^m)(U)$. There are two cases to consider: that where $E \cap R_0$ and that where $E \cap R_0$.

**Case 1:** Assume $E \cap R_0$. Since $R_0$ is abelian of rank 3, we must have $E \cap hU; \mathcal{A}$, the 2-torsion subgroup of $R_0$, and $C_3(E) = R_1$. Also, by (2), neither $R_0$ nor $R_1$ is isomorphic to any other subgroup of $S$; and hence \[ \text{Aut}_F(R_i) = \text{Aut}_{\text{Spin}}(R_i); \text{Aut}_F(R_i) \quad \text{for i = 0, 1}. \hspace{1cm} (4) \]

By Proposition A.8, $\text{Aut}_{\text{Spin}}(E)$ is the group of all automorphisms of $E$ which send $z$ to itself. In particular, since $H(q^m) = C_{\text{Spin}}(U)$, $\text{Aut}_{H(q^m)}(E)$ is the group of all automorphisms of $E$ which are the identity on $U$. Also, $\Gamma = \text{Inn}(H(q^m)) B_3$, where $B_3$ sends $\mathcal{A} = [A; A; A]$ to itself and permutes the non-trivial elements of $U = [1; 1; 1]g$. Hence $\text{Aut}_F(E)$ is the group of all
automorphisms which send $U$ to itself. So if we identify $\text{Aut}(E) = \text{GL}_3(\mathbb{Z} \otimes \mathbb{Z})$ via the basis $f; z_1; \mathcal{A}g$, then

$$\text{Aut}_{\text{Spin}}(E) = T_1 \overset{\text{def}}{=} \text{GL}_3(\mathbb{Z} \otimes \mathbb{Z}) = (a_{ij}) \, 2 \, \text{GL}_3(\mathbb{Z} \otimes \mathbb{Z}) \, j \, a_{21} = a_{31} = 0$$

and

$$\text{Aut}_{\text{F}}(E) = T_2 \overset{\text{def}}{=} \text{GL}_3(\mathbb{Z} \otimes \mathbb{Z}) = (a_{ij}) \, 2 \, \text{GL}_3(\mathbb{Z} \otimes \mathbb{Z}) \, j \, a_{31} = a_{32} = 0 :$$

By (2) (and since $E$ is the 2-torsion in $R_0$),

$$\text{N}_{\text{Spin}}(E) = \text{N}_{\text{Spin}}(R_0) \quad \text{and} \quad f \gamma 2 \Gamma j \gamma(E) = E g = f \gamma 2 \Gamma j \gamma(R_0) = R_0 g :$$

Since $C_{\text{Spin}}(E) = C_{\text{Spin}}(R_0) \mathcal{H} \mathcal{I}$, the only nonidentity element of $\text{Aut}_{\text{Spin}}(R_0)$ or of $\text{Aut}_{\text{F}}(R_0)$ which is the identity on $E$ is conjugation by $\mathcal{B}$, which is $-1$. Hence restriction from $R_0$ to $E$ induces isomorphisms

$$\text{Aut}_{\text{Spin}}(R_0) \not\rightarrow \text{I} \, g = \text{Aut}_{\text{Spin}}(E) \quad \text{and} \quad \text{Aut}_{\text{F}}(R_0) \not\rightarrow \text{I} \, g = \text{Aut}_{\text{F}}(E) :$$

Upon identifying $\text{Aut}(R_0) = \text{GL}_3(\mathbb{Z} \otimes \mathbb{Z})$ via the basis $f x_1; x_2; x_3 g$, these can be regarded as sections

$$\iota : T_1 \longrightarrow \text{GL}_3(\mathbb{Z} \otimes \mathbb{Z}) \not\rightarrow \text{I} \, g = \text{SL}_3(\mathbb{Z} \otimes \mathbb{Z}) \quad \text{I} \, j \, 2 \, (\mathbb{Z} \otimes \mathbb{Z}) \, g = f \, I \, g$$

of the natural projection from $\text{GL}_3(\mathbb{Z} \otimes \mathbb{Z})$ to $\text{GL}_3(\mathbb{Z} \otimes \mathbb{Z})$, which agree on the group $T_0 = T_1 \setminus T_2$ of upper triangular matrices.

We claim that 1 and 2 both map trivially to the second factor. Since this factor is abelian, it suffices to show that $T_0$ is generated by $[T_1; T_1] \setminus T_0$ and $[T_2; T_2] \setminus T_0$, and that each $T_i$ is generated by $[T_i; T_i]$ and $T_0 \setminus -1$ and this is easily checked. (Note that $T_1 = T_2 = 4$.)

By carrying out the above procedure over the field $\mathbb{F}^{\text{Spin}}_q$, we see that both of these sections $\iota$ can be lifted further to $\text{SL}_3(\mathbb{Z} \otimes \mathbb{Z})$ (still agreeing on $T_0$). So by Lemma A.10, there is a section

$$\tau : \text{GL}_3(\mathbb{Z} \otimes \mathbb{Z}) \longrightarrow \text{SL}_3(\mathbb{Z} \otimes \mathbb{Z})$$

which extends both 1 and 2. By (4), $\text{Aut}_{\text{F}}(R_0) = \text{Im}(\tau) \setminus \text{I} \, h = 1$.

We next identify $\text{Aut}_{\text{F}}(R_1)$. By Lemma 2.8(a), $E \overset{\text{def}}{=} \text{H} \, z_1; \mathcal{A} \mathcal{B} \mathcal{I}$ is a subset of rank 4 and type 4. So by Proposition A.8, $\text{Aut}_{\text{Spin}}(E)$ contains all automorphisms of $E = C_2^4$ which send $z 2 Z(\text{Spin})$ to itself. Hence for any $x 2 N_{\text{Spin}}(R_1)$, since $c_{x^1}(z) = z$, there is $x_1 2 N_{\text{Spin}}(E)$ such that $c_{x^1} = c_{x_1} \mathcal{B} = c_{x_1} \mathcal{B} (\text{ie}, [x_1; \mathcal{B}] = 1)$. Set $x_2 = xx_1^{-1}$. Since $C_{\text{Spin}}(U) = H(q') \setminus \text{Im}(\tau)$, we see that $C_{\text{Spin}}(E) = K_0 \mathcal{H} \mathcal{I}$, where $K_0 = ! (C_{\text{Spin}}(E) / \mathcal{B} \mathcal{I}) \setminus \text{Spin}$.
is abelian, \( R_0 \trianglelefteq Syl_2(K_0) \), and \( \mathfrak{H} \) acts on \( K_0 \) by inversion. Upon replacing \( x_1 \) by \( \mathfrak{H}x_1 \) and \( x_2 \) by \( x_2\mathfrak{H}^{-1} \) if necessary, we can assume that \( x_2 \trianglelefteq K_0 \). Then
\[
[x_2; \mathfrak{H}] = x_2 (\mathfrak{H}x_2\mathfrak{H}^{-1})^{-1} = x_2^2;
\]
while by the original choice of \( x; x_1 \) we have
\[
[x_2; \mathfrak{H}] = [xx_1^{-1}; \mathfrak{H}] = [x; \mathfrak{H}] 2 R_0:
\]
Thus \( x_2^2 \trianglelefteq R_0 \trianglelefteq Syl_2(K_0) \), and hence \( x_2 \trianglelefteq R_0 \trianglelefteq R_1 \). Since \( x = x_2x_1 \) was an arbitrary element of \( N_{\text{Spin}}(R_1) \), this shows that \( N_{\text{Spin}}(R_1) R_1 \trianglelefteq C_{\text{Spin}}(\mathfrak{H}) \), and hence that
\[
\text{Aut}_{\text{Spin}}(R_1) = \text{Inn}(R_1) f' 2 \text{Aut}_{\text{Spin}}(R_1) j' (\mathfrak{H}) = \mathfrak{H} g;
\]
(5)
Since \( \text{Aut}_F(R_1) \) is generated by its intersection with \( \text{Aut}_{\text{Spin}}(R_1) \) and the group \( \mathfrak{B}_3 \), which permutes the three factors in \( H(q^2) \) (and since the elements of \( \mathfrak{B}_3 \) all \( x \mathfrak{H} \)), we also have
\[
\text{Aut}_F(R_1) = \text{Inn}(R_1) f' 2 \text{Aut}_F(R_1) j' (\mathfrak{H}) = \mathfrak{H} g:
\]
Together with (4) and (5), this shows that \( \text{Aut}_F(R_1) \) is generated by \( \text{Inn}(R_1) \) together with certain automorphisms of \( R_1 = R_0 \trianglelefteq \mathfrak{H} \) which send \( \mathfrak{H} \) to itself. In other words,
\[
\text{Aut}_F(R_1) = \text{Inn}(R_1) f' 2 \text{Aut}_F(R_1) j' (\mathfrak{H}) = \mathfrak{H} g;
\]
(4)
Thus
\[
\text{Aut}_F(R_1) j' (z) = z
\]
\[
= \text{Inn}(R_1) f' 2 \text{Aut}_F(R_1) j' (\mathfrak{H}) = \mathfrak{H} g;
\]
the last equality by (5); and (1) now follows.

**Case 2:** Now assume that \( E \nsubseteq R_0 \). By assumption, \( U \trianglelefteq E \) (hence \( E \trianglelefteq C_5(U) \)), and \( C_5(E) \) is a Sylow subgroup of \( C_{\text{Spin}}(E) \). Since \( C_5(E) \) is not isomorphic to \( R_1 = C_5(hz; z_2; A\mathfrak{H}) \) (by (2)), this shows that \( E \) is not \( \text{Spin} \)-conjugate to \( hz; z_2; A\mathfrak{H} \). By Proposition A.8, \( \text{Spin} \) contains exactly two conjugacy classes of rank 3 subgroups containing \( z \), and thus \( E \) must have type II. Hence by Proposition A.8(d), \( C_5(E) \) is elementary abelian of rank 4, and also has type II.

Let \( C \) be the \( \text{Spin}_7(q^3) \) {conjugacy class of the subgroup \( E = hU; A\mathfrak{H}; \mathfrak{H}i = C_4 \) which by Lemma 2.8(a) has type I. Let \( E^0 \) be the set of all subgroups of \( S \) which
are elementary abelian of rank 4, contain $U$, and are not in $C$. By Lemma 2.8(e), for any $2 \text{Iso}_T(E^0, E^0)$ and any $E^0 \leq E^0$, $E^0 \overset{\text{def}}{=} (E^0) 2 \text{E}^0$, and sends $x_C(E^0)$ to $x_C(E^0)$. The same holds for $2 \text{Iso}_{\text{Spin}}(E^0, E^0)$ by definition of the elements $x_C(-)$ (Proposition A.9). Since $C_E(E) 2 \text{E}^0$, this shows that all elements of $\text{Aut}_E(C_E(E))$ send the element $x_C(C_E(E))$ to itself. By Proposition A.9(c), $\text{Aut}_{\text{Spin}}(C_E(E))$ is the group of automorphisms which are the identity on the rank two subgroup $\text{h} x_C(C_E(E)); z i$; and (1) now follows.

One more technical result is needed.

**Lemma 2.10** Fix $n = 1$, and let $E; E^0$ $S(q^3)$ be two elementary abelian subgroups of rank three which contain $U$, and which are $\Gamma_n \{\text{conjugate}\}$. Then $E$ and $E^0$ are $\text{Spin}_7(q^3) \{\text{conjugate}\}$.

**Proof** By [23, 3.6.3(ii)], $-1$ is the only element of order 2 in $SL_2(q)$. Consider the sets

$$J_1 = X 2 SL_2(q^3) X^2 = -1$$

and

$$J_2 = X 2 SL_2(q^3) q^3(X) = -X; X^2 = -1 :$$

Here, as usual, $q^3$ is induced by the field automorphism $(x \mapsto x^{q^3})$. All elements in $J_1$ are $SL_2(q) \{\text{conjugate}\}$ (this follows, for example, from [23, 3.6.23]), and we claim the same is true for elements of $J_2$.

Let $SL_2(q^3)$ be the group of all elements $X 2 SL_2(q^3)$ such that $q^3(X) = ^X X$. This is a group which contains $SL_2(q^3)$ with index 2. Let $k$ be such that the Sylow 2-subgroups of $SL_2(q^3)$ have order $2^k$; then $k = 3$ since $|SL_2(q^3)| = q^3(q^{2n} - 1)$. Any $S 2 \text{Syl}_2(SL_2(q^3))$ is quaternion of order $2^{k+1}$ [16 (see [15, Theorem 2.8.3]) and its intersection with $SL_2(q^3)$ is quaternion of order $2^k$, so all elements in $S \setminus J_2$ are $S \{\text{conjugate}\}$. It follows that all elements of $J_2$ are $SL_2(q^3) \{\text{conjugate}\}$. If $X; X^0$ $J_2$ and $X^0 = gXg^{-1}$ for $g 2 SL_2(q^3)$, then either $g 2 SL_2(q^3)$ or $gX 2 SL_2(q^3)$, and in either case $X$ and $X^0$ are conjugate by an element of $SL_2(q^3)$.

By Proposition 2.5(a),

$$E; E^0 \text{Spin}_7(q^3)(U) = H(q^3) \overset{\text{def}}{=} (SL_2(q^3))^2 \setminus \text{Spin}_7(q^3):$$

Thus $E = \text{hz}; z_1; [X_1; X_2; X_3]$ and $E^0 = \text{hz}; z_1; [X_0; X_2; X_3]$, where the $X_i$ are all in $J_1$ or all in $J_2$, and similarly for the $X_i$. Also, since $E$ and $E^0$ are $\Gamma_n \{\text{conjugate}\}$ (and each element of $\Gamma_n$ leaves $U = \text{hz}; z_1$ invariant), the $X_i$ and $X^0$ must all be in the same set $J_1$ or $J_2$. Hence they are all $SL_2(q^3) \{\text{conjugate}\}$, and so $E$ and $E^0$ are $\text{Spin}_7(q^3) \{\text{conjugate}\}$. 

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We are now ready to show that the fusion systems $F_n$ are saturated, and satisfy the conditions listed in Theorem 2.1.

**Proposition 2.11** For a fixed odd prime power $q$, let $S(q^n) = S(q^1)$ be as defined above. Let $z$ be the central element of order 2. Then for each $n$, $F_n = F_{Sol}(q^n)$ is saturated as a fusion system over $S(q^n)$, and satisfies the following conditions:

(a) For all $P;Q \in S(q^n)$ which contain $z$, if $2 \text{Hom}(P;Q)$ is such that $(z) = z$, then $2 \text{Hom}_{F_n}(P;Q)$ if and only if $2 \text{Hom}_{Spin_7(q^n)}(P;Q)$.

(b) $C_{F_n}(z) = F_{S(q^n)}(Spin_7(q^n))$ as fusion systems over $S(q^n)$.

(c) All involutions of $S(q^n)$ are $F_n$ conjugate.

Furthermore, $F_m = F_n$ for $m \leq n$. The union of the $F_n$ is thus a category $F_{Sol}(q^1)$ whose objects are the finite subgroups of $S(q^1)$.

**Proof** We apply Proposition 1.2, where $p = 2$, $G = Spin_7(q^n)$, $S = S(q^n)$, $Z = hzi = Z(G)$; and $U$ and $C_G(U) = H(q^n)$ are as defined above. Also, $\Gamma = \Gamma_n = Aut(H(q^n))$. Condition (a) in Proposition 1.2 (all noncentral involutions in $G$ are conjugate) holds since all subgroups in $E_2$ are conjugate (Proposition A.8), and condition (b) holds by definition of $\Gamma$. Condition (c) holds since

$$f \gamma, 2 \Gamma \gamma(\gamma(z) = zg = Inn(H(q^n)) \cdot i = Aut_{N_G(U)}(H(q^n))$$

by definition, since $H(q^n) = C_G(U)$, and by Proposition 2.5(b). Condition (d) was shown in Proposition 2.9, and condition (e) in Lemma 2.10. So by Proposition 1.2, $F_n$ is a saturated fusion system, and $C_{F_n}(Z) = F_{S(q^n)}(Spin_7(q^n))$.

The last statement is clear.

## 3 Linking systems and their automorphisms

We next show the existence and uniqueness of centric linking systems associated to the $F_{Sol}(q)$, and also construct certain automorphisms of these categories analogous to the automorphisms $q$ of the group $Spin_7(q^n)$. One more technical lemma about elementary abelian subgroups, this time about their $F_n$ conjugacy classes, is first needed.

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Lemma 3.1 Set $F = F_{S_0}(q)$. For each $r \geq 3$, there is a unique $F$ {conjugacy class of elementary abelian subgroups $E \in S(q)$ of rank $r$. There are two $F$ {conjugacy classes of rank four elementary abelian subgroups $E \in S(q)$: one is the set $C$ of subgroups $Spin_7(q)$ {conjugate to $E = hz; z_1; A; [B; B; B]$}, while the other contains the other conjugacy class of type I subgroups as well as all type II subgroups. Furthermore, $Aut_F(E) = Aut(E)$ for all elementary abelian subgroups $E \in S(q)$ except when $E$ has rank four and is not $F$ {conjugate to $E$, in which case

$$Aut_F(E) = f \ 2 \ Aut(E) \ j (x_C(E)) = x_C(E)g.$$

Proof By Lemma 2.8(d), the three subgroups

$$E = hz; z_1; A; [B; B; B]; E_{001} = hz; z_1; A; [B; B; X B]; E_{100} = hz; z_1; A; [X B; B; B]$$

(where $X$ is a generator of $C(q)$) represent the three $Spin_7(q)$ {conjugacy classes of rank four subgroups. Clearly, $E_{100}$ and $E_{001}$ are $\Gamma_1$ {conjugate, hence $F$ {conjugate; and by Lemma 2.8(e), neither is $\Gamma_1$ {conjugate to $E$. This proves that there are exactly two $F$ {conjugacy classes of such subgroups.

Since $E$ and $E_{001}$ both are of type I in $Spin_7(q)$, their $Spin_7(q)$ {automorphism groups contain all automorphisms which $xz$ (see Proposition A.8). By Lemma 2.8(e), $z$ is fixed by all $\Gamma$ {automorphisms of $E_{001}$, and so $Aut_F(E_{001})$ is the group of all automorphisms of $E_{001}$ which send $z = x_C(E_{001})$ to itself. On the other hand, $E$ contains automorphisms (induced by permuting the three coordinates of $H$) which permute the three elements $z; z_1; z_2$, and these together with $Aut_{Spin}(E)$ generate $Aut(E)$.

It remains to deal with the subgroups of smaller rank. By Proposition A.8 again, there is just one $Spin_7(q)$ {conjugacy class of elementary abelian subgroups of rank one or two. There are two conjugacy classes of rank three subgroups, those of type I and those of type II. Since $E_{100}$ is of type II and $E_{001}$ of type I, all rank three subgroups of $E_{001}$ have type I, while some of the rank three subgroups of $E_{100}$ have type II. Since $E_{001}$ is $F$ {conjugate to $E_{100}$, this shows that some subgroup of rank three and type II is $F$ {conjugate to a subgroup of type I, and hence all rank three subgroups are conjugate to each other. Finally, $Aut_F(E) = Aut(E)$ whenever $rk(E) \geq 3$ since any such group is $F$ {conjugate to a subgroup of $E$ (and we have just seen that $Aut_F(E) = Aut(E)$).

To simplify the notation, we now define

$$F_{Spin}(q^i) \overset{def}{=} F_{S(q^i)}(Spin_7(q^i))$$

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for all 1 \leq n \leq 1: the fusion system of the group Spin_7(q') at the Sylow subgroup S(q'). By construction, this is a subcategory of F_{Sol}(q'). We write

$$O_{Sol}(q') = O(F_{Sol}(q')) \quad \text{and} \quad O_{Spin}(q') = O(F_{Spin}(q'))$$

for the corresponding orbit categories: both of these have as objects the subgroups of S(q'), and have as morphism sets

$$\text{Mor}_{O_{Sol}(q')}(P;Q) = \text{Hom}_{F_{Sol}(q')}(P;Q) = \text{Inn}(Q) \cdot \text{Rep}(P;Q)$$

and

$$\text{Mor}_{O_{Spin}(q')}(P;Q) = \text{Hom}_{F_{Spin}(q')}(P;Q) = \text{Inn}(Q):$$

Let O_{Sol}(q') \to O_{Spin}(q') be the centric orbit categories; i.e., the full subcategories whose objects are the F_{Sol}(q') \{ or F_{Spin}(q') \} centric subgroups of S(q'). (We will see shortly that these in fact have the same objects.)

The obstructions to the existence and uniqueness of linking systems associated to the fusion systems F_{Sol}(q'), and to the existence and uniqueness of certain automorphisms of those linking systems, lie in certain groups which were identified in [6] and [5]. It is these groups which are shown to vanish in the next lemma.

**Lemma 3.2** Fix a prime power q, and let

$$Z_{Sol}(q): O_{Sol}(q) \to \text{Ab} \quad \text{and} \quad Z_{Spin}(q): O_{Spin}(q) \to \text{Ab}$$

be the functors Z(P) = Z(P). Then for all i \geq 0,

$$\lim_{\to} (Z_{Sol}(q)) = 0 = \lim_{\to} (Z_{Spin}(q))$$

**Proof** Set F = F_{Sol}(q) for short. Let P_1; \ldots; P_k be F \{centric subgroups P_i \ S(q), arranged such that jP_i j \not\sim jP_j j for i \neq j. For each i, let Z_i \subseteq Z_{Sol}(q) be the subfunctor defined by setting Z_i(P) = Z_{Sol}(q)(P) if P is conjugate to P_j for some j \neq i and Z_i(P) = 0 otherwise. We thus have a filtration

$$0 = Z_0 \subseteq Z_1 \subseteq \ldots \subseteq Z_k = Z_{Sol}(q)$$

of Z_{Sol}(q) by subfunctors, with the property that for each i, the quotient functor Z_i = Z_{i-1} vanishes except on the conjugacy class of P_i (and such that (Z_i = Z_{i-1})(P_i) = Z_{Sol}(q)(P_i)). By [6, Proposition 3.2],

$$\lim_{\to} (Z_i = Z_{i-1}) = (\text{Out}_F(P_i); Z(P_i))$$

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for each \( i \). Here, \((\Gamma; \mathcal{M})\) are certain graded groups, defined in [16, section 5] for all finite groups \( \Gamma \) and all finite \( \mathcal{M} \{ \text{modules} \ mathcal{M} \). We will show that
\[
(\text{Out}_F(\mathcal{P}_i); Z(\mathcal{P}_i)) = 0 \quad \text{except when} \quad \mathcal{P}_i = S(q) \text{ or } S_0(q) \quad \text{(see Definition 2.6)}.
\]

Fix an \( F \} \{ \text{centric subgroup} \mathcal{P} \mapsto S(q) \). For each \( j \in \{ 1 \), let \( \Omega_j(\mathcal{Z}(\mathcal{P})) = f(g_2 \mathcal{Z}(\mathcal{P}) \cdot g_2^{-1} = 1g, \text{and set} \ E = \Omega_j(\mathcal{Z}(\mathcal{P})) \mid \ 2\text{torsion in the center of} \ \mathcal{P} \). For each \( j \), let \( \Omega_j(\mathcal{Z}(\mathcal{P})) = f(g_2 \mathcal{Z}(\mathcal{P}) \cdot g_2^{-1} = 1g, \text{and set} \ E = \Omega_j(\mathcal{Z}(\mathcal{P})) \mid \ 2\text{torsion in the center of} \ \mathcal{P} \). We can assume \( E \) is fully centralized in \( F \) (otherwise replace \( \mathcal{P} \) and \( E \) by appropriate subgroups in the same \( F \} \{ \text{conjugacy classes} \).

Assume rst that \( Q \mapsto C_{\Sigma(q)}(E) \supseteq \mathcal{P} \), and hence that \( N_\Omega(\mathcal{P}) \supseteq \mathcal{P} \). Then any \( x \in N_\Omega(\mathcal{P}) \) centralizes \( E = \Omega_1(\mathcal{Z}(\mathcal{P})) \). Hence for each \( j \), \( x \) acts trivially on \( \Omega_j(\mathcal{Z}(\mathcal{P})) = \Omega_{j-1}(\mathcal{Z}(\mathcal{P})) \), since multiplication by \( p^{j-1} \) sends this group \( N_\Omega(\mathcal{P}) \mapsto \Omega_j(\mathcal{Z}(\mathcal{P})) \) linearly and monomorphically to \( E \). Since \( c_x \) is a nontrivial element of \( \text{Out}_F(\mathcal{P}) \) of \( p \} \{ \text{power order} \),
\[
(\text{Out}_F(\mathcal{P}); \Omega_j(\mathcal{Z}(\mathcal{P})) = \Omega_{j-1}(\mathcal{Z}(\mathcal{P}))) = 0
\]
for all \( j \) by [16, Proposition 5.5], and thus
\[
(\text{Out}_F(\mathcal{P}); \Omega_j(\mathcal{Z}(\mathcal{P})) = \Omega_{j-1}(\mathcal{Z}(\mathcal{P}))) = 0.
\]

Now assume that \( \mathcal{P} = C_{\Sigma(q)}(E) = \mathcal{P} \), the centralizer in \( S(q) \) of a fully \( F \} \{ \text{centralized elementary abelian subgroup} \). Since there is a unique conjugacy class of elementary abelian subgroup of any rank \( 3 \), \( C_{\Sigma(q)}(E) \) always contains a subgroup \( C_{2^4} \), and hence \( \mathcal{P} \) contains a subgroup \( C_{2^4} \) which is self centralizing by Proposition A.8(a). This shows that \( \mathcal{Z}(\mathcal{P}) \) is elementary abelian, and hence that \( \mathcal{Z}(\mathcal{P}) = E \).

We can assume \( \mathcal{P} \) is fully normalized in \( F \), so
\[
\text{Aut}_{\Sigma(q)}(\mathcal{P}) \leq \text{Syl}_2(\text{Aut}_F(\mathcal{P}))
\]
by condition (1) in the definition of a saturated fusion system. Since \( \mathcal{P} = C_{\Sigma(q)}(E) \) (and \( E = \mathcal{Z}(\mathcal{P}) \)), this shows that
\[
\text{Ker} \text{Out}_F(\mathcal{P}) \longrightarrow \text{Aut}_F(\mathcal{E})
\]
has odd order. Also, since \( E \) is fully centralized, any \( F \} \{ \text{automorphism of} \ E \) extends to an \( F \} \{ \text{automorphism of} \ \mathcal{P} = C_{\Sigma(q)}(E) \), and thus this restriction map between automorphism groups is onto. By [16, Proposition 6.1(i,iii)], it now follows that
\[
i(\text{Out}_F(\mathcal{P}); \mathcal{Z}(\mathcal{P})) = i(\text{Aut}_F(\mathcal{E}); \mathcal{E}):
\]
(1)

By Lemma 3.1, \( \text{Aut}_F(\mathcal{E}) = \text{Aut}(\mathcal{E}) \), except when \( E \) lies in one certain \( F \} \{ \text{conjugacy class} \} \mathcal{G} = C_{2^4} \); and in this case \( \mathcal{P} = E \) and \( \text{Aut}_F(\mathcal{E}) \) is

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the group of automorphisms fixing the element \( x_C(E) \). In this last (exceptional) case, \( O_2(\text{Aut}_F(E)) \cong 1 \) (the subgroup of elements which are the identity on \( E = x_C(E)i \)), so

\[
(\text{Out}_F(P); Z(P)) = (\text{Aut}_F(E); E) = 0
\]

by [16, Proposition 6.1(ii)]. Otherwise, when \( \text{Aut}_F(E) = \text{Aut}(E) \), by [16, Proposition 6.3] we have

\[
\begin{align*}
8 
&\geq 2 & \text{if } \text{rk}(E) = 2, i = 1 \\
1(\text{Aut}_F(E); E) 
&\geq 2 & \text{if } \text{rk}(E) = 1, i = 0 \\
&> 0 & \text{otherwise}
\end{align*}
\]

By points (1), (2), and (3), the groups \( (\text{Out}_F(P); Z(P)) \) vanish except in the two cases \( E = \mathbb{Z}i \) or \( E = U \), and these correspond to \( P = S(q) \) or \( P = N_S(U) = S_0(q) \).

We can assume that \( P_k = S(q) \) and \( P_{k-1} = S_0(q) \). We have now shown that \( \lim_{i\to\infty}(Z_{k-2}) = 0 \), and thus that \( Z_S(q) \) has the same higher limits as \( Z_k = Z_{k-2} \). Hence \( \lim_i(Z_S(q)) = 0 \) for all \( j \geq 2 \), and there is an exact sequence

\[
0 \longrightarrow \lim^0(Z_S(q)) \longrightarrow \lim^0(Z_{k-1}) \longrightarrow \lim^1(Z_{k-1}) \longrightarrow \lim^1(Z_S(q)) \longrightarrow 0.
\]

One easily checks that \( \lim^0(Z_S(q)) = 0 \), and hence we also get \( \lim^1(Z_S(q)) = 0 \).

The proof that \( \lim^1(Z_{\text{Spin}}(q)) = 0 \) for all \( i \geq 1 \) is similar, but simpler. If \( F = F_{\text{Spin}}(q) \), then for any \( F \) {centric subgroup} \( P \subseteq S(q) \), there is an element \( x \in 2 N_S(P) \setminus P \) such that \( [x; P] = \mathbb{Z}i \), and \( c_x \) is a nontrivial element of \( O_2(\text{Out}_F(P)) \). Thus

\[
(\text{Out}_F(P); Z(P)) = 0
\]

for all such \( P \) by [16, Proposition 6.1(ii)] again.

We are now ready to construct classifying spaces \( B S(q) \) for these fusion systems \( F_S(q) \). The following proposition finishes the proof of Theorem 2.1, and also contains additional information about the spaces \( B S(q) \).

To simplify notation, we write \( L^C_{\text{Spin}}(q^i) = L^C_{S(q^i)}(\text{Spin}_n(q^i)) \) (\( n \geq 1 \)) to denote the centric linking system for the group \( \text{Spin}_n(q^i) \). The end automorphism \((x \forall x^q)\) induces an automorphism of \( \text{Spin}_n(q^i) \) which sends \( S(q^i) \) to itself; and this in turn induces automorphisms \( \frac{q}{F} = \frac{q}{F}(S(q)) \), \( \frac{q}{F}(\text{Spin}) \), and \( \frac{q}{F}(\text{Spin}) \) of the fusion systems \( F_S(q^i) \) and of the linking system \( L^C_{\text{Spin}}(q^i) \).

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Proposition 3.3  Fix an odd prime $q$, and let $n \geq 1$. Let $S = S(q^n)$.

Proposition 2.5: Let $z$ be the central element of order 2. Then there is a centric linking system

$$L = L_{\text{Sol}}(q^n) \to F_{\text{Sol}}(q^n)$$

associated to the saturated fusion system $F \overset{\text{def}}{=} F_{\text{Sol}}(q^n)$ over $S$, which has the following additional properties.

(a) A subgroup $P$ of $S$ is $F$-centric if and only if it is $F_{\text{Spin}}(q^n)$-centric.

(b) $L_{\text{Sol}}(q^n)$ contains $L_{\text{Spin}}(q^n)$ as a subcategory, in such a way that $j_{L_{\text{Spin}}(q^n)}$ is the usual projection to $F_{\text{Spin}}(q^n)$, and that the distinguished monomorphisms

$$P \to \text{Aut}_L(P)$$

for $L = L_{\text{Sol}}(q^n)$ are the same as those for $L_{\text{Spin}}(q^n)$.

(c) Each automorphism of $L_{\text{Spin}}(q^n)$ which covers the identity on $F_{\text{Spin}}(q^n)$ extends to an automorphism of $L_{\text{Sol}}(q^n)$ which covers the identity on $F_{\text{Sol}}(q^n)$. Furthermore, such an extension is unique up to composition with the functor

$$C_z : L_{\text{Sol}}(q^n) \to L_{\text{Sol}}(q^n)$$

which is the identity on objects and sends $2 \text{Mor}_{L_{\text{Sol}}(q^n)}(P; Q)$ to $b b^{-1}$ (conjugation by $z^n$).

(d) There is a unique automorphism $q_i L_{\text{Sol}}(q^n)$ which covers the automorphism of $F_{\text{Sol}}(q^n)$ induced by the element automorphism $(x \mapsto x^q)$, which extends the automorphism of $L_{\text{Spin}}(q^n)$ induced by the element automorphism, and which is the identity on $-1(F_{\text{Sol}}(q^n))$.

Proof  By Proposition 2.11, $F = F_{\text{Sol}}(q^n)$ is a saturated fusion system over $S = S(q^n)$ 2 Syl$_2$(Spin$_7(q^n))$, with the property that $C_F(z) = F_{\text{Spin}}(q^n)$. Point (a) follows as a special case of [6, Proposition 2.5(a)].

Since $\lim_{i=0}^1 \text{Z}_{\text{Sol}}(q^n) = 0$ for $i = 2; 3$ by Lemma 3.2, there is by [6, Proposition 3.1] a centric linking system $L = L_{\text{Sol}}(q^n)$ associated to $F$, which is unique up to isomorphism (an isomorphism which commutes with the projection to $F_{\text{Sol}}(q^n)$ and with the distinguished monomorphisms). Furthermore, $-1(F_{\text{Spin}}(q^n))$ is a linking system associated to $F_{\text{Spin}}(q^n)$, such a linking system is unique up to isomorphism since $\lim_{i=0}^2 \text{Z}_{\text{Spin}}(q^n) = 0$ (Lemma 3.2 again), and this proves (b).
By [5, Theorem 6.2] (more precisely, by the same proof as that used in [5]), the vanishing of \(\lim^i(Z_{Sol}(q^n))\) for \(i = 1; 2\) (Lemma 3.2) shows that each automorphism of \(F = F_{Sol}(q^n)\) lifts to an automorphism of \(L\), which is unique up to a natural isomorphism of functors; and any such natural isomorphism sends each object \(P\) to an isomorphism \(g\) for some \(g \in Z(P)\). Similarly, the vanishing of \(\lim^i(Z_{Spin}(q^n))\) for \(i = 1; 2\) shows that each automorphism of \(F_{Spin}(q^n)\) lifts to an automorphism of \(L_{Spin}^c(q^n)\), also unique up to a natural isomorphism of functors. Since \(L_{Sol}^c(q^n)\) and \(L_{Spin}^c(q^n)\) have the same objects by (a), this shows that each automorphism of \(L_{Spin}^c(q^n)\) which covers the identity on \(F_{Spin}^c(q^n)\) extends to a unique automorphism of \(L_{Sol}^c(q^n)\) which covers the identity on \(F_{Sol}(q^n)\).

It remains to show, for any \(2 \text{ Aut}(L_{Sol}^c(q^n))\) which covers the identity on \(F_{Spin}^c(q^n)\) and such that \(j_{L_{Spin}^c(q^n)} = \text{Id}\), that the identity or conjugation by \(z\). We have already noted that \(z\) must be naturally isomorphic to the identity; i.e., there are elements \(\gamma(P) \in Z(P)\), for all \(P \in L_{Sol}^c(q^n)\), such that

\[
\gamma(Q) = \gamma(P)^{-1} \quad \text{for all} \quad 2 \text{ Mor}(L_{Sol}^c(q^n); P; Q), \text{ all } P; Q.
\]

Since \(z\) is the identity on \(L_{Spin}^c(q^n)\), the only possibilities are \(\gamma(P) = 1\) for all \(P\) (hence \(z = \text{Id}\)), or \(\gamma(P) = z\) for all \(P\) (hence \(z\) is conjugation by \(z\)).

Now consider the automorphism \(\delta L\) 2 \text{ Aut}(F_{Sol}(q^n))\) induced by the \(\delta L\) automorphism \((x \ 7 \ x^3)\) of \(F_{Sol}(q^n)\). We have just seen that this lifts to an automorphism \(\delta L\) of \(L_{Sol}^c(q^n)\), which is unique up to natural isomorphism of functors. The restriction of \(\delta L\) to \(L_{Spin}^c(q^n)\), and the automorphism \(\delta L_{Spin}^c(q^n)\) of \(L_{Spin}^c(q^n)\) induced directly by the \(\delta L\) automorphism, are two liftings of \(\delta L_{Spin}^c(q^n)\), and hence different by a natural isomorphism of functors which extends to a natural isomorphism of functors on \(L_{Sol}^c(q^n)\). Upon composing with this natural isomorphism, we can thus assume that \(\delta L\) does restrict to the automorphism of \(L_{Spin}^c(q^n)\) induced by the \(\delta L\) automorphism.

Now consider the action of \(\delta L\) on \(\text{Aut}_L(S_0(q))\), which by assumption is the identity on \(\text{Aut}_L(S_0(q))\), and in particular on \((S_0(q))\) itself. Thus, with respect to the extension

\[
1 \longrightarrow S_0(q) \longrightarrow \text{Aut}_L(S_0(q)) \longrightarrow \text{Aut}_L(S_0(q)) \longrightarrow 1;
\]

\(\delta L\) is the identity on the kernel and on the quotient, and hence is described by a cocycle

\[
2 \text{ Z}(S_0(q)) = Z^1(3; Z_0^1) = Z^1(3; (Z_0^2)^2);
\]

Since 1(3; (Z_0^2)^2) = 0, \(\delta L\) must be a coboundary, and thus the action of \(\delta L\) on \(\text{Aut}_L(S_0(q))\) is conjugation by an element of \(Z(S_0(q))\). Since it is the identity...
on $\text{Aut}_{L_{\text{Spin}}}(S_0(q))$, it must be conjugation by 1 or $z$. If it is conjugation by $z$, then we can replace $q_L$ (on the whole category $L$) by its composite with $z$; i.e., by its composite with the functor which is the identity on objects and sends $2\text{Mor}_L(P;Q)$ to $b$.

In this way, we can assume that $q_L$ is the identity on $\text{Aut}_L(S_0(q))$. By construction, every morphism in $F_{\text{Sol}}(q)$ is a composite of morphisms in $F_{\text{Spin}}(q)$ and restrictions of automorphisms in $F_{\text{Sol}}(q)$ of $S_0(q)$. Since $q_L$ is the identity on $\Omega_{-1}(F_{\text{Spin}}(q))$, this shows that it is the identity on $\Omega_{-1}(F_{\text{Sol}}(q))$.

It remains to check the uniqueness of $q_L$. If $0$ is another functor with the same properties, then by (e), $(0)^{-1} q_L$ is either the identity or conjugation by $z$; and the latter is not possible since conjugation by $z$ is not the identity on $\Omega_{-1}(F_{\text{Sol}}(q))$.

This finishes the construction of the classifying spaces $B\text{Sol}(q) = jL_{\text{Sol}}(q)_J^2$ for the fusion systems constructed in Section 2. We end the section with an explanation of why these are not the fusion systems of finite groups.

**Proposition 3.4** For any odd prime power $q$, there is no finite group $G$ whose fusion system is isomorphic to that of $F_{\text{Sol}}(q)$.

**Proof** Let $G$ be a finite group, $x S 2 \text{Syl}_2(G)$, and assume that $S = S(q) 2 \text{Syl}_2(\text{Spin}_7(q))$, and that the fusion system $F_S(G)$ satisfies conditions (a) and (b) in Theorem 2.1. In particular, all involutions in $G$ are conjugate, and the centralizer of any involution $z 2 G$ has the fusion system of Spin$_7(q)$. When $q \equiv 3 \pmod{8}$, Solomon showed [22, Theorem 3.2] that there is no finite group whose fusion system has these properties. When $q \equiv 1 \pmod{8}$, he showed in the same theorem that there is no such $G$ such that $\Phi \trianglelefteq C_G(z) = O_{2^6}(C_G(z))$ is isomorphic to a subgroup of $\text{Aut}(\text{Spin}_7(q))$ which contains $\text{Spin}_7(q)$ with odd index. (Here, $O_{2^6}(-)$ means largest odd order normal subgroup.)

Let $G$ be a finite group whose fusion system is isomorphic to $F_{\text{Sol}}(q)$, and again set $\Phi \trianglelefteq C_G(z) = O_{2^6}(C_G(z))$ for some involution $z 2 G$. Set $H = O_{2^6}(\Phi z)$: the smallest normal subgroup of $\Phi z$ of odd index. Then $H$ has the fusion system of $\Omega_7(q) = \text{Spin}_7(q) - Z(\text{Spin}_7(q))$. We will show that $H = \Omega_7(q)$ for some odd prime power $q^0$. It then follows that $O_{2^6}(\Phi) = \text{Spin}_7(q^0)$, thus contradicting Solomon's theorem and proving our claim.

The following classification free argument for proving that $H = \Omega_7(q)$ for some $q^0$ was explained to us by Solomon. We refer to the appendix for general

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results about the groups Spin$_n(q)$ and $\Omega_n(q)$. Fix $S 
shortleftharpoonup$ Syl$_2(H)$. Thus $S$ is isomorphic to a Sylow 2-subgroup of $\Omega_7(q)$, and has the same fusion.

We first claim that $H$ must be simple. By definition ($H = O^2(\Phi_2^\ast)$), $H$ has no proper normal subgroup of odd index, and $H$ has no proper normal subgroup of odd order since any such subgroup would lift to an odd order normal subgroup of $\Phi = C_G(z) = O_2^2(C_G(z))$. Hence for any proper normal subgroup $N < H$, $Q \leftarrow N \setminus S$ is a proper normal subgroup of $S$, which is strongly closed in $S$ with respect to $H$ in the sense that no element of $Q$ can be $H$-conjugate to an element of $S \setminus Q$. Using Lemma A.4(a), one checks that the group $\Omega_7(q)$ contains three conjugacy classes of involutions, classified by the dimension of their $(-1)$-eigenspace. It is not hard to see (by taking products) that any subgroup of $S$ which contains all involutions in one of these conjugacy classes contains all involutions in the other two classes as well. Furthermore, $S$ is generated by the set of all of its involutions, and this shows that there are no proper subgroups which are strongly closed in $S$ with respect to $H$. Since we have already seen that the intersection with $S$ of any proper normal subgroup of $H$ would have to be such a subgroup, this shows that $H$ is simple.

Fix an isomorphism

$$S \xrightarrow{=} S^0 \big|_{\text{Syl}_2(\Omega_7(q))}$$

which preserves fusion. Choose $x^0 \big|_{\text{Syl}_2(\Omega_7(q))}$ whose $(-1)$-eigenspace is $4$-dimensional, and such that $h^x$ is fully centralized in $F_{S^0}(\Omega_7(q))$. Then

$$C_{\Omega_7(q)}(x^0) = O_4^+(q) \big|_{\text{Syl}_2(\Omega_7(q))}$$

by Lemma A.4(c). Since $\Omega_7(q)$ has index $4$, $O_4^+(q)$ and $\Omega_3(q)$ both have index $4$, $C_{\Omega_7(q)}(x^0)$ is isomorphic to a subgroup of $O_4^+(q) \big|_{\text{Syl}_2(\Omega_7(q))}$, and contains a normal subgroup $K^0 = \Omega_3(q) \big|_{\text{Syl}_2(\Omega_7(q))}$ of index $4$. Since $h^x$ is fully centralized, $C_{S^0}(x^0)$ is a Sylow 2-subgroup of $C_{\Omega_7(q)}(x^0)$, and hence $S^0 \big|_{\text{Syl}_2(\Omega_7(q))}$ is a Sylow 2-subgroup of $K^0$.

Set $x = x^{-1}(x^0) \big|_{\text{Syl}_2(\Omega_7(q))} \big|_{\text{Syl}_2(\Omega_7(q))}$. Since $S = S^0$ have the same fusion in $H$ and $\Omega_7(q)$, $C_S(x) = C_{S^0}(x^0)$ have the same fusion in $C_{H(x)}$ and $C_{\Omega_7(q)}(x^0)$. Hence

$$H_1(C_H(x); \mathbb{Z}_2) = H_1(C_{\Omega_7(q)}(x^0); \mathbb{Z}_2)$$

(homology is determined by fusion), both have order $4$, and thus $C_H(x)$ also has a unique normal subgroup $K < H$ of index $4$. Set $S_0 = K \setminus S$. Thus $S_0 = S^0$, and using Alperin's fusion theorem one can show that this isomorphism is fusion preserving with respect to the inclusions of Sylow subgroups $S_0 \big|_{\text{Syl}_2(\Omega_7(q))}$ and $S^0 \big|_{\text{Syl}_2(\Omega_7(q))} = K^0$.

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Using the isomorphisms of Proposition A.5:
\[
\Omega^1(q) = SL_2(q) \rtimes_1 S_2(q) \quad \text{and} \quad \Omega^3(q) = PSL_2(q);
\]
we can write \( K^0 = K_1^0 \rtimes_1 K_1^0 \), where \( K_1^0 = SL_2(q) \) and \( K_2^0 = SL_2(q) \) \( PSL_2(q) \). Set \( S_0^0 = S_2 \backslash K_1^0 \backslash 2Syl_2(K_1^0) \); thus \( S_0^0 = S_1^0 \rtimes_1 0 S_2^0 \). Set \( S_1^0 = \langle \Omega^1(S_0^0) \rangle \), so that \( S_0^0 = S_1^0 \rtimes_1 S_2^0 \) is normal of index 4 in \( C_5(x) \). The fusion system of \( K \) thus splits as a central product of fusion systems, one of which is isomorphic to the fusion system of \( SL_2(q) \).

We now apply a theorem of Goldschmidt, which says very roughly that under these conditions, the group \( K \) also splits as a central product. To make this more precise, let \( K_1 \) be the normal closure of \( S_1 \) in \( K \triangleleft C_5(x) \). By [14, Corollary A2], since \( S_1 \) and \( S_2 \) are strongly closed in \( S_0 \), with respect to \( K \),
\[
[K_1;K_2] \ltimes_2 O_2^3(K);
\]
Using this, it is not hard to check that \( S_1 \triangleright 2Syl_2(K_1) \). Thus \( K_1 \) has same fusion as \( SL_2(q) \) and is subnormal in \( C_5(x) \) \( (K_1 \triangleleft K \triangleleft C_5(x)) \), and an argument similar to that used above to prove the simplicity of \( H \) shows that \( K_1 = \langle \Omega^1(S_0^0) \rangle \) is simple. Hence \( K_1 \) is a 2-component of \( C_5(x) \) in the sense described by Aschbacher in [1]. By [1, Corollary III], this implies that \( H \) must be isomorphic to a Chevalley group of odd characteristic, or to \( M_{11} \). It is now straightforward to check that among these groups, the only possibility is that \( H = \Omega^1(q^3) \) for some odd prime power \( q^3 \).

\[\square\]

4 Relation with the Dwyer-Wilkerson space

We now want to examine the relation between the spaces \( BSol(q) \) which we have just constructed, and the space \( BDI(4) \) constructed by Dwyer and Wilkerson in [9]. Recall that this is a 2-complete space characterized by the property that its cohomology is the Dickson algebra in four variables over \( \mathbb{F}_2 \); i.e, the ring of invariants \( \mathbb{F}_2[x_1; x_2; x_3; x_4^{SL_2(2)}] \). We show, for any odd prime power \( q \), that \( BDI(4) \) is homotopy equivalent to the 2-completion of the union of the spaces \( BSol(q^4) \), and that \( BSol(q) \) is homotopy equivalent to the homotopy fixed point set of an Adams map from \( BDI(4) \) to itself.

We would like to define a transitive linking system \( L_{Sol}^{Sol}(q^4) \) as the union of the finite categories \( L_{Sol}^{Sol}(q^4) \), and then set \( BSol(q^4) = SL_{Sol}(q^4) j_{Sol}^{Sol}(q^4) \). The difficulty with this approach is that a subgroup which is centric in the fusion system \( F_{Sol}(q^4) \) need not be centric in a larger fusion system \( F_{Sol}(q^4) \) (for \( qj \)). To get around this problem, we define \( L_{Sol}^{Sol}(q^4) \) \( L_{Sol}^{Sol}(q^4) \) to be the full subcategory

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whose objects are those subgroups of $S(q^n)$ which are $F_{\text{Sol}}(q^n)$ (centric; or equivalently $F_{\text{Sol}}(q^n)$ (centric for all $k \geq 2$). Similarly, we define $L_{\text{Spin}}(q^n)$ to be the full subcategory of $L_{\text{Spin}}(q^n)$ whose objects are those subgroups of $S(q^n)$ which are $F_{\text{Spin}}(q^n)$ (centric). We can then define $L_{\text{Sol}}^c(q^n)$ and $L_{\text{Spin}}^c(q^n)$ to be the unions of these categories.

For these definitions to be useful, we must first show that $jL_{\text{Sol}}^c(q^n)$ has the same homotopy type as $jL_{\text{Sol}}^c(q^n)$, and $jL_{\text{Spin}}^c(q^n)$ has the same homotopy type as $jL_{\text{Spin}}^c(q^n)$. This is done in the following lemma.

**Lemma 4.1** For any odd prime power $q$ and any $n \geq 1$, the inclusions $jL_{\text{Sol}}^c(q^n) \subset jL_{\text{Sol}}^c(q^n)$ and $jL_{\text{Spin}}^c(q^n) \subset jL_{\text{Spin}}^c(q^n)$ are homotopy equivalences.

**Proof** It clearly suffices to show this when $n = 1$. Recall, for a fusion system $F$ over a $p$-group $S$, that a subgroup $P$ of $S$ is $F$-radical if $\text{Out}_F(P)$ is $F$-reduced; i.e., if $O_p(\text{Out}_F(P)) = 1$. We will show that

$$\text{all } F_{\text{Sol}}(q) \{\text{centric} \} \text{ are } F_{\text{Sol}}(q^n) \{\text{centric} \} \quad (1)$$

and similarly

$$\text{all } F_{\text{Spin}}(q) \{\text{centric} \} \text{ are } F_{\text{Spin}}(q^n) \{\text{centric} \} \quad (2)$$

In other words, (1) says that for each $P \leq S(q)$ which is a $F_{\text{Sol}}(q)$ (centric) subgroup of $S(q)$, but not of $L_{\text{Sol}}^c(q)$, $O_2 \text{Out}_{F_{\text{Sol}}(q)}(P)$ is different from 1. By [9, Proposition 6.1(ii)], this implies that $$(\text{Out}_{F_{\text{Sol}}(q)}(P); H \text{ (} B P ; \mathbb{F}_2 )) = 0.$$ Hence by [5, Propositions 3.2 and 2.2] (and the spectral sequence for a homotopy colimit), the inclusion $L_{\text{Sol}}^c(q) \to L_{\text{Sol}}^c(q)$ induces an isomorphism $H \to jL_{\text{Sol}}^c(q); \mathbb{F}_2 \to H \to jL_{\text{Sol}}^c(q); \mathbb{F}_2$.

and thus $jL_{\text{Sol}}^c(q); \mathbb{F}_2 \to jL_{\text{Sol}}^c(q); \mathbb{F}_2$. The proof that $L_{\text{Spin}}^c(q); \mathbb{F}_2 \to jL_{\text{Spin}}^c(q); \mathbb{F}_2$ is similar, using (2).

Point (2) is shown in Proposition A.12, so it remains only to prove (1). Set $F = F_{\text{Sol}}(q)$, and set $F_K = F_{\text{Sol}}(q_K)$ for all $1 \leq k \leq 1$. Let $E \to Z(P)$ be the $(2\text{torsion in the center of } P)$, so that $P \to C_{Z(q)}(E)$. Set

$$E^0 = \begin{cases} \mathbb{F}_2^{\text{hi}} & \text{if } \text{rk}(E) = 1 \\ \mathbb{F}_2^{\text{hi}}; z_i & \text{if } \text{rk}(E) = 2 \\ \mathbb{F}_2^{\text{hi}}; z_i, \mathbb{F}_2^h & \text{if } \text{rk}(E) = 3 \\ \mathbb{F}_2^{\text{hi}}; z_i; \mathbb{F}_2^h & \text{if } \text{rk}(E) = 4 \end{cases}$$

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in the notation of Definition 2.6. In all cases, \( E \) is \( F \) conjugate to \( E_0 \) by Lemma 3.1. We claim that \( E_0 \) is fully centralized in \( F_k \) for all \( k < 1 \). This is clear when \( \text{rk}(E_0) = 1 \) (\( E_0 = Z(S(q^k)) \)), follows from Proposition 2.5(a) when \( \text{rk}(E_0) = 2 \), and from Proposition A.8(a) (all rank 4 subgroups are self centralizing) when \( \text{rk}(E_0) = 4 \). If \( \text{rk}(E_0) = 3 \), then by Proposition A.8(d), the centralizer in \( \text{Spin}_7(q^k) \) (hence in \( S(q^k) \)) of any rank 3 subgroup has an abelian subgroup of index 2; and using this (together with the construction of \( S(q^k) \) in Deinition 2.6), one sees that \( E_0 \) is fully centralized in \( F_k \).

If \( E_0 \not\subseteq E \), choose \( \leq 2 \) Hom\(_F \)(E; S(q)) such that \( \langle E \rangle = E_0 \); then \( \langle E \rangle \) extends to \( \leq 2 \) Hom\(_F \)(C\(_S(q)(E); S(q) \)) by condition (II) in the definition of a saturated fusion system, and we can replace \( P \) by \( \langle P \rangle \) and \( E \) by \( \langle E \rangle \). We can thus assume that \( E \) is fully centralized in \( F_k \) for each \( k < 1 \). So by [6, Proposition 2.5(a)], \( P \) is \( F_k \) {centric if and only if it is \( C_{F_k}(E) \) {centric; and this also holds when \( k = 1 \). Furthermore, since \( \text{Out}_{CF}(E)(P) \vartriangleleft \text{Out}_F(P) \), \( O_2(\text{Out}_{CF}(E)(P)) \) is a normal 2{subgroup of \( \text{Out}_F(P) \), and thus

\[
O_2 \text{Out}_{CF}(E)(P) = O_2(\text{Out}_F(P));
\]

Hence \( P \) is \( C_{F}(E) \) {radical if it is \( F \) {radical. So it remains to show that all \( C_{F}(E) \) {centric \( C_{F}(E) \) {radical subgroups of \( S(q) \) are also \( C_{F_1}(E) \) {centric.

(3)

If \( \text{rk}(E) = 1 \), then \( C_{F}(E) = F_{\text{Spin}}(q) \) and \( C_{F_1}(E) = F_{\text{Spin}}(q^k) \), and (3) follows from (2). If \( \text{rk}(E) = 4 \), then \( P = E = C_{S(q^k)(E)} \) by Proposition A.8(a), so \( P \) is \( F_1 \) {centric, and the result is clear.

If \( \text{rk}(E) = 3 \), then by Proposition A.8(d), \( C_{F}(E) = C_{F_1}(E) \) are the fusion systems of a pair of semidirect products \( A \rtimes C_2 \) \( A_1 \rtimes C_2 \), where \( A \) \( A_1 \) are abelian and \( C_2 \) acts on \( A_1 \) by inversion. Also, \( E \) is the full 2{torsion subgroup of \( A_1 \), since otherwise \( \text{rk}(A_1) > 3 \) would imply \( A_1 \rtimes C_2 \) \( \text{Spin}_7(q^k) \) contains a subgroup \( C_2^5 \) (contradicting Proposition A.8). If \( P = A \), then either \( \text{Out}_{CF}(E)(P) \) has order 2, which contradicts the assumption that \( P \) is radical; or \( P \) is elementary abelian and \( \text{Out}_{CF}(E)(P) = 1 \), in which case \( P = Z(A \rtimes C_2) \) is not centric. Thus \( P \not\subseteq A \); \( P \backslash A \) \( E \) contains all 2{torsion in \( A_1 \), and hence \( P \) is centric in \( A_1 \rtimes C_2 \).

If \( \text{rk}(E) = 2 \), then by Proposition 2.5(a), \( C_{F_1}(E) \) and \( C_{F}(E) \) are the fusion systems of the groups

\[
H(q^k) = SL_2(q^k) \not\cong (1;1;1)g
\]

and

\[
H(q) = H(q^2) \backslash \text{Spin}_7(q) \quad H_0(q) \cong SL_2(q) \not\cong (1;1;1)g.
\]
If \( P \) is centric and radical in the fusion system of \( H(q) \), then by Lemma A.11(c), its intersection with \( H_0(q) = \text{SL}_2(q)^3 \) is centric and radical in the fusion system of that group. So by Lemma A.11(a,f),
\[
P \setminus H_0(q) = (P_1 \cap P_2 \cap P_3) = (I;I;I)g
\]
for some \( P_i \) which are centric and radical in the fusion system of \( \text{SL}_2(q) \). Since the Sylow 2-subgroups of \( \text{SL}_2(q) \) are quaternion [15, Theorem 2.8.3], the \( P_i \) must be nonabelian and quaternion, so each \( P_i = (I;I;I)g \) by (5), and so \( P \) is centric in \( H(q) \) by (4).

We would like to be able to regard \( \text{B Spin}_7(q) \) as a subcomplex of \( \text{B Sol}(q) \), but there is no simple natural way to do so. Instead, we set
\[
\text{B Spin}_0^0(q) = jL_{\text{Spin}}^\infty(q)_2 \cdot jL_{\text{Sol}}^\infty(q)_2 \cdot \text{B Sol}(q);
\]
then \( \text{B Spin}_0^0(q) \) by [5, Proposition 1.1] and Lemma 4.1. Also, we write
\[
\text{B Sol}^0(q) = jL_{\text{Sol}}^\infty(q)_2 \cdot \text{B Sol}(q) \quad \text{def} \quad jL_{\text{Sol}}^\infty(q)_2;
\]
to denote the subcomplex shown in Lemma 4.1 to be equivalent to \( \text{B Sol}(q) \); and set
\[
\text{B Spin}_0^0(q^1) = jL_{\text{Spin}}^\infty(q^1)_2;
\]
From now on, when we talk about the inclusion of \( \text{B Spin}_7(q) \) into \( \text{B Sol}(q) \), as long as it need only be well defined up to homotopy, we mean the composite
\[
\text{B Spin}_7(q) \rightarrow \text{B Spin}_0^0(q) \rightarrow \text{B Sol}(q)
\]
(for some choice of homotopy equivalence). Similarly, if we talk about the inclusion of \( \text{B Sol}(q^m) \) into \( \text{B Sol}(q^n) \) (for \( m \neq n \)) where it need only be defined up to homotopy, we mean these spaces identified with their equivalent subcomplexes \( \text{B Sol}^0(q^m) \rightarrow \text{B Sol}^0(q^n) \).

**Lemma 4.2** Let \( q \) be any odd prime. Then for all \( n \geq 1 \),
\[
\begin{array}{c}
H(\text{B Sol}(q^n); \mathbb{F}_2) \rightarrow H(\text{B H}(q^n); \mathbb{F}_2)^{C_3} \\
\# & \# \\
H(\text{B Spin}_7(q^n); \mathbb{F}_2) \rightarrow H(\text{B H}(q^n); \mathbb{F}_2)
\end{array}
\]
(with all maps induced by inclusions of groups or spaces) is a pullback square.
Proof By [6, Theorem B], \( H(B\text{Sol}(q^n); F_2) \) is the ring of elements in the cohomology of \( S(q^n) \) which are stable relative to the fusion. By the construction in Section 2, the fusion in \( \text{Sol}(q^n) \) is generated by that in \( \text{Spin}_7(q^n) \), together with the permutation action of \( C_3 \) on the subgroup \( H(q^n) \text{Spin}_7(q^n) \), and hence (1) is a pullback square.

\[
\begin{array}{c}
\text{BSol}(q^n) \\
\downarrow \text{#} \\
\text{BDI}(4)
\end{array}
\]

\[
\begin{array}{c}
\text{BSpin}(q^n) \\
\downarrow \text{#} \\
\text{BDI}(4)
\end{array}
\]

\[
\begin{array}{c}
\text{BSpin}(7) \\
\downarrow \text{#} \\
\text{BDI}(4)
\end{array}
\]

Proposition 4.3 For each odd prime \( q \), there is a category \( L^c_{\text{Sol}}(q^n) \), together with a functor

\[
L^c_{\text{Sol}}(q^n) \rightarrow \text{Sol}(q^n)
\]

such that the following hold:

(a) For each \( n \geq 1 \), \( -1(F_{\text{Sol}}(q^n)) = L^c_{\text{Sol}}(q^n) \).

(b) There is a homotopy equivalence

\[
B\text{Sol}(q^n) \xrightarrow{\text{def}} jL^c_{\text{Sol}}(q^n) \xrightarrow{j^2} B\text{DI}(4)
\]

such that the following square commutes up to homotopy

\[
\begin{array}{c}
B\text{Spin}(q^n) \xrightarrow{(q^n)} B\text{Sol}(q^n) \\
\downarrow \text{#} \\
B\text{Spin}(7) \xrightarrow{\text{#}} B\text{DI}(4)
\end{array}
\]

Here, \( _0 \) is the homotopy equivalence of [13], induced by some fixed choice of embedding of the Witt vectors for \( \overline{F}_q \) into \( \mathbb{C} \), while \( (q^n) \) is the union of the inclusions \( jL^c_{\text{Spin}}(q^n)j^2 \), \( jL^c_{\text{Sol}}(q^n)j^2 \), and \( b \) is the inclusion arising from the construction of \( B\text{DI}(4) \) in [9].

Furthermore, there is an automorphism \( L^c_L \) of categories which satisfies the conditions:

(c) the restriction of \( L^c_L \) to each subcategory \( L^c_{\text{Sol}}(q^n) \) is equal to the restriction of \( L^c_L \) as defined in Proposition 3.3(d);

(d) \( L^c_L \) covers the automorphism \( L^c_F \) of \( F_{\text{Sol}}(q^n) \) induced by the field automorphism \( (x, y) \rightarrow (x^q, y) \); and

(e) for each \( n \), \( L^c_L^n \) is \( L^c_{\text{Sol}}(q^n) \).

Proof By Proposition 2.11, the inclusions \( \text{Spin}_7(q^n) \xrightarrow{\text{Spin}_7(q^n)} \text{Sol}(q^n) \) for all \( m \) induce inclusions of fusion systems \( F_{\text{Sol}}(q^n) \xrightarrow{F_{\text{Sol}}(q^n)} F_{\text{Sol}}(q^n) \). Since the restriction of

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a linking system over $F_{Sol}^c(q^n)$ is a linking system over $F_{Sol}^c(q^m)$, the uniqueness of linking systems (Proposition 3.3) implies that we get inclusions $L_{Sol}^c(q^m) \subseteq L_{Sol}^c(q^n)$. We define $L_{Sol}^c(q^n)$ to be the union of the finite categories $L_{Sol}^c(q^n)$. (More precisely, fix a sequence of positive integers $n_1, n_2, n_3, \ldots$ such that every positive integer divides some $n_i$, and set

$$L_{Sol}^c(q^n) = \bigsqcup_{i=1}^\infty L_{Sol}^c(q^{n_i}).$$

Then by uniqueness again, we can identify $L_{Sol}^c(q^n)$ for each $n$ with the appropriate subcategory.)

Let $L_{Sol}^c(q^n) \to F_{Sol}(q^n)$ be the union of the projections from $L_{Sol}^c(q^n)$ to $F_{Sol}(q^n)$. Condition (a) is clearly satisfied. Also, using Proposition 3.3(d) and Lemma 4.1, we see that there is an automorphism $\phi$ of $L_{Sol}^c(q^n)$ which satisfies conditions (c,d,e) above. (Note that by the fusion theorem as shown in [6, Theorem A.10], morphisms in $L_{Sol}^c(q^n)$ are generated by those between radical subgroups, and hence by those in $L_{Sol}^c(q^n)$.)

It remains only to show that $jL_{Sol}^c(q^n)$ BDI(4), and to show that square (1) commutes. The space BDI(4) is 2-complete by its construction in [9]. By Lemma 4.1,

$$H(BSol(q^n); F_2) = \lim_n H(jL_{Sol}^c(q^n)); F_2 = \lim_n H(BSol(q^n); F_2).$$

Hence by Lemma 4.2 (and since the inclusions BSpin$_7(q^n) \to BSol(q^n)$ commute with the maps induced by inclusions of fields $F_{q^n}$), there is a pullback square

$$\begin{array}{ccc}
H(BSol(q^n); F_2) & \to & H(BSpin(q^n); F_2) \\
\downarrow & & \downarrow \\
H(BSpin(q^n); F_2) & \to & H(BSpin(q^n); F_2). \\
\end{array}$$

(2)

Also, by [13, Theorem 1.4], there are maps

$$BSpin_7(q^n) \to BSpin(7) \quad \text{and} \quad BH(q^n) \to SU(2)^3 \cong SU(2)^3 \cong SU(2).$$

which induce isomorphisms of $F_2$-cohomology, and hence homotopy equivalences after 2-completion. So by Propositions 4.7 and 4.9 (or more directly by the computations in [9, section 3]), the pullback of the above square is the ring of Dickson invariants in the polynomial algebra $H(BC_2^2; F_2)$, and thus isomorphic to $H(BDI(4); F_2)$.

Point (b), including the commutativity of (1), now follows from the following lemma.
Lemma 4.4 Let $X$ be a 2-complete space such that $H(X; \mathbb{F}_2)$ is the Dickson algebra in 4 variables. Assume further that there is a map $B\text{Spin}(7) \rightarrow X$ such that $H(f_{BC_2}; \mathbb{F}_2)$ is the inclusion of the Dickson invariants in the polynomial algebra $H(BC_2; \mathbb{F}_2)$. Then $X \rightarrow BDI(4)$. More precisely, there is a homotopy equivalence between these spaces such that the composite

$$B\text{Spin}(7) \rightarrow X \rightarrow BDI(4)$$

is the inclusion arising from the construction in [9].

Proof In fact, Notbohm [18, Theorem 1.2] has proven that the lemma holds even without the assumption about $B\text{Spin}(7)$ (but with the more precise assumption that $H(X; \mathbb{F}_2)$ is isomorphic as an algebra over the Steenrod algebra to the Dickson algebra). The result as stated above is much more elementary (and also implicit in [9]), so we sketch the proof here.

Since $H(X; \mathbb{F}_2)$ is a polynomial algebra, $H(\Omega X; \mathbb{F}_2)$ is isomorphic as a graded vector space to an exterior algebra on the same number of variables, and in particular is finite. Hence $X$ is a 2-complete group. By [11, Theorem 8.1] (the centralizer decomposition for a p-compact group), there is an $\mathbb{F}_2$-homology equivalence

$$\text{hocolim}( ) \rightarrow X$$

Here, $A$ is the category of pairs $(V, ')$, where $V$ is a nontrivial elementary abelian 2-group, and $': BV \rightarrow X$ makes $H(BV; \mathbb{F}_2)$ into a finitely generated module over $H(X; \mathbb{F}_2)$ (see [10, Proposition 9.11]). Morphisms in $A$ are defined by letting $\text{Mor}_A((V, '); (V_0, '))$ be the set of monomorphisms $V \rightarrow V_0$ of groups which make the obvious triangle commute up to homotopy. Also,

$$A^{op} \rightarrow \text{Top}$$

is the functor $(V, ')$ = $\text{Map}(BV; X)$:

By [9, Lemma 1.6(1)] and [17, Theoreme 0.4], $A$ is equivalent to the category of elementary abelian 2-groups $E$ with $1 \leq \text{rk}(E) \leq 4$, whose morphisms consist of all group monomorphisms. Also, if $BC_2 \rightarrow X$ is the restriction of $f$ to any subgroup $C_2 \rightarrow \text{Spin}(7)$, then in the notation of Lannes,

$$T_{C_2}(H(X; \mathbb{F}_2); ')) = H(B\text{Spin}(7); \mathbb{F}_2)$$

by [9, Lemmas 16.(3), 3.10 and 3.11], and hence

$$H(\text{Map}(BC_2; X); \mathbb{F}_2) = H(B\text{Spin}(7); \mathbb{F}_2)$$
Construction of 2-finite groups by Lannes [17, Theoreme 3.2.1]. This shows that

\[ \text{Map}(\text{BC}_2; X) \cong \text{BSpin}(7)^2; \]

and thus that the centralizers of other elementary abelian 2-groups are the same as their centralizers in \( \text{BSpin}(7)^2 \). In other words, \( \text{BSpin}(7)^2 \) is equivalent in the homotopy category to the diagram used in [9] to define \( \text{BDI}(4) \). By [9, Proposition 7.7] (and the remarks in its proof), this homotopy functor has a unique homotopy lifting to spaces. So by definition of \( \text{BDI}(4) \),

\[ X \xrightarrow{\text{hocolim}(\ )} \text{BDI}(4): \]

Set \( B^q \overset{\text{def}}{=} j^q \), a self homotopy equivalence of \( B\text{Sol}(q^1) \) to \( \text{BDI}(4) \). By construction, the restriction of \( B^q \) to the maximal torus of \( B\text{Sol}(q^1) \) is the map induced by \( x \mapsto x^q \), and hence this is an \( \text{Adams map} \) as defined by Notbohm [18]. In fact, by [18, Theorem 3.5], there is an Adams map from \( \text{BDI}(4) \) to itself, unique up to homotopy, of degree any 2-adic unit.

Following Benson [3], we define \( \text{BDI}_4(q) \) for any odd prime power \( q \) to be the homotopy fixed point set of \( B\text{Sol}(q^1) \) by the Adams map \( B^q \). By \( \text{homotopy fixed point set} \), in this situation, we mean that the following square is a homotopy pullback:

\[ \begin{array}{ccc}
B\text{Di}_4(q) & \longrightarrow & B\text{Sol}(q^1) \\
\downarrow \# & & \downarrow \#
\end{array} \]

\[ \begin{array}{ccc}
\text{BSol}(q^1) & \longrightarrow & \text{Bsol}(q^1)^2 \\
(\text{id}; B^q) & & \text{id}
\end{array} \]

The actual pullback of this square is the subspace \( B\text{Sol}(q) \) of elements fixed by \( B^q \), and we thus have a natural map \( B\text{Sol}(q) \rightarrow \text{BDI}_4(q) \).

**Theorem 4.5** For any odd prime power \( q \), the natural map

\[ \begin{array}{ccc}
B\text{Sol}(q) & \longrightarrow & B\text{Di}_4(q) \\
\end{array} \]

is a homotopy equivalence.

**Proof** Since \( \text{BDI}(4) \) is simply connected, the square used to define \( \text{BDI}_4(q) \) remains a homotopy pullback square after 2-completion by [4, II.5.3]. Thus \( \text{BDI}_4(q) \) is 2-complete. Also, \( B\text{Sol}(q) \overset{\text{def}}{=} j L_{\text{Sol}}(q)^2 \) is 2-complete since \( j L_{\text{Sol}}(q) \) is 2-good [6, Proposition 1.12], and hence it suffices to prove that the map
between these spaces is an \( F_2 \) cohomology equivalence. By Lemma 4.2, this means showing that the following commutative square is a pullback square:

\[
\begin{array}{ccc}
H(B D I_4(q); F_2) & \longrightarrow & H(B H(q); F_2)^G \\
\downarrow & & \downarrow \\
H(B \text{Spin}_7(q); F_2) & \longrightarrow & H(B H(q); F_2)
\end{array}
\]  

(1)

Here, the maps are induced by the composite

\[
B \text{Spin}_7(q) \to B \text{Spin}_7(F_q) \to B \text{Sol}(q)
\]

and its restriction to \( B H(q) \). Also, by Proposition 4.3(b), the following diagram commutes up to homotopy:

\[
\begin{array}{ccc}
B \text{Spin}_7(q) & \longrightarrow & B \text{Spin}_7(F_q) \\
(q) \downarrow & & \downarrow \\
B \text{Sol}(q) & \longrightarrow & B \text{Sol}(F_q)
\end{array}
\]  

(2)

By [12, Theorem 12.2], together with [13, section 1], for any connected reductive Lie group \( G \) and any algebraic epimorphism on \( G(F_q) \) with finite kernel, there is a homotopy pullback square

\[
\begin{array}{ccc}
B(G(F_q)) & \longrightarrow & B G(F_q)^G \\
\downarrow & & \downarrow \\
B G(F_q) & \longrightarrow & B G(F_q)
\end{array}
\]  

(3)

We need to apply this when \( G = \text{Spin}_7 \) or \( G = H = (\text{SL}_2)^3 \rtimes (I; I; I)g \). In particular, if \( q = q^3 \) is the automorphism induced by the field automorphism \( (x \mapsto x^q) \), then \( \text{Spin}_7(F_q) = \text{Spin}_7(q) \) by Lemma A.3, and \( H(F_q) = H(q) \equiv H(F_q) \setminus \text{Spin}_7(q) \). We thus get a description of \( B \text{Spin}_7(q) \) and \( B H(q) \) as homotopy pullbacks.

By [13, Theorem 1.4], \( B G(F_q)^G \to B G(C)^G \). Also, we can replace the complex Lie groups \( \text{Spin}_7(C) \) and \( H(C) \) by maximal compact subgroups \( \text{Spin}(7) \) and \( H \equiv \text{SU}(2)^3 \rtimes (I; I; I)g \), since these have the same homotopy type.

If we set \( R = H(B G(F_q); F_2) = H(B G(C); F_2) \), then there are Eilenberg-Moore spectral sequences

\[
E_2 = \text{Tor}_{R \otimes R}(R; R) \Rightarrow H(B G(F_q); F_2)
\]
where the \((R \otimes R^{\text{op}})\) module structure on \(R\) is defined by setting \(a \otimes b \cdot x = a \cdot b \cdot x\). When \(G = \text{Spin}_7\) or \(H\), then \(R\) is a polynomial algebra by Proposition 4.7 and the above remarks, and \(B\) acts on \(R\) via the identity. The above spectral sequence thus satisfies the hypotheses of [20, Theorem II.3.1], and hence collapses. (Alternatively, note that in this case, \(E_2\) is generated multiplicatively by \(E_0^2\) and \(E_2^{-1}\) by (5) below.) Similarly, when \(R = H(B D I_4(q); F_2)\), there is an analogous spectral sequence which converges to \(H(B D I_4(q); F_2)\), and which collapses for the same reason. By the above remarks, these spectral sequences are natural with respect to the inclusions \(H(B H(-); B \text{Spin}_7(-))\), and (using the naturality of \(q\) shown in Proposition 3.3(d)) of \(B \text{Spin}_7(-)\) into \(B \text{Sol}(-)\) or \(B D I(4)\).

To simplify the notation, we now write

\[
\begin{align*}
A & \equiv H(B D I_4(4); F_2); \\
B & \equiv H(B \text{Spin}(7); F_2); \\
C & \equiv H(H; F_2)
\end{align*}
\]

to denote these cohomology rings. The Frobenius automorphism \(q\) acts via the identity on each of them. We claim that the square

\[
\begin{array}{ccc}
\text{Tor}_{A \otimes A^{\text{op}}} & \longrightarrow & \text{Tor}_{C \otimes C^{\text{op}}} \\
\downarrow \# & & \downarrow \#
\end{array}
\]

is a pullback square. Once this has been shown, it then follows that in each degree, square (1) has a finite filtration under which each quotient is a pullback square. Hence (1) itself is a pullback.

For any commutative \(F_2\)-algebra \(R\), let \(\Omega_{\mathfrak{R} \otimes F_2}\) denote the \(R\) module generated by elements \(dr\) for \(r \in R\) with the relations \(dr = 0\) if \(r \in F_2\),

\[
d(r + s) = dr + ds \quad \text{and} \quad d(rs) = r \cdot ds + s \cdot dr.
\]

Let \(\Omega_{\mathfrak{R} \otimes F_2}\) denote the ring of Kähler differentials: the exterior algebra (over \(\mathfrak{R}\)) of \(\Omega_{\mathfrak{R} \otimes F_2} = \Omega_{\mathfrak{R} \otimes F_2}^1\). When \(\mathfrak{R}\) is a polynomial algebra, there are natural identifications

\[
\text{Tor}_{\mathfrak{R} \otimes \mathfrak{R}^{\text{op}}} = HH(\mathfrak{R}) = \Omega_{\mathfrak{R} \otimes F_2}^1.
\]

The first isomorphism holds for arbitrary algebras, and is shown, e.g., in [25, Lemma 9.1.3]. The second holds for smooth algebras over an algebra [25, Theorem 9.4.7] (and polynomial algebras are smooth as shown in [25, section 9.3.1]). In particular, the isomorphisms (5) hold for \(\mathfrak{R} = A; B; C\), which are shown to be polynomial algebras in Proposition 4.7 below. Thus, square (4) is isomorphic...
to the square
\[
\begin{array}{c}
\Omega_{\mathfrak{g} \oplus \mathfrak{g}} \quad \Omega_{\mathfrak{g} \oplus \mathfrak{g}}^C \\
\downarrow \quad \downarrow \\
\Omega_{\mathfrak{g} \oplus \mathfrak{g}} \quad \Omega_{\mathfrak{g} \oplus \mathfrak{g}}^C,
\end{array}
\]
which is shown to be a pullback square in Propositions 4.7 and 4.9 below. □

It remains to prove that square (6) in the above proof is a pullback square. In what follows, we let \(D_i(x_1;\ldots;x_n)\) denote the \(i\)-th Dickson invariant in variables \(x_1;\ldots;x_n\). This is the \((2^n - 2^{n-i})\)-th symmetric polynomial in the elements (equivalently in the nonzero elements) of the vector space \(\mathfrak{h} x_1;\ldots;x_n\mathfrak{l}_{\mathbb{F}_2}\).

We refer to [26] for more detail. Note that what he denotes \(c_{n,i}\) is what we call \(D_{n-i}(x_1;\ldots;x_n)\).

**Lemma 4.6** For any \(n\),
\[
D_1(x_1;\ldots;x_{n+1}) = \sum_{x \in \mathfrak{h} x_1;\ldots;x_n \mathfrak{l}_{\mathbb{F}_2}} (x_{n+1} + x) + D_1(x_1;\ldots;x_n)^2
\]
\[
= x_{n+1}^{2^n} + \sum_{i=1}^{n-1} x_{n+1}^{2^n-i} D_i(x_1;\ldots;x_n) + D_1(x_1;\ldots;x_n)^2.
\]

**Proof** The first equality is shown in [26, Proposition 1.3(b)]; here we prove them both simultaneously. Set \(V_n = \mathfrak{h} x_1;\ldots;x_n \mathfrak{l}_{\mathbb{F}_2}\). Since \(i(V_n) = 0\) whenever \(2^n - i\) is not a power of 2 (cf [26, Proposition 1.1]),
\[
D_1(x_1;\ldots;x_{n+1}) = \sum_{i=0}^{n} i(V_n) x_{n+1}^{2^n-i} (x_{n+1} + V_n)
\]
\[
= \sum_{i=1}^{n-1} D_i(x_1;\ldots;x_n) x_{n+1}^{2^n-i} (x_{n+1} + V_n):
\]

Also, since \(i(V_n) = 0\) for \(0 < i < 2^n - 1\) as noted above,
\[
k(x_{n+1} + V_n) = \sum_{i=0}^{n} x_{n+1}^{k-i} x_{n+1}^{2^n-i} i(V_n) = 0 \quad \text{if} \quad 0 < k < 2^n - 1
\]
\[
D_1(x_1;\ldots;x_n) = 0 \quad \text{if} \quad k = 2^n - 1.
\]

This proves the first equality, and the second follows since
\[
\sum_{x \in \mathfrak{h} x_1;\ldots;x_n \mathfrak{l}_{\mathbb{F}_2}} (x_{n+1} + x) = \sum_{i=1}^{n-1} x_{n+1}^{2^n-i} i(V_n) = \sum_{i=1}^{n-1} x_{n+1}^{2^n-i} D_i(x_1;\ldots;x_n):
\]

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Thus, as defined above, together with the identification $H \cong F_2$ with the polynomial ring $F_2$, in the following proposition (and throughout the rest of the section), we work with the polynomial ring $F_2$. Let

$$\text{GL}_2(F_2); \text{GL}_3(F_2); \text{GL}_4(F_2)$$

be the subgroups of automorphisms of $V$ defined by $h; y; z; w_i$ which leave invariant the subspaces $h; y_i$ and $h; y_i; z_i$, respectively. Also, let $\text{GL}_2(F_2); \text{GL}_3(F_2)$ be the subgroup of automorphisms which are the identity modulo $h; y_i$. Thus, when described in terms of block matrices (with respect to the given basis $f; y; z; w_i$),

$$\text{GL}_3(F_2) = \begin{pmatrix} A & X \\ 0 & 1 \end{pmatrix} \quad \text{GL}_2(F_2) = \begin{pmatrix} B & Y \\ 0 & C \end{pmatrix} \quad \text{GL}_2(F_2) = \begin{pmatrix} B & Y \\ 0 & C \end{pmatrix}$$

for $A, B, C \in \text{Gl}_2(F_2)$, $X$ a column vector, $B; C \in \text{Gl}_2(F_2)$, and $Y \in \text{M}_2(F_2)$. We need to make more precise the relation between $V$ (or the polynomial ring $F_2[x; y; z; w]$) and the cohomology of $\text{Spin}(7)$. To do this, let $W = \text{Spin}(7)$ be the inverse image of the elementary abelian subgroup

$$\text{diag}(-1; -1; -1; -1; 1; 1); \text{diag}(-1; -1; 1; -1; -1; 1); \text{SO}(7): \text{diag}(-1; -1; -1; 1; -1; 1)$$

Thus, $W = C_2$. Fix a basis $f; 0; g$ for $W$, where $2 \text{Z}(\text{Spin}(7))$ is the nontrivial element. Identify $V = \text{W}$ in such a way that $f; y; z; w_i$ and $V$ is the dual basis to $f; 0; g$. This gives an identification $H = \text{B}(B; W; F_2) = F_2[x; y; z; w]$ arranged such that the action of $N_{\text{Spin}(7)}(W) \cong W$ on $V = h; y; z; w_i$ consists of all automorphisms which leave invariant, and thus can be identified with the action of $\text{GL}_3(F_2)$. Finally, set

$$\text{H} = C_{\text{Spin}(7)}(\text{W}) = \text{Spin}(4) \quad c_1 \text{Spin}(3) = \text{SU}(2)3 \text{F} (1; 1; 1)$$

and (the central product). Then in the same way, the action of $N_{\text{H}}(W) \cong W$ on $H = \text{B}(B; W; F_2)$ can be identified with that of $\text{GL}_2(F_2)$.

**Proposition 4.7** The inclusions

$$\text{B}(B; W; F_2) \rightarrow \text{B}(W; F_2) \rightarrow \text{B}(\text{Spin}(7); F_2) \rightarrow \text{B}(\text{DI}(4); F_2)$$

as defined above, together with the identification $H = \text{B}(B; W; F_2) = F_2[x; y; z; w]$, induce isomorphisms

$$a \overset{\text{def}}{=} H = \text{B}(\text{DI}(4); F_2) = F_2[x; y; z; w]^\text{GL}_4(F_2) = F_2[a_8; a_{12}; a_{14}; a_{15}]$$

$$b \overset{\text{def}}{=} H = \text{B}(\text{Spin}(7); F_2) = F_2[x; y; z; w]^\text{GL}_2(F_2) = F_2[b_1; b_2; b_3; b_4]$$

$$c \overset{\text{def}}{=} H = \text{B}(H; F_2) = F_2[x; y; z; w]^\text{GL}_2(F_2) = F_2[c_2; c_3; c_4]$$

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where
\[ a_8 = D_1(x; y; z; w); \quad a_{12} = D_2(x; y; z; w); \]
\[ a_{14} = D_3(x; y; z; w); \quad a_{15} = D_4(x; y; z; w); \]
\[ b_4 = D_1(x; y; z); \quad b_6 = D_2(x; y; z); \quad b_7 = D_3(x; y; z); \quad b_8 = Y (w + ); \]

and
\[ c_2 = D_1(x; y); \quad c_3 = D_2(x; y); \quad c_4^0 = Y (z + ); \quad c_4^0 = Y (w + ); \]

Furthermore,

(a) the natural action of \( H = SU(2)^3 \) on \( C \) which takes \( c_2, c_3 \) and permutes \( f c_0^0, c_0^0 + c_0^0 \) and
(b) the above variables satisfy the relations
\[
\begin{align*}
    a_8 &= b_8 + b_4^2 \quad a_{12} = b_4 b_6 + b_6^2 \quad a_{14} = b_6 b_8 + b_7^2 \quad a_{15} = b_8 b_7 \\
    b_4 &= c_0^0 + c_2 \quad b_6 = c_2 c_0^0 + c_3 \quad b_7 = c_3 c_0^0 \quad b_8 = c_0^0 (c_0^0 + c_0^0) \
\end{align*}
\]

**Proof** The formulas for \( \mathfrak{A} = H (BDI(4); F_2) \) are shown in [9]. From [9, Lemmas 3.10 and 3.11], we see there are (some) identifications

\[
H (B Spin(7); F_2) = F_2[x; y; z; w]^{GL_1(F_2)} \quad \text{and} \quad H (B H; F_2) = F_2[x; y; z; w]^{GL_2(F_2)}
\]

From the explicit choices of subgroups \( W \) and \( Spin(7) \) as described above (and by the descriptions in Proposition A.8 of the automorphism groups), the images of \( H (B Spin(7); F_2) \) and \( H (B H; F_2) \) in \( F_2[x; y; z; w] \) are seen to be contained in the rings of invariants, and hence these isomorphisms actually are equalities as claimed.

We next prove the equalities in ( ) between the given rings of invariants and polynomial algebras. The following argument was shown to us by Larry Smith. If \( k \) is a field and \( V \) is an \( n \)-dimensional vector space over \( k \), then a system of parameters in the polynomial algebra \( k[V] \) is a set of \( n \) homogeneous elements \( f_1, \ldots, f_n \) such that \( k[V] = \langle f_1, \ldots, f_n \rangle \) is finite dimensional over \( k \). By [21, Proposition 5.5.5], if \( V \) is an \( n \)-dimensional \( k[G] \) representation, and \( f_1, \ldots, f_n \) \( 2 k[V] \) is a system of parameters the product of whose degrees is equal to \( |G| \), then \( k[V] \) is a polynomial algebra with \( f_1, \ldots, f_n \) as generators. By [21, Proposition 8.1.7], \( F_2[x; y; z; w] \) is a free, finitely generated
module over the ring generated by its Dickson invariants (this holds for polynomial algebras over any $F_p$), and thus $F_2[x; y; z; w]$ is finite. (This can also be shown directly using the relation in Lemma 4.6.) So assuming the relations in point (b), the quotients $F_2[x; y; z; w] = (b_1; b_2; b_3; b_4)$ and $F_2[x; y; z; w]/(c_2; c_3; c_4; c_5)$ are also finite. In each case, the product of the degrees of the generators is clearly equal to the order of the group in question, and this finishes the proof of the last equality in the second and third lines of (1).

It remains to prove points (a) and (b). Using Lemma 4.6, the $c_i$ are expressed as polynomials in $x; y; z; w$ as follows:

$$c_2 = D_1(x; y) = x^2 + xy + y^2$$
$$c_3 = D_2(x; y) = xy(x + y)$$
$$c_4^0 = D_1(x; y; z) + D_1(x; y)^2 = z^4 + z^2c_1 + z^2c_2 + zc_3$$
$$c_4^0 = D_1(x; y; w) + D_1(x; y)^2 = w^4 + w^2c_1 + wc_2 + wc_3.$$  

In particular,

$$c_4^0 + c_4^0 = (z + w)^4 + (z + w)^2D_1(x; y) + (z + w)D_2(x; y) = Y \quad \text{for } z + w \neq 0 \quad (2)$$

Furthermore, by (1), we get

$$Sq^2(c_2) = c_3$$
$$Sq^1(c_3) = Sq^1(c_4^0) = Sq^1(c_4^0) = 0$$
$$Sq^2(c_3) = x^2y^2(x + y) + xy(x + y)^3 = c_2c_3$$
$$Sq^2(c_4^0) = z^4c_2 + z^2c_4^0 + zc_3c_3 = c_4c_4$$

The permutation action of $G = SU(2)^3$ on $\tilde{H} = SU(2)^3$ permutes the three elements $a_1; a_1; a_1; a_1$ of $Z(\tilde{H})$, and thus (via the identification $V = W$ described above) induces the identity on $x; y 2 V$ and permutes the elements $fz; wz + wg$ modulo $x; y$. Hence the induced action of $G$ on $c = F_2[V]^G$ is the restriction of the action on $F_2[V]/F_2[x; y; z; w]$ which permutes $c_2$ and permutes the set $c_3^0; c_4^0; c_5^0$. This proves (a).

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Lemma 4.8

It remains to prove the formulas in (b). From (1) and (3) we get

\[
\begin{align*}
    b_1 &= D_1(x; y; z) = c_1^4 + c_2^4; \\
    b_2 &= D_2(x; y; z) = Sq^2(b_1) = c_2 c_4^0 + c_3^4; \\
    b_7 &= D_3(x; y; z) = Sq^1(b_3) = c_3 c_1^4.
\end{align*}
\]

Also, by (1) and (2),

\[
\begin{align*}
    b_8 &= Y (w + ) = Y (w + ) (w + z + ) = c_3^4 (c_3 + c_4^0); \\
    b_9 &= 2h: (w + ) (w + ) (w + z + ) = c_3^4 (c_3 + c_4^0); \\
    b_{10} &= 2h: (w + ) (w + ) (w + z + ) = c_3^4 (c_3 + c_4^0);
\end{align*}
\]

This proves the formulas for the \( b_i \) in terms of \( c \). Finally, we have

\[
\begin{align*}
    a_8 &= D_3(x; y; z; w) = b_8 + b_9; \\
    a_{12} &= D_2(x; y; z; w) = Sq^1(b_8 + b_9) = Sq^1(c_3^4 (c_3 + c_4^0) + (c_3 + c_4^2)^2) \\
    &= c_4^0 (c_4^0 + c_4^0) + c_2^0 (c_4^0 + c_4^0) + c_2^0 (c_4^0 + c_3^4) + c_3^4 = b_8 b_9 + b_{10}; \\
    a_{14} &= D_3(x; y; z; w) = Sq^1(a_{12}) = c_3^4 (c_3^4 (c_3 + c_4^0) + (c_3 + c_4^2)^2) \\
    &= c_3^4 (c_3^4 (c_3 + c_4^0) + c_3^4) + c_3^4 (c_3 + c_4^0) + c_4^0 c_2^3 + c_2^0 c_2^3 + c_2^3 c_2^3 = b_8 b_9 + b_{10}; \\
    a_{15} &= D_4(x; y; z; w) = Sq^1(a_{14}) = c_3^4 (c_3^4 (c_3 + c_4^0) + c_3^4) = b_8 b_9;
\end{align*}
\]

and this finishes the proof of the proposition. \( \square \)

**Lemma 4.8** Let \( 2 \text{ Aut}(c) \) be the algebra involution which exchanges \( c_1^4 \) and \( c_2^4 \) and leaves \( c_2 \) and \( c_3 \) xed. An element of \( c \) will be called \( \{ \text{invariant}\} \) if it is xed by this involution. Then the following hold:

(a) If \( 2 \, \otimes \) is \( \{ \text{invariant}\} \), then \( 2 \, \otimes \).

(b) If \( 2 \, \otimes \) is such that \( c_2^4 \) is \( \{ \text{invariant}\} \), then \( 0 \, \otimes \, b_8 \) for some \( 0 \, \otimes \).

**Proof** Point (a) follows from Proposition 4.7 upon regarding \( \otimes \, \otimes \, \otimes \) as the xed subrings of the groups \( GL_4(F_2) \), \( GL_3(F_2) \) and \( GL_2(F_2) \) acting on \( F_2[x; y; z; w] \), but also follows from the following direct argument. Let \( m \) be the degree of \( c_3 \) as a polynomial in \( b_8 \); we argue by induction on \( m \). Write \( 0 \, \otimes \, b_8 = 0 \, \otimes \, b_8 \), where \( 1 \, 2 F_2[b_8; b_8; b_8] \), and where \( 0 \) has degree \( < m \) (as a polynomial in \( b_8 \)). If \( m = 0 \), then \( 0 \, \otimes \, 1 \, F_2[b_8; b_8; b_8; F_2[c_2; c_3; c_3^4] \), and hence \( 0 \, \otimes \, F_2[c_2; c_3] \) since it is \( \{ \text{invariant}\} \). But from the formulas in Proposition 4.7(b), we see that \( F_2[b_8; b_8; b_8; F_2[c_2; c_3]] \) contains only constant polynomials (hence it is contained in \( \otimes \)).

Now assume \( m = 1 \). Then, expressed as a polynomial in \( c_2; c_3; c_2^4; c_2^4 \), the largest power of \( c_2^4 \) which occurs in \( 0 \, \otimes \) is \( c_2^{4m} \). Since \( 0 \, \otimes \) is \( \{ \text{invariant}\} \), the
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highest power of \( c_j \) which occurs is \( c_j^{Q_2} \); and hence by Proposition 4.7(b), the total degree of each term in \( 1 \) (its degree as a polynomial in \( b_i; b_0; b_7 \)) is at most \( m \). So for each term \( b_i b_j b_k \) in \( 1 \),

\[
b_i b_j b_k = a_i^{m-r-s} a_j^{44} a_k^{15}
\]
is a sum of terms which have degree < \( m \) in \( b_i \), and thus lies in \( \mathfrak{A} \) by the induction hypothesis.

To prove (b), note rst that since \( c_4^j \) is \{invariant and divisible by \( c_4^i \), it must also be divisible by \( c_4^j \), and hence \( c_4^j \). Furthermore, by Proposition 4.7, all elements of \( \mathfrak{B} \) as well as \( c_4^i \) are invariant under the involution which xes \( c_4^i \) and sends \( c_4^j \). Thus \( (c_4^i + c_4^j)^\mathfrak{B} \). Since \( b_0 = c_4^i(c_4^i + c_4^j) \), we can now write \( = 0 b_5 \) for some \( 2 \mathfrak{B} \). Finally, since

\[
c_4^j = 0 c_4^i c_4^a (c_4^i + c_4^j)
\]
is \{invariant, \( 0 \) is also \{invariant, and hence \( 0 \mathfrak{A} \) by (a).

\[
\begin{array}{c}
\Omega_{\mathfrak{A}, \mathbb{F}_2} \Downarrow \Omega_{\mathfrak{C}, \mathbb{F}_2} \\
\# \quad \# \\
\Omega_{\mathfrak{B}, \mathbb{F}_2} \Downarrow \Omega_{\mathfrak{C}, \mathbb{F}_2}
\end{array}
\]

Note that \( C_3 \) is \( \mathbb{F}_2(\mathbb{F}_2) \) act on \( \mathfrak{C} = \mathbb{F}_2[x; y; z; w] : \) via the action of the group \( \mathbb{F}_2^2(\mathbb{F}_2) = \mathbb{F}_2^2(\mathbb{F}_2) \), or equivalently by permuting \( c_4^4, c_4^5, \) and \( c_4^6, c_4^7 \) (and xing \( c_2; c_3 \)). Thus \( \mathfrak{A} = \mathfrak{B} \setminus \mathfrak{C}^3 \), since \( GL_4(\mathbb{F}_2) \) is generated by the subgroups \( GL_2^3(\mathbb{F}_2) \) and \( GL_2^3(\mathbb{F}_2) \). This is also shown directly in the following lemma.

**Proposition 4.9** The following square is a pullback square, where all maps are induced by inclusions between the subrings of \( \mathbb{F}_2[x; y; z; w] \):

\[
\begin{array}{c}
\Omega_{\mathfrak{C}, \mathbb{F}_2} \Downarrow \Omega_{\mathfrak{B}, \mathbb{F}_2} \\
\# \quad \#
\end{array}
\]

**Proof** Let \( \# \) be the involution of Lemma 4.8: the algebra involution of \( \mathfrak{C} \) which exchanges \( c_4^4 \) and \( c_4^5 \), and leaves \( c_2 \) and \( c_3 \) xed. By construction, all elements in the image of \( \Omega_{\mathfrak{B}, \mathbb{F}_2} \) are xed by \( C_3 \) if and only if they are xed by \( \mathbb{F}_2 \), and thus it will suffice to show that all of the above maps are injective, and that all \{invariant elements in the image of \( \Omega_{\mathfrak{B}, \mathbb{F}_2} \) lie in the image of \( \Omega_{\mathfrak{A}, \mathbb{F}_2} \). The injectivity is clear, and the square is a pullback for \( \Omega^0_{\mathfrak{C}, \mathbb{F}_2} \) by Lemma 4.8.

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Fix an invariant element

\[ I = P_1 \, db_1 + P_2 \, db_6 + P_3 \, db_7 + P_4 \, db_8 \]

\[ = P_2 c_4^3 \, dc_2 + P_3 c_3^3 \, dc_3 + P_4 c_2^3 \, dc_4^3 + (P_1 + P_2 c_2 + P_3 c_3 + P_4 c_4) \, dc_4^3 \]

\[ \Omega_1^{2} \]  \hspace{1cm} (1)

where \( P_i \neq 0 \) for each \( i \). By applying \( \Omega_1^{2} \) to (1) and comparing the coefficients of \( dc_2 \) and \( dc_3 \), we see that \( P_2 c_4^3 \) and \( P_3 c_3^3 \) are invariant. Also, upon comparing the coefficients of \( dc_1 \), we get the equation

\[ P_1 + P_2 c_2 + P_3 c_3 + P_4 c_4 = (P_4) c_4^3. \]  \hspace{1cm} (2)

So by Lemma 4.8, \( P_2 = P_2^g b \) and \( P_3 = P_3^g b \) for some \( P_2^g, P_3^g \neq 0 \). Upon subtracting

\[ P_2^g db_1 + P_3^g db_1 = P_2 \, db_6 + P_3 \, db_7 + (P_2^g db_6 + P_3^g db_7) \, db_8 \]

from \( I \) and introducing an appropriate modification to \( P_4 \), we can now assume that \( P_2 = P_3 = 0 \). With this assumption and (2), we have

\[ P_1 + P_4 c_4^3 = (P_4 c_4^3) = (P_4) c_4^3; \]

so that

\[ P_1 c_4^3 = (P_4 + (P_4) c_4^3) \]  \hspace{1cm} (3)

is invariant. This now shows that \( P_1 = P_1^g b \) for some \( P_1^g \neq 0 \), and upon subtracting \( P_1^g db_1 \) from \( I \) we can assume that \( P_1 = 0 \). This leaves \( I = P_4 \, db_6 = P_4 \, db_8 \). By (3) again, \( P_4 \) is invariant, so \( P_4 \neq 0 \) by Lemma 4.8 again, and thus \( 2 \, \Omega_1^{2} \).

The remaining cases are proved using the same techniques, and so we sketch them more briefly. To prove the result in degree two, \( x \) a invariant element

\[ I = P_1 \, db_1 \, db_6 \, db_7 + P_2 \, db_1 \, db_6 \, db_7 + P_3 \, db_1 \, db_7 \, db_8 + P_4 \, db_6 \, db_7 \, db_8 \]

\[ = P_4 c_4^2 \, dc_2 \, dc_3 + (P_1 c_4^3 + P_4 c_3^3 + P_5 c_2^3 \, dc_4^3) \, dc_4^3 + P_5 c_2^3 \, dc_4^3 \]

\[ + (P_2 c_4^3 + P_4 c_2^3 \, dc_3 + P_6 c_2^3 \, dc_4^3 + P_6 c_2^3 \, dc_4^3 + P_6 c_2^3 \, dc_4^3) \]

\[ + (P_3 c_4^3 + P_5 c_2^3 \, dc_2 \, dc_4^3 + P_6 c_2^3 \, dc_2 \, dc_4^3) \]

\[ \Omega_3^{2} \]  \hspace{1cm} (2)

Using Lemma 4.8, we see that \( P_4 = P_4^g b \), and hence can assume that \( P_4 = 0 \). One then eliminates \( P_1 \) and \( P_2 \), then \( P_5 \) and \( P_6 \), and finally \( P_3 \).

If

\[ I = P_1 \, db_1 \, db_6 \, db_7 + P_2 \, db_1 \, db_6 \, db_7 + P_3 \, db_1 \, db_7 \, db_8 + P_4 \, db_6 \, db_7 \, db_8 \]

\[ = (P_1 c_4^2 + P_4 c_4^2 \, dc_4^3) \, dc_2 \, dc_3 \, dc_4^3 + (P_2 c_4^2 + P_4 c_3^2 \, dc_4^3) \, dc_2 \, dc_4^3 \]

\[ + (P_3 c_4^2 + P_4 c_2^2 \, dc_3 \, dc_4^3 + P_4 c_2^3 \, dc_2 \, dc_4^3) \]

\[ \Omega_3^{2} \]  \hspace{1cm} (3)
Construction of 2-finite groups

is invariant, then we eliminate successively $P_1$, then $P_4$, then $P_2$ and $P_3$.

Finally, if

$$1 = P \sigma \delta \theta = P \xi^4 \pi \sigma^3 \rho \sigma^2 \rho \sigma^3 \rho \sigma^2 \xi^4 \Omega^4_{2 \sigma \xi^4}$$

is invariant, then $P = P_0 \sigma$ for some $P_0 \sigma \xi^4$ by Lemma 4.8 again, and so

$$1 = P_0 \sigma \delta \theta \xi^4 \Omega^4_{2 \sigma \xi^4}.$$ 

\[ \square \]

A Appendix: Spinor groups over finite fields

Let $F$ be any field of characteristic $\neq 2$. Let $V$ be a vector space over $F$, and let $b: V \to F$ be a nonsingular quadratic form. As usual, $O(V; b)$ denotes the group of isometries of $(V; b)$, and $SO(V; b)$ the subgroup of isometries of determinant 1. We will be particularly interested in elementary abelian 2-subgroups of such orthogonal groups.

**Lemma A.1** Fix an elementary abelian 2-subgroup $E \subseteq O(V; b)$. For each irreducible character $\chi' : E \to \mathbb{C}$, let $V' \in V$ denote the corresponding eigenspace, the subspace of elements $v \in V$ such that $g(v) = (g)\cdot v$ for all $g \in E$. Then the restriction of $\chi'$ to each subspace $V'$ is nonsingular, and $V'$ is the orthogonal direct sum of the $V'$.

**Proof** Elementary.

We give a very brief sketch of the definition of spinor groups via Clifford algebras; for more details we refer to [8, section II.7] or [2, section 22]. Let $T(V)$ denote the tensor algebra of $V$, and set

$$C(V; b) = T(V) \oplus (v \otimes v) - b(v) i :$$

the Clifford algebra of $(V; b)$. To simplify the notation, we regard $F$ as a subring of $C(V; b)$, and $V$ as a subgroup of its additive group; thus the class of $v_1 \otimes \cdots \otimes v_k$ will be written $v_1 \cdots v_k$. Note that $vw + vw = 0$ if $v; w \in V$ and $v \not= w$. Hence if $\dim_F V = n$, and $f v_1; \cdots; v_n,g$ is an orthogonal basis, then the set of $1$ and all $v_{i_1} \cdots v_{i_k}$ for $i_1 < \cdots < i_k (1 \leq k \leq n)$ is an $F$-basis for $C(V; b)$.

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Write $C(V; b) = C_0 \oplus C_1$, where $C_0$ and $C_1$ consist of classes of elements of even or odd degree, respectively. Let $G = C(V; b)$ denote the group of invertible elements $u$ such that $uV u^{-1} = V$, and let $G \to \text{O}(V; b)$ be the homomorphism

$$
(u) = \begin{cases} 
(v \not\equiv -uvu^{-1}) & \text{if } u \in C_1 \\
(v \not\equiv uvu^{-1}) & \text{if } u \in C_0.
\end{cases}
$$

In particular, for any nonisotropic element $v \not\in V$ (i.e., $b(v) \neq 0$), $v \in G$ and $(v)$ is the reflection in the hyperplane $v^\perp$. By [8, section II.7], $\text{Ker}(\ ) = F$.

Let $J$ be the antiautomorphism of $C(V; b)$ induced by the antiautomorphism $v_1 \otimes \cdots \otimes v_k \not\equiv v_k \otimes \cdots \otimes v_1$ of $T(V)$. Since $\text{O}(V; b)$ is generated by hyperplane reflections, $G$ is generated by $F$ and nonisotropic elements $v \not\in V$. In particular, for any $u = v_1 \otimes \cdots \otimes v_k \in G$,

$$
J(u) = \begin{cases} 
v_k & v_1 \not\equiv v_k \\
2 & v_1 \not\equiv v_k
\end{cases}
$$

implying that $J(u) = (u)^{-1}$ for all $u \in G$. There is thus a homomorphism $\Theta: G \to F$ defined by $\Theta(u) = u J(u)$.

In particular, $\Theta( ) = 2$ for any $u \in G$, while for any set of nonisotropic elements $v_1, \ldots, v_k$ of $V$,

$$
\Theta(v_1, v_k) = (v_1 v_k)(v_k v_1) = b(v_1) b(v_k);
$$

Hence $\Theta$ factors through a homomorphism

$$
\nu: \text{O}(V; b) \to F \to F = \Theta( ) = u J(u);
$$

called the spinor norm.

Set $G^+ = \text{SO}(V; b)^{-1} = G \setminus C_0$, and define

$$
\text{Spin}(V; b) = \text{Ker}(\Theta|_{G^+}) \quad \text{and} \quad \Omega(V; b) = \text{Ker}(\nu|_{\text{SO}(V; b)});
$$

In particular, $\Omega(V; b)$ has index 2 in $\text{SO}(V; b)$ if $F$ is a finite field, and $\Omega(V; b) = \text{SO}(V; b)$ if $F$ is algebraically closed (all units are squares). We thus get a commutative diagram

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where all rows and columns are short exact, and where all columns are central extensions of groups. If \( \dim(V) = 3 \) (or if \( \dim(V) = 2 \) and the form \( b \) is hyperbolic), then \( \Omega(V;b) \) is the commutator subgroup of \( SO(V;b) \) [8, section II.8].

The following lemma follows immediately from this description of \( \text{Spin}(V;b) \), together with the analogous description of the corresponding spinor group over the algebraic closure of \( F \).

**Lemma A.3** Let \( \overline{F} \) be the algebraic closure of \( F \), and set \( \overline{V} = \overline{F} \otimes_F V \) and \( \overline{b} = \text{Id}_F \otimes b \). Then \( \text{Spin}(V;b) \) is the subgroup of \( \text{Spin}(\overline{V};\overline{b}) \) consisting of those elements fixed by all Galois automorphisms \( 2 \text{Gal}(\overline{F}/F) \).

For any nonsingular quadratic form \( b \) on a vector space \( V \), the discriminant of \( b \) (or of \( V \)) is the determinant of the corresponding symmetric bilinear form \( B \), related to \( b \) by the formulas

\[
b(v) = B(v;v) \quad \text{and} \quad B(v;w) = \frac{1}{2} b(v+w) - b(v) - b(w)
\]

Note that the discriminant is well defined only modulo squares in \( F \). When \( W \) is a subspace, then we define the discriminant of \( W \) to mean the discriminant of \( b|_W \). In what follows, we say that the discriminant of a quadratic form is a square or a nonsquare to mean that it is the identity or not in the quotient group \( F/\mathbb{F}_2 \).

**Lemma A.4** Fix an involution \( \times 2 \) \( SO(V;b) \), and let \( V = V_+ \oplus V_- \) be its eigenspace decomposition. Then the following hold.

(a) \( \times 2 \Omega(V;b) \) if and only if the discriminant of \( V_- \) is a square.
We will need explicit isomorphisms which describe Spin\(_9\)Ran Levi and Bob Oliver

terminant is a nonsingular quadratic form on \(M\).

Let \(v\) be an orthogonal basis for \(V\). Then \(x = (v_1 \ldots v_k)\) in the above notation, since \((v_i)\) is the reflection in the hyperplane \(v_i\). Hence by the commutativity of Diagram (A.2),

\[
V; (x) b(v_1) b(v_k) = \det(b_{ij}) \pmod{2}:
\]

Thus \(x \in \Omega(V; b) = \ker(\Omega(V; b))\) if and only if \(V\) has square discriminant.

In particular, if \(x \in \Omega(V; b)\), then the product of the \(b(v_i)\) is a square, and hence (upon replacing \(v_1\) by a scalar multiple) we can assume that \(b(v_1) = \ldots = b(v_k) = 1\).

Then \(e \overset{\text{def}}{=} v_1 \ldots v_k \in \text{Spin}(V; b) = \ker(\Omega).\) Since \(v w = -wv\) in the Clifford algebra whenever \(v \neq w\), and since \((v_i)^2 = b(v_i)\) for each \(i\),

\[
g^2 = (-1)^{k(k-1)/2} (v_1)^2 = (-1)^{k(k-1)/2} = \begin{cases} 1 & \text{if } k \equiv 0 \pmod{4} \\ -1 & \text{if } k \equiv 2 \pmod{4}. \end{cases}
\]

This proves (b).

It remains to prove (c). The first statement \((\ldots)^{-1} = \ldots^{-1}\) is clear. Fix liftings \(e \in \text{Spin}(V; b)\) with an element of \(C(V; b)\), we regard the groups \(C(V; b)\) as (commuting) subgroups of \(\text{Spin}(V; b)\), and set

\[
e = e_+ e_- e_+ e_+ 2 \text{ Spin}(V; b):
\]

Let \(e = v_1 \ldots v_k\) be as above. Clearly, \(e\) commutes with all elements of \(C(V; b)\). Since

\[
(v_1 \ldots v_k) v_i = (-1)^{k-1} v_i (v_1 \ldots v_k) = -v_i (v_1 \ldots v_k)
\]

for \(i = 1; \ldots; k\), we have \(e = (-1)^i e\) for all \(2 C_i(V; b) (i = 0; 1)\). In particular, since \([e_+; e_-] = 1, [e; e] = [e; e_-] = \det(\ldots)\), and this finishes the proof.

We will need explicit isomorphisms which describe \(\text{Spin}_3(F)\) and \(\text{Spin}_4(F)\) in terms of \(\text{SL}_2(F)\). These are constructed in the following proposition, where \(M_2(F)\) denotes the vector space of matrices of trace zero. Note that the determinant is a nonsingular quadratic form on \(M_2(F)\) and on \(M_2^0(F)\), in both cases with square discriminant.
Proposition A.5 Define 
\[ 3 : \text{SL}_2(F) \rightarrow \Omega(M^0_2(F); \det) \]
and 
\[ 4 : \text{SL}_2(F) \rightarrow \Omega(M_2(F); \det) \]
by setting 
\[ 3(A)(X) = AXA^{-1} \quad \text{and} \quad 4(A;B)(X) = AXB^{-1}. \]
Then 3 and 4 are both epimorphisms, and lift to unique isomorphisms 
\[ \text{SL}_2(F) \rightarrow \text{Spin}(M^0_2(F); \det) \]
and 
\[ \text{SL}_2(F) \rightarrow \text{Spin}(M_2(F); \det); \]

Proof See [24, pages 142, 199] for other ways of defining these isomorphisms.

By Lemma A.3, it suffices to prove this (except for the uniqueness of the lifting) when F is algebraically closed. In particular,
\[ \Omega(M^0_2(F); \det) = \text{SO}(M^0_2(F); \det) \quad \text{and} \quad \Omega(M_2(F); \det) = \text{SO}(M_2(F); \det) \]
in this case.

For general V and b, the group \( \text{SO}(V;b) \) is generated by reflections fixing nonisotropic subspaces (i.e., of nonvanishing discriminant) of codimension 2 (cf [8, section II.6(1)]). Hence to see that 3 and 4 are surjective, it suffices to show that such elements lie in their images. A codimension 2 reflection in \( \text{SO}(M^0_2(F); \det) \) is of the form \( R_X \) (the reflection fixing the line generated by X) for some \( X \in M^0_2(F) \) which is nonisotropic (i.e., det(X) \( \neq 0 \)). Since F is algebraically closed, we can assume \( X \in 2 \text{SL}_2(F) \). Then \( X^2 = -I \) (since \( \text{Tr}(X) = 0 \) and \( \det(X) = 1 \)), and \( R_X = 3(X) \) since it has order 2 and fixes X. Thus 3 is onto.

Similarly, any 2-dimensional nonisotropic subspace \( W \subseteq V \) has an orthonormal basis \( fY; Zg \), and \( ZY^{-1} \) and \( Y^{-1}Z \) have trace zero (since they are orthogonal to the identity matrix) and determinant one. Hence their square is \( -I \), and one repeats the above argument to show that \( R_W = 4(ZY^{-1}; Y^{-1}Z) \). So 4 is onto.

The liftings \( e_m \) exist and are unique since \( \text{SL}_2(F) \) is the universal central extension of \( \text{PSL}_2(F) \) (or universal among central extensions by 2-groups if \( F = \mathbb{F}_3 \)).
We now restrict to the case $F = \mathbb{F}_q$ where $q$ is an odd prime power. We refer to [2, section 21] for a description of quadratic forms in this situation, and the notation for the associated orthogonal groups. If $n$ is odd and $b$ is any nonsingular quadratic form on $\mathbb{F}^n_q$, then every nonsingular quadratic form is isomorphic to $ub$ for some $u \in \mathbb{F}_q^*$, and hence one can write $SO_n(q) = SO(\mathbb{F}^n_q; b) = SO(\mathbb{F}^n_q; ub)$ without ambiguity. If $n$ is even, then there are exactly two isomorphism classes of quadratic forms on $\mathbb{F}^n_q$; and one writes $SO^+_n(q) = SO(\mathbb{F}^n_q; b)$ when $b$ is the hyperbolic form (equivalently, has discriminant $(-1)^{n/2}$ modulo squares), and $SO^-_n(q) = SO(\mathbb{F}^n_q; b)$ when $b$ is not hyperbolic (equivalently, has discriminant $(-1)^{n/2} u$ for $u \in \mathbb{F}_q^*$ not a square). This notation extends in the obvious way to $\Omega_n(q)$ and $\text{Spin}_n(q)$.

The following lemma does, in fact, hold for orthogonal representations over arbitrary fields of characteristic $\neq 2$. But to simplify the proof (and since we were unable to find a reference), we state it only in the case of finite fields.

**Lemma A.6** Assume $F = \mathbb{F}_q$, where $q$ is a power of an odd prime. Let $V$ be a vector space, and let $b$ be a nonsingular quadratic form on $V$. Let $P O(V; b)$ be a 2-subgroup which is orthogonally irreducible, i.e., such that $V$ has no splitting as an orthogonal direct sum of nonzero invariant subspaces. Then $\dim F(V)$ is a power of 2; and if $\dim(V) > 1$ then $b$ has square discriminant.

**Proof** This means showing that each orthogonal group $O(\mathbb{F}^n_q; b)$, such that either $n$ is not a power of 2, or $n = 2^k$ and the quadratic form $b$ has nonsquare discriminant, contains some subgroup $O_m(q) \cong O_{n-m}(q)$ (for $0 \leq m < n$) of odd index. We refer to the standard formulas for the orders of these groups (see [24, p.165]): if $n = 1$ then

$$j_{O_{2n}}(q) = 2q^{(n-1)(q^n - 1)} (q^{2i} - 1) \quad \text{and} \quad j_{O_{2n+1}}(q) = 2q^n (q^{2i} - 1).$$

We will also use repeatedly the fact that for all $0 < i < 2^k$ ($k \geq 1$), the largest powers of 2 dividing $(q^{2^k+i} - 1)$ and $(q^{2i} - 1)$ are the same. In other words, $(q^{2^k+i} - 1) = q^i (q^{2i} - 1)$ is invertible in $\mathbb{Z}_2$.

For any $n \geq 1$,

$$\frac{j_{O_{2n+1}}(q)}{j_{O_{2n}}(q)j_{O_1}(q)} = q^n \frac{q^n + 1}{2}$$

is odd for an appropriate choice of $q$. Thus, there are no irreducibles of odd dimension.
Assume \( n \) is not a power of 2, and write \( n = 2^k + m \) where \( 0 < m < 2^k \) and \( k \geq 1 \). Then
\[
\frac{jO_{2n}(q)}{jO_{2k+1}(q)jO_{2m}(q)} = q^{m2k+1} \prod_{i=1}^{\nu-1} \frac{q^{2(2^k+i)} - 1}{q^{2i} - 1} \frac{q^{2k+m} - 1}{q^{m} - 1} \frac{q^k + 1}{2} ;
\]
and each factor is invertible in \( \mathbb{Z}_2(q) \). When \( n = 2m = 2^k \) and \( k \geq 1 \), then \( O^-_{2n}(q) \) is the orthogonal group for the quadratic form with nonsquare discriminant, and
\[
\frac{jO^-_{2n}(q)}{jO^+_{2m}(q)jO^-_{2m}(q)} = q^{2m^2} \prod_{i=1}^{\nu-1} \frac{q^{2(m+i)} - 1}{q^{2i} - 1} \frac{q^{2m} + 1}{2} ;
\]
and again each factor is invertible in \( \mathbb{Z}_2(q) \). Finally,
\[
\frac{jO_{2}(q)}{jO_{1}(q)jO_{2}(q)} = q - \frac{2}{2}
\]
is odd whenever \( q \equiv 1 \pmod{4} \) and \( -1 \), or \( q \equiv 3 \pmod{4} \) and \( +1 \); and these are precisely the cases where the quadratic form on \( \mathbb{F}_q^2 \) has nonsquare discriminant.

We must classify the conjugacy classes of those elementary abelian 2-subgroups of \( \text{Spin}_7(q) \) which contain its center. The following definition will be useful when doing this.

**Definition A.7** Fix an odd prime power \( q \). Identify \( \text{SO}_7(q) = \text{SO}(\mathbb{F}_q^7, b) \) and \( \text{Spin}_7(q) = \text{Spin}(\mathbb{F}_q^7, b) \), where \( b \) is a nonsingular quadratic form with square discriminant. An elementary abelian 2-subgroup of \( \text{SO}_7(q) \) or of \( \text{Spin}_7(q) \) will be called of type I if its eigenspaces all have square discriminant (with respect to \( b \)), and of type II otherwise. Let \( E_n \) be the set of elementary abelian 2-subgroups in \( \text{Spin}_7(q) \) which contain \( \mathbb{Z}(\text{Spin}_7(q)) = C_2 \) and have rank \( n \). Let \( E^I_n \) and \( E^II_n \) be the subsets of \( E_n \) consisting of those subgroups of types I and II, respectively.

In the following two propositions, we collect together the information which will be needed about elementary abelian 2-subgroups of \( \text{Spin}_7(q) \). We define \( \text{Spin}_7(q) \) as \( \text{Spin}(V; b) \), where \( V = \mathbb{F}_q^7 \), and \( b \) is a nonsingular quadratic form with square discriminant. Let \( 2 \mathbb{Z}(\text{Spin}_7(q)) \) be the generator. For any subgroup \( H \leq \text{Spin}_7(q) \) or any element \( g \in 2 \text{Spin}_7(q) \), let \( \bar{H} \) and \( \bar{g} \) denote their images in \( \Omega_7(q) \twoheadrightarrow \text{SO}_7(q) \). For each elementary abelian 2-subgroup \( E \leq \text{Spin}_7(q) \), and each character \( 2 \text{Hom}(E; f g) \), \( V \twoheadrightarrow V \) denotes the...
eigenspace of \( (and \ V_1 \ denotes \ the \ eigenspace \ of \ the \ trivial \ character) \). Also \( \text{when } z \in \mathbb{E} \), \( \text{Aut}(\mathbb{E}; z) \) denotes the group of all automorphisms of \( \mathbb{E} \) which send \( z \) to itself.

**Proposition A.8** For any odd prime power \( q \), the following table describes the numbers of \( \text{Spin}_7(q) \) {conjugacy classes in each of the sets \( E_1^I \) and \( E_1^II \), the dimensions and discriminants of the eigenspaces of subgroups in these sets, and indicates in which cases \( \text{Aut}_{\text{Spin}_7(q)}(\mathbb{E}) \) contains all automorphisms which \( \times \) \( z \).

<table>
<thead>
<tr>
<th>Set of subgroups</th>
<th>( E_2 )</th>
<th>( E_3 )</th>
<th>( E_3^I )</th>
<th>( E_4^I )</th>
<th>( E_4^II )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nr. conj. classes</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( \dim(V_1) )</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \dim(V), \in \mathfrak{g} )</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \text{discr}(V_1; b) )</td>
<td>square</td>
<td>square</td>
<td>nonsq.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \text{discr}(V; b), \in \mathfrak{g} )</td>
<td>square</td>
<td>square</td>
<td>nonsq.</td>
<td>square</td>
<td>both</td>
</tr>
<tr>
<td>( \text{Aut}_{\text{Spin}_7(q)}(\mathbb{E}) = \text{Aut}(\mathbb{E}; z) )</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
</tr>
</tbody>
</table>

There are no subgroups in \( E_2 \) of type II, and no subgroups of rank 5. Furthermore, we have:

(a) For all \( E_2 E_4 \), \( C_{\text{Spin}_7(q)}(E) = E \).

(b) If \( E; E^0 \in E_4^I \), then \( E^0 = gEg^{-1} \) for some \( g \in \text{SO}_7(q) \), and \( E \) and \( E^0 \) are \( \text{Spin}_7(q) \) {conjugate if and only if \( g \in \Omega_7(q) \).

(c) If \( E_2 E_4^I \), then there is a unique element \( 1 \otimes \mathfrak{g} = \mathfrak{r}(E) \in \mathbb{E} \) with the property that for \( 1 \otimes 2 \in \text{Hom}(\mathbb{E}; f \otimes 1g) \), \( V \) has square discriminant if \( \langle x \rangle = 1 \) and nonsquare discriminant if \( \langle x \rangle = -1 \). Also, the image of \( \text{Aut}_{\text{Spin}_7(q)}(\mathbb{E}) \) in \( \text{Aut}(\mathbb{E}) \) is the group of all automorphisms which send \( \mathfrak{r} \) to itself; and if \( X \in \mathbb{E} \) denotes the inverse image of \( \Omega_i \in \mathbb{E} \), then \( \text{Aut}_{\text{Spin}_7(q)}(\mathbb{E}) \) contains all automorphisms of \( \mathbb{E} \) which are the identity on \( X \) and the identity modulo \( \mathfrak{h} \).

(d) If \( E_2 E_3 \), then \( C_{\text{Spin}_7(q)}(E) = A \times C_2 \), where \( A \) is abelian and \( C_2 \) acts on \( A \) by inversion. If \( E_2 E_4^I \), then the Sylow 2-subgroups of \( C_{\text{Spin}_7(q)}(E) \) are elementary abelian of rank 4 (and type II).

**Proof** Write \( \text{Spin} = \text{Spin}_7(q) \) for short. Fix an elementary abelian subgroup \( E \in \text{Spin} \) such that \( z \in \mathbb{E} \).

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**Step 1** We first show that \( \text{rk}(E) = 4 \), and that the dimensions of the eigenspaces \( V \) for \( \text{Hom}(E; f \cdot g) \) are as described in the table.

By Lemma A.4, every involution in \( E \) has a 4-dimensional \((-1)-\)eigenspace. In particular, if \( \text{rk}(E) = 2 \), \((E = C_2)\), then \( \dim(V) = 4 \) for \( 1 \in \text{Hom}(E; f \cdot 1g) \), while \( \dim(V_1) = 3 \).

Now assume \( \text{rk}(E) = n \) for some \( n > 2 \). Assume we have shown, for all \( E \in \mathcal{E}_{n-1} \), that the eigenspace of the trivial character of \( E \) is \( r \)-dimensional. For each \( 1 \in \text{Hom}(E; f \cdot 1g) \), let \( E \in \mathcal{E}_{n-1} \) be the subgroup such that \( E = \text{ker}(\cdot) \); then \( V_1 \) is the eigenspace of the trivial character of \( E = \text{ker}(\cdot) \), and thus \( \dim(V_1) + \dim(V) = r \). Hence all nontrivial characters of \( E \) have eigenspaces of the same dimension. Since there are \( 2^{n-1} - 1 \) nontrivial characters of \( E \), we have \( \dim(V_1) + (2^{n-1} - 1) \dim(V) = 7 \), and these two equations completely determine \( \dim(V_1) \) and \( \dim(V) \). Using this procedure, the dimensions of the eigenspaces are shown inductively to be equal to those given by the table. Also, this shows that \( \text{rk}(E) = 4 \), since otherwise \( \text{rk}(E) = 4 \), so the \( V \) for \( 1 \in 1 \) must be trivial (they cannot all have dimension 1), so \( E \) acts on \( V \) via the identity, which contradicts the assumption that \( E = \text{Spin}_7(q) \).

**Step 2** We next show that \( E \in \mathcal{E}_1 \) = \( ; \), describe the discriminants of the eigenspaces of characters of \( E \) for \( E \in \mathcal{E}_n \) (for all \( n \)), and show that subgroups of rank 4 are self centralizing. In particular, this proves (a) together with the first statement of (c).

If \( E \in \mathcal{E}_2 \), then \( E = h \cdot g i \) for some noncentral involution \( g \in \text{Spin}_7(q) \), and the eigenspaces of \( E = h \cdot g i \) have square discriminant by Lemma A.4(a) (and since the ambient space \( V \) has square discriminant by assumption). Thus \( E \in \mathcal{E}_2 \) = \( ; \).

If \( E \in \mathcal{E}_3 \), then the sum of any two eigenspaces of \( E \) is an eigenspace of \( g \) for some \( g \in \mathcal{E}_3 \). Hence the sum of any two eigenspaces of \( E \) has square discriminant, so either all of the eigenspaces have square discriminant \( (E \in \mathcal{E}_3) \), or all of the eigenspaces have nonsquare discriminant \( (E \in \mathcal{E}_3) \).

Assume \( \text{rk}(E) = 4 \). We have seen that all eigenspaces of \( E \) are 1-dimensional. By Lemma A.4(c), for each \( 1 \in \mathcal{E}(E) \), \( \text{ker}(V) = V \) for each \( \in 1 \), and since \( \dim(V) = 1 \) it must act on each \( V \) via \( \text{Id} \). Thus \( 1 \in \Omega_7(q) \) has order 2; let \( V \) be its eigenspaces. Then \( \dim(V_\cdot) \) is even since \( \det(\cdot) = 1 \), and \( V_\cdot \) has square discriminant by Lemma A.4(a). If \( \dim(V_\cdot) = 4 \), then \( \text{j}\in 2 \) (Lemma A.4(b)), and hence a \( E \) since otherwise \( hE; \cdot i \) would have rank 5. If \( \dim(V_\cdot) = 2 \), then \( V_\cdot \) is the sum of the eigenspaces of two distinct characters \( 1; 2 \) of \( E \), there is some \( g \in \mathcal{E} \) such that \( 1(g) \cdot 2(g) \), hence \( \det(g_{V_\cdot}) = \)
\( g(1) = -1 \), so \([g; a] = z\) by Lemma A.4(c), and this contradicts the assumption that \([a; E] = 1\). If \( \dim V_\omega = 6\), then \( V_\omega \) is the sum of the eigenspaces of all but one of the nontrivial characters of \( \bar{E} \), and this gives a similar contradiction to the assumption \([a; E] = 1\). Thus, \( C_{\text{Spin}_7(q)}(E) = E \).

Now assume that \( E \cong E_{14} \), and let \( \pi \cong O_7(q) \) be the element which acts via \(-\text{Id}\) on eigenspaces with nonsquare discriminant, and via the identity on those with square discriminant. Since \( b \) has square discriminant on \( V \), the number of eigenspaces of \( \bar{E} \) on which the discriminant is nonsquare is even, so \( \pi \cong O_7(q) \) by Lemma A.4(a), and lifts to an element \( \pi \cong \text{Spin}_7(q) \). Also, for each \( g \in E \), the \((-1)\) eigenvalue of \( g \) has square discriminant (Lemma A.4(a) again), hence contains an even number of eigenspaces of \( \bar{E} \) of nonsquare discriminant, and by Lemma A.4(c) this shows that \([g; x] = 1\). Thus \( \pi \cong C_{\text{Spin}_7(q)}(E) = E \), and this proves the first statement in (c).

**Step 3** We next check the numbers of conjugacy classes of subgroups in each of the sets \( E_{14} \) and \( E_{I4} \), and describe \( \text{Aut}_{\text{Spin}}(E) \) in each case. This finishes the proof of (b) and (c), and of all points in the above table.

From the above description, we see immediately that if \( E \) and \( E^0 \) have the same rank and type, then any isomorphism \( \pi \cong \text{Isom}(E; E^0) \), such that \( \pi(E) = E^0 \), has the property that for all \( \text{Hom}(E^0, f \cong 1_g) \), \( V \) and \( V^0 \) have the same dimension and the same discriminant (modulo squares).

Hence for any such \( \pi \), there is an element \( g \cong O_7(q) \) such that \( g(V^0) = V \) for all \( \pi \), and \( g = c_{\pi} \cong \text{Isom}(E; E^0) \) for such \( g \). Upon replacing \( g \) by \( -g \) if necessary, we can assume that \( g \cong \text{SO}_7(q) \). This shows that \( E \cong E^0 \) have the same rank and type implies \( E \) and \( E^0 \) are \( \text{SO}_7(q) \) conjugate

and also that

\[
\text{Aut}_{\text{SO}_7(q)}(E) = \begin{cases} 
\text{Aut}(E) & \text{if } E \cong E_{14} \\
\text{Aut}(E; \pi(E)) & \text{if } E \cong E_{I4}.
\end{cases}
\]

We next claim that

\[
E \cong E_{14} \implies 9 \gamma \cong \text{SO}_7(q) \setminus \Omega_7(q) \text{ such that } [\gamma; E] = 1:
\]

To prove this, choose 1-dimensional nonisotropic summands \( W \) and \( W^0 \) of \( V \), where \( \omega \) are two distinct characters of \( E \), and where \( W \) has square discriminant and \( W^0 \) has nonsquare discriminant. Let \( \gamma \cong \text{SO}_7(q) \) be the involution with \((-1)\) eigenvalue \( W \) and \( W^0 \). Then \([\gamma; E] = 1\), since \( \gamma \) sends...
each eigenspace of $\bar{E}$ to itself, and $\gamma \in \Omega_7(q)$ since its $(-1)$-eigenspace has nonsquare discriminant (Lemma A.4(a)).

If $E$ has rank 4 and type I, then $\text{Aut}(E) = \text{GL}_3(\mathbb{F}_2)$ is simple, and in particular has no subgroup of index 2. Hence by (2), each element of $\text{Aut}(\bar{E})$ is induced by conjugation by an element of $\Omega_7(q)$. Also, if $g \in \text{SO}_7(q)$ centralizes $\bar{E}$, then $g(V) = V$ for all $2 \text{ Hom}(\bar{E}; f \ 1g)$, $g$ acts via $-1\text{Id}$ on an even number of eigenspaces (since it has determinant $+1$), and hence $g \in \Omega_7(q)$ by Lemma A.4(a). Thus

$$E \cong E_4 \implies N_{\text{SO}_7(q)}(\bar{E}) \Omega_7(q)$$

(4)

If $E \cong E_4$, then by (3), for any $g \in \text{SO}_7(q)$, there is a $\gamma \in \text{SO}_7(q) \setminus \Omega_7(q)$ such that $c_{gE} = c_{\gamma E}$, and either $g$ or $\gamma g$ lies in $\Omega_7(q)$. Thus $\text{ISO}_{\text{SO}_7(q)}(\bar{E}; E_7) = \text{ISO}_{\text{SO}_7(q)}(\bar{E}; E_7)$ for any $E$. Together with (1), this shows that $E$ is Spin conjugate to all other subgroups of the same rank and type, and together with (2) it shows that

$$\text{Im \ Aut}_{\text{Spin}}(E) \to \text{Aut}(\bar{E}) = \begin{cases} \text{Aut}(\bar{E}) & \text{if } E \cong E_4 \\ \text{Aut}(\bar{E}; x(E)) & \text{if } E \cong E_4 \end{cases}$$

(5)

If $E \cong E_4$, then by (4) and (2), $\text{Aut}_{\text{SO}_7(q)}(\bar{E}) = \text{Aut}_{\text{SO}_7(q)}(\bar{E}) = \text{Aut}(\bar{E})$, and so (5) also holds in this case. Furthermore, for any $g \in \text{SO}_7(q) \setminus \Omega_7(q)$, $\bar{E}$ and $g\bar{E}g^{-1}$ are representatives for two distinct $\Omega_7(q)$-conjugacy classes, since by (4), no element of the coset $g \Omega_7(q)$ normalizes $\bar{E}$.

We have now determined in all cases the number of conjugacy classes of subgroups of a given rank and type, and the image of $\text{Aut}_{\text{Spin}}(E)$ in $\text{Aut}(\bar{E})$. We next claim that if $\text{rk}(E) < 4$ or $E \cong E_4$, then

$$E \cong E_4 \implies \text{Aut}_{\text{Spin}}(E) \cong \text{Aut}(E) \quad (z = \text{Id} \mod \text{Id})$$

(6)

Together with (5), this will finish the proof that $\text{Aut}_{\text{Spin}}(E)$ is the group of all automorphisms of $E$ which send $z$ to itself. We also claim that

$$E \cong E_4 \implies \text{Aut}_{\text{Spin}}(E) \cong \text{Aut}(E) \quad j_X = \text{Id} \mod \text{Id}$$

(7)

where $X$ denotes the inverse image of $\text{HR}(E)$ in $\bar{E}$, and this will finish the proof of (c).

We prove (6) and (7) together. Fix $2 \text{ Aut}(E)$ ( $\not\in \text{Id}$) which sends $z$ to itself, induces the identity on $\bar{E}$, and such that $j_X = \text{Id}$ if $E \cong E_4$. Then there is $1 \not\in 2 \text{ Aut}(\bar{E}; f \ 1g)$ such that $(g) = g$ when $(g) = 1$ and $(g) = zg$.
when \((\mathfrak{g}) = -1\). Choose any character \(\chi\) such that \(V \in \mathfrak{g}\) and \(V \in \mathfrak{g}\), and let \(W = V + W^0\) be \(1\)-dimensional nonisotropic subspaces with the same discriminant (this is possible when \(E \cong E_1^1\) since \(x(E) \cong 2\mathcal{K}(\mathfrak{g})\)). Let \(\mathfrak{g} \to O_7(q)\) be the involution whose \((-1)\)-eigenspace is \(W \cong W^0\). Then \(\mathfrak{g} \to \Omega_7(q)\) by Lemma A.4(a), so \(\mathfrak{g}\) lifts to \(\mathfrak{g} \to \text{Spin}_7(q)\), and using Lemma A.4(c) one sees that \(c_3 = 1\).

**Step 4** It remains to prove (d). Assume \(E \cong E_3\). Let \(1 = 1; 2; 3; 4\) be the four characters of \(E\), and set \(V_i = V_i\). Then \(\dim(V_i) = 1\), \(\dim(V_j) = 2\) for \(i = 2; 3; 4\), and the \(V_i\) either all have square discriminant or all have nonsquare discriminant. For each \(g \in \text{Spin}(E)\), we can write \(g = \prod_{i=1}^4 g_i\), where \(g_i \in O(V_i; b_i)\). For each pair of distinct indices \(i; j \neq 2; 3; 4\), there is some \(g \in E\) whose \((-1)\)-eigenspace is \(V_i \cap V_j\), and hence \(\det(g; g) = 1\) by Lemma A.4(c). This shows that the \(g_i\) all have the same determinant. Let \(A \subset \text{Spin}(E)\) be the subgroup of index 2 consisting of those \(g\) such that \(\det(g) = 1\) for all \(i\).

Now, \(SO_1(\mathbb{F}_q) = 1\), while \(SO_2(\mathbb{F}_q) = \mathbb{F}_q^*\) is the group of diagonal matrices of the form \(\text{diag}(u; u^{-1})\) with respect to a hyperbolic basis of \(\mathbb{F}_q^2\). Thus \(A\) is contained in a central extension of \(C_2\) by \((\mathbb{F}_q^*)^3\), and any such extension is abelian since \(H_2((\mathbb{F}_q^*)^3) = 0\). Hence \(A\) is abelian. The groups \(O_2(q)\) are all dihedral (see [24, Theorem 11.4]). Hence for any \(g \in \text{Spin}(E)\), \(\mathfrak{g}\) has order 2 and \((-1)\)-eigenspace of dimension 4 (its intersection with each \(V_i\) is \(1\)-dimensional), and hence \(\mathfrak{g}\) is an \(E\). Thus all elements of \(\text{Spin}(E)\) have order 2, so the centralizer must be a semisimple product of \(A\) with a group of order 2 which acts on it by inversion.

Now assume that \(E \cong E_1^1\); i.e., that the \(V_i\) all have nonsquare discriminant. Then for \(i = 2; 3; 4\), \(SO(V_i; b_1)\) has order \(q - 1\), which is not a multiple of 4 (see [24, Theorem 11.4] again). Thus if \(g \in A \subset \text{Spin}(E)\) has 2-power order, then \(g = 1\) for each \(i\), the number of \(i\) for which \(g_i = 1\) is even (since the \((-1)\)-eigenspace of \(\mathfrak{g}\) has square discriminant), and hence \(g \in E\). In other words, \(E \cong Syl_2(A)\). A Sylow 2-subgroup of \(\text{Spin}(E)\) is thus generated by \(E\) together with an element of order 2 which acts on \(E\) by inversion; this is an elementary abelian subgroup of rank 4, and is necessarily of type II.

We also need some more precise information about the subgroups of \(\text{Spin}_7(q)\) of rank 4 and type II. Let \(q^2 \in \text{Aut}(\text{Spin}_7(\mathbb{F}_q))\) denote the automorphism induced by the field automorphism \((x \mapsto x^q)\). By Lemma A.3, \(\text{Spin}_7(q)\) is precisely the subgroup of elements fixed by \(q\).
Proposition A.9

Fix an odd prime power \( q \), and let \( z \in \mathbb{Z}^{\text{Spin}}_7(q) \) be the central involution. Let \( C \) and \( C^0 \) denote the two conjugacy classes of subgroups \( \text{Spin}_7(q) \) of rank 4 and type I. Then the following hold.

(a) For each \( E \in 2\text{E}_4 \), there is an element \( a \in \text{Spin}_7(F_q) \) such that \( aEa^{-1} \subset 2\text{C} \).

For any such \( a \), if we set
\[
 x_C(E) \overset{\text{def}}{=} a^{-1} q(a);
\]
then \( x_C(E) \in 2\text{E} \) and is independent of the choice of \( a \).

(b) \( E \in 2\text{C} \) if and only if \( x_C(E) = 1 \), and \( E \in 2\text{C}^0 \) if and only if \( x_C(E) = z \).

(c) Assume \( E \in 2\text{E}_4^1 \), and set \( (E) = hz; x_C(E)i \). Then \( \text{rk}(E) = 2 \), and
\[
 \text{Aut}_{\text{Spin}_7}((E)) = 2 \text{Aut}(E) \quad j(E) = 1d : \]
The four eigenspaces of \( E \) contained in the \((-1)\)-eigenspace of \( x_C(E) \) all have nonsquare discriminant, and the other three eigenspaces all have square discriminant.

Proof (a) For all \( E \in 2\text{E}_4 \), \( E \) has type I as a subgroup of \( \text{Spin}_7(q^2) \) since all elements of \( F_q \) are squares in \( F_{q^2} \). Hence by Proposition A.8(b), for all \( E \in 2\text{C} \), there is a \( 2\text{SO}_7(q^2) \) \( \Omega_7(q^2) \) such that \( aEa^{-1} = E^0 \). Upon lifting \( a \) to a \( 2\text{Spin}_7(q^4) \), this proves that there is a \( 2\text{Spin}_7(F_q) \) such that \( aEa^{-1} \subset 2\text{C} \).

Fix any such \( a \), and set
\[
 x = x_C(E) = a^{-1} q(a);
\]
For all \( g \in E \), \( q(g) = g \) and \( q(aga^{-1}) = aga^{-1} \) since \( E ; aEa^{-1} \in \text{Spin}_7(q) \), and hence
\[
 aga^{-1} = q(a) g \quad q(a^{-1}) = a(xgx^{-1})a^{-1} ;
\]
Thus, \( x \in 2\text{C}_{\text{Spin}_7(F_q)}(E) \), and so \( x \in 2\text{E} \) since it is self centralizing in each \( \text{Spin}_7(q^2) \) (Proposition A.8(a)).

We next check that \( x_C(E) \) is independent of the choice of \( a \). Assume \( a;b \in 2\text{Spin}_7(F_q) \) are such that \( aEa^{-1} \subset 2\text{C} \) and \( bEb^{-1} \subset 2\text{C} \). Then by Proposition A.8(b), there is a \( 2\text{Spin}_7(q) \) such that \( gbE(gb)^{-1} = aEa^{-1} \). Set \( E^0 = aEa^{-1} \subset 2\text{C} \), then \( gb^{-1}2\text{N}_{\text{Spin}_7(F_q)}(E^0) \). Furthermore, since \( \text{Aut}_{\text{Spin}_7(q)}(E^0) \) contains all automorphisms which send \( z \) to itself, and since \( E^0 \) is self centralizing in each of the groups \( \text{Spin}_7(q) \) (both by Proposition A.8 again), we see that \( \text{N}_{\text{Spin}_7(F_q)}(E^0) \) is contained in \( \text{Spin}_7(q) \). Thus, \( ba^{-1}2\text{Spin}_7(q) \), so \( q(ba^{-1}) = \)
and this proves that \( x_C(E) = a^{-1} q(a) = b^{-1} q(b) \) is independent of the choice of \( a \).

(b) If \( E \not\subset C \), then we can choose \( a = 1 \), and so \( x_C(E) = 1 \).

If \( E \subset C \), then by Proposition A.8(b), there is a 2-\( \text{Spin}_7(q) \) such that \( a E \rO_7(q) \) and \( a \not\subset 2 \ C \). Then \( q(a) \not\subset 2 \text{Spin}_7(q) \) (Proposition A.3), but \( q(a) = a \) since \( a \not\subset 2 \text{SO}_7(q) \). Thus, \( x_C(E) = a^{-1} q(a) = z \) in this case.

We have now shown that \( x_C(E) \not\subset hz \) if \( E \) has type I, and it remains to prove the converse. Fix a 2-\( \text{Spin}_7(F_q) \) such that \( a E a^{-1} \not\subset C \). If \( x_C(E) \not\subset hz \), then \( q(a) \not\subset f a z a \), so \( q(a) = a \), and hence \( a \not\subset 2 \text{SO}_7(q) \). Conjugation by an element of \( \text{SO}_7(q) \) sends eigenspaces with square discriminant to eigenspaces with square discriminant, so all eigenspaces of \( E \) must have square discriminant since all eigenspaces of \( a E a^{-1} \) do. Hence \( E \) has type I.

(c) Now write \( \text{Spin} = \text{Spin}_7(q) \) for short. Assume \( E \not\subset E_4 \), and set \( x = x_C(E) \) and \( (E) = hz; xi \). Then \( x \not\subset hz \) by (b), and thus \( (E) \) has rank 2.

By (a) (the uniqueness of \( x \) having the given properties), each element of \( \text{Aut}_{\text{Spin}}(E) \) restricts to the identity on \( (E) \). We have already seen (Proposition A.8(c) again) that \( \text{Aut}_{\text{Spin}}(E) \) contains all automorphisms which are the identity on \( (E) \) and the identity modulo \( hz \), this finishes the proof that \( \text{Aut}_{\text{Spin}}(E) \) is the group of all automorphisms which are the identity on \( (E) \). The last statement (about the discriminants of the eigenspaces) follows directly from the first statement of Proposition A.8(c).

Throughout the rest of the section, we collect some more technical results which will be needed in Sections 2 and 4.

**Lemma A.10** Fix \( k \geq 2 \). Let \( A = e_{13}(2^{k-2}) \) \( 2 \text{GL}_3(Z_{2^k}) \) be the elementary matrix which has one diagonal entry \( 2^{k-1} \) in position \((1, 3)\). Let \( T_1 \) and \( T_2 \) be the two maximal parabolic subgroups of \( \text{GL}(2) \):

\[
T_1 = \text{GL}_2^1(Z_{2^k}) = (a_{ij}) 2 \text{GL}_3(2) j a_{21} = a_{31} = 0
\]

and

\[
T_2 = \text{GL}_3^2(Z_{2^k}) = (a_{ij}) 2 \text{GL}_3(2) j a_{31} = a_{32} = 0
\]
Set $T_0 = T_1 \setminus T_2$: the group of upper triangular matrices in $GL_3(2)$. Assume that

$$i: T_1 \longrightarrow \ SL_3(\mathbb{Z}/2^k)$$

are lifts of the inclusions (for $i = 1, 2$) such that $j_{T_0} = j_{T_0}$. Then there is a homomorphism

$$: GL_3(2) \longrightarrow \ SL_3(\mathbb{Z}/2^k)$$

such that $j_{T_1} = 1$, and either $j_{T_2} = 2$, or $j_{T_2} = 2$.

**Proof** We first claim that any two liftings $0: T_2 \longrightarrow \ SL_3(\mathbb{Z}/2^k)$ are conjugate by an element of $SL_3(\mathbb{Z}/2^k)$. This clearly holds when $k = 1$, and so we can assume inductively that $0 \equiv 2^{k-1} \pmod{2^{k-1}}$. Let $M^0_3(\mathbb{F}_2)$ be the group of $3 \times 3$ matrices of trace zero, and define $T_2 \rightarrow M^0_3(\mathbb{F}_2)$ via the formula

$$q(B) = (I + 2^{k-1} B) \ (B)$$

for $B \in T_2$. Then is a 1-cocycle. Also, $H^1(T_2; M^0_3(\mathbb{F}_2)) = 0$ by [9, Lemma 4.3] (the module is $F_2[T_2]$ (projective), so is the coboundary of some $X \Rightarrow M^0_3(\mathbb{F}_2)$, and 0 differs by conjugation by $I + 2^{k-1}X$.

By [9, Theorem 4.1], there exists a section needed on $GL_3(2)$ such that $j_{T_1} = 1$. Let $B \in SL_3(\mathbb{Z}/2^k)$ be such that $j_{T_2} = c_B \ 2$. Since $j_{T_0} = 2$, $B$ must commute with all elements in $(T_0)$, and one easily checks that the only such elements are $A = e_{13}(2^{k-1})$ and the identity. \qed

Recall that a $p$-subgroup $P$ of a finite group $G$ is $p$-radical if $N_G(P) = P$ is $p$-reduced; i.e, if $O_p(N_G(P) \subseteq P) = 1$. (Here, $O_p(-)$ denotes the largest normal $p$-subgroup.) We say here that $P$ is $F_p(G)$ radical if $Out_G(P) = Out_{F_p(G)}(P)$ is $p$-reduced. In Section 4, some information will be needed involving the $F_2(Spin_7(q))$ radical subgroups of $Spin_7(q)$ which are also 2-centric. We rst note the following general result.

**Lemma A.1** Fix a finite group $G$ and a prime $p$. Then the following hold for any $p$-subgroup $P \triangleleft G$ which is $p$-centric and $F_p(G)$ {radical}.

(a) If $G = G_1 \times G_2$, then $P = P_1 \times P_2$, where $P_i$ is $p$-centric in $G_i$ and $F_p(G_i)$ {radical}.

(b) If $P \triangleleft G$, then $P$ is $p$-centric in $H$ and $F_p(H)$ {radical}.

(c) If $H \triangleleft G$ has $p$-power index, then $P \setminus H$ is $p$-centric in $H$ and $F_p(H)$ {radical}.
(d) If $G < G$ has $p\{power index\}$, then $P = G \setminus \overline{P}$ for some $\overline{P} \supset G$ which is $p\{centric in $G$ and $F_p(G)\}$. 

(e) If $Q < G$ is a central $p\{subgroup\}$, then $Q \cap P = Q$ is $p\{centric in $G=Q$ and $F_p(G=Q)\}$. 

(f) If $\mathcal{G} ightarrow G$ is an epimorphism such that $\ker(\mathcal{G}) = Z(\mathcal{G})$, then $-1(P)$ is $p\{centric in $\mathcal{G}$ and $F_p(\mathcal{G})\}$. 

Proof Point (a) follows from [16, Proposition 1.6(ii)]: $P = P_1 \cap P_2$ for $P_i \subset G_i$ since $P$ is $p\{radical\}$, and $P_i$ must be $p\{centric in $G_i$ and $F_p(G_i)\}$. 

Point (b) holds since $C_{\mathcal{H}}(P) = C_{\mathcal{G}}(P) = O_p(Out_{\mathcal{H}}(P))外包O_p(Out_{\mathcal{G}}(P))$. 

It remains to prove the other four points. 

(e) Fix a central $p\{subgroup\} Q < Z(G)$. Then $P = P_1 \cap P_2$ for $P_i \subset G_i$ since $P$ is $p\{radical\}$, and $P_i$ must be $p\{centric in $G_i$ and $F_p(G_i)\}$. 

Point (b) holds since $C_{\mathcal{H}}(P) = C_{\mathcal{G}}(P) = O_p(Out_{\mathcal{H}}(P))外包O_p(Out_{\mathcal{G}}(P))$. 

It remains only to prove that $P = Q$ is $F_p(G=Q)$. 

Equivalently, since $P = Q$ and $P$ are $p\{centric\}$, we must show that 

$\frac{N_{G=Q}(P=Q)}{C_{G=Q}(P=Q)} = \frac{N_G(P)}{C_G(P)}$; 

and this is clear once we have shown that 

$C_G(P) = C_{G=Q}(P=Q)$.

Any $x \in C_G(P)$ lifts to an element $x \in 2G$ of order prime to $p$, whose conjugation action on $P$ induces the identity on $Q$ and on $P = Q$. By [15, Corollary 5.3.3], all such automorphisms of $P$ have $p\{power order\}$, and thus $x$ centralizes $P$. Since $Q$ is a $p\{group\}$ and $C_{G=Q}(P=Q)$ has order prime to $p$, this shows that the projection modulo $Q$ sends $C_{G=Q}(P=Q)$ isomorphically to $C_G(P)$.
(f) Let \( \mathcal{G} \twoheadrightarrow G \) be an epimorphism whose kernel is central. Clearly, \(-1P\) is \( p \)-centric in \( \mathcal{G} \). It remains only to prove that \(-1P\) is \( F_p(\mathcal{G}) \) \{radical, and to do this it suffices to show that

\[
\text{Out}_\mathcal{G}( -1P ) = \text{Out}_G(P);
\]

Equivalently, since \( P \) and \(-1(P)\) are \( p \)-centric, we must show that

\[
\frac{N_G( -1P )}{C_G( -1P )} -1P = \frac{N_G(P)}{C_G(P) -P};
\]

and this is clear once we have shown that

\[
C_G( -1P ) = C_G(P);
\]

This follows by exactly the same argument as in the proof of (e).

(c) Set \( P^0 = P \setminus H \) for short. Let

\[
N_H(P^0) \twoheadrightarrow \text{Out}_H(P^0) = N_H(P^0)\text{\{centric in }H\) ;
\]

be the natural projection, and set

\[
K = \{ -1(O_P(\text{Out}_H(P^0))) | N_H(P^0) \};
\]

Then \( \text{Out}_P(N_H(P^0)) \) is an extension of \( C_H(P^0) P^0 \) by \( O_P(\text{Out}_H(P^0)) \). It suffices to show that \( p \mid [K : P^0] \), since this implies that \( O_P(\text{Out}_H(P^0)) = 1 \) (ie, \( P^0 \) is \( F_P(H) \) \{radical, and that any Sylow \( p \)-subgroup of \( C_H(P^0) \) is contained in \( P^0 \) (hence \( P^0 \) is \( p \)-centric in \( H \)).

Assume otherwise that \( p \mid [K : P^0] \). Note that \( P^0 \triangleleft N_G(P) \), and that \( N_G(P) \triangleright N_G(K) \); ie, \( N_G(P) \) normalizes \( P^0 \) and \( K \). The first statement is obvious, and the second is verified by observing directly that \( N_G(P) \) normalizes \( N_H(P^0) \) and \( C_H(P^0) \). Thus the action of \( N_G(P) \) on \( K \) induces an action of \( N_G(P) \), and in particular of \( P \), on \( K = P^0 \). Let \( K_0 = P^0 \) denote the fixed subgroup of this action of \( P \). Since \( p \mid [K : P^0] \) by assumption, and since \( P \) is a \( p \)-group, \( p \mid K_0 \). A straightforward check also shows that \( K_0 \triangleleft N_G(P) \), and therefore that \( P K_0 \triangleleft N_G(P) \). Also, since \( P^0 \triangleleft K_0 \triangleleft H \),

\[
P K_0 = K_0 = (P \setminus K_0) = K_0 = P^0
\]

is a normal subgroup of \( N_G(P) = P \) of order a multiple of \( p \). Since \( P \) is \( p \)-centric in \( G \) by assumption,

\[
\text{Out}_G(P) = N_G(P)\text{\{centric in }G\text{\})} = N_G(P)\text{\{centric in }G\text{\})} = (C_G(P) -P);
\]

and hence the image of \( P K_0 = P \) in \( \text{Out}_G(P) \) is a normal subgroup which also has order a multiple of \( p \).

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By definition of $K$ as an extension of $C_H(P) P^0$ by a $p$-group, if $x \in K$ has order prime to $p$, then $x \in C_H(P)$. Hence if $x \in K_0$ has order prime to $p$, then for every $z \in P$, $[x,z] \in P^0$, so $x$ acts trivially on $P=P^0$. Since $x$ also centralizes $P^0$, it follows that $x$ centralizes $P$. This shows that the image of $PK_0=\Phi$ in $\text{Out}_G(P)$ is a $p$-group, thus a nontrivial normal $p$-subgroup of $\text{Out}_G(P)$, and this contradicts the original assumption that $P$ is $F_p(G)$-radical.

**(d)** Let $G \triangleleft \bar{G}$ be a normal subgroup of $p$-power index and let $P \triangleleft G$ be a $p$-centric and $F_p(G)$-radical subgroup. Let

$$
N_{\bar{G}}(P) \rightarrow \text{Out}_{\bar{G}}(P) = N_{\bar{G}}(P) \cdot C_{\bar{G}}(P) \cdot P
$$

be the natural surjection, and set

$$
K = -1 \text{Op}(\text{Out}_{\bar{G}}(P)) \cdot N_{\bar{G}}(P):
$$

Then $K$ is an extension of $C_{\bar{G}}(P) P$ by $\text{Op}(\text{Out}_{\bar{G}}(P))$. Fix any $\Phi \in \text{Syl}_p(K)$. We will show that $\Phi \backslash G = P$, and that $\Phi$ is $p$-centric in $\bar{G}$ and $F_p(\bar{G})$-radical.

For each $x \in K \backslash G$, $N_G(P)$,

$$(x) \ 2 \text{Op}(\text{Out}_{\bar{G}}(P)) \backslash \text{Out}_{\bar{G}}(P) \ 	ext{Op}(\text{Out}_{\bar{G}}(P)) = 1.$$

Hence

$x \in \text{Ker} N_G(P) \rightarrow \text{Out}_{\bar{G}}(P) = (C_{\bar{G}}(P) P) \backslash G = C_{\bar{G}}(P) P = C_{\bar{G}}^0(P) P$;

where $C_{\bar{G}}^0(P)$ is of order prime to $p$. Since the opposite inclusion is obvious, this shows that $K \backslash G = C_{\bar{G}}^0(P) P$, and hence (since $\Phi \in \text{Syl}_p(K)$) that $\Phi \backslash G = P$.

Next, note that $(K \backslash G) \triangleleft K$ and $K/(K \backslash G) = \bar{G}$, and hence $K=C_{\bar{G}}^0(P)$ has $p$-power order. Since $\Phi \in \text{Syl}_p(K)$, $\Phi$ is an extension of $P$ by $K=(K \backslash G)$, and $N_G(\Phi)$ is an extension of a subgroup of $(K \backslash G) = (C_{\bar{G}}^0(P) P) / K = (K \backslash G)$.

Also, an element $x \in C_{\bar{G}}^0(P)$ normalizes $\Phi$ if and only if $[x;\Phi] 2 \Phi \backslash C_{\bar{G}}^0(P) = 1$. Hence

$$
N_K(\Phi) = C_K(\Phi) \Phi = C_{\bar{G}}^0(\Phi) \Phi;
$$

where $C_{\bar{G}}^0(\Phi) = C_{\bar{G}}^0(\Phi) \backslash C_{\bar{G}}(\Phi)$ has order prime to $p$ and is normal in $N_K(\Phi)$.

Since $C_{\bar{G}}(\Phi)$ is $\bar{G}$, (1) shows that $C_{\bar{G}}(\Phi) C_{\bar{G}}^0(\Phi) \Phi$, and hence that $\Phi$ is $p$-centric in $\bar{G}$.

It remains to show that $\Phi$ is $F_p(\bar{G})$-radical. Note that $K \triangleleft N_G(P)$ by construction, so for any $x \in N_G(P)$, $x^\Phi x^{-1} \in \text{Syl}_p(K)$. Since $K$ is an
extension of $C_G^0(P)\to P$ by the $p$-group $K = (K \setminus G)$, and since $C_G^0(P) \triangleleft K$, it follows that $K$ is a split extension of $C_G^0(P)$ by $P$. Hence for any $x \in 2N_G(P)$, $xP^{-1} = yPy^{-1}$ for some $y \in C_G^0(P)$. Consequently, the restriction map

$$N_G(P) = C_G(P) = \text{Aut}_G(P) \longrightarrow \text{Aut}_G(P) = C_G^0(P) \quad (2)$$

is surjective. Also, if $x \in C_G^0(P)$, $K$ normalizes $P$, then $x \in N_K(P) = P$. Thus the kernel of the map in (2) is contained in $\text{Inn}(P)$. Consequently,

$$\text{Out}_G(P) = \text{Out}_G(P) = \text{Inn}(P) = \text{Out}_G(P) = \text{Out}_G(P) = \text{Out}_G(P) = \text{Out}_G(P) = \text{Out}_G(P) = \text{Out}_G(P) = \text{Out}_G(P).$$

and it follows that $P$ is $F_p(G)$-radical.

This is now applied to show the following:

**Proposition A.12** Fix an odd prime power $q$, and let $P \in \text{Spin}_7(q)$ be any subgroup which is 2-centric and $F_2(\text{Spin}_7(q))$-radical. Then $P$ is centric in $\text{Spin}_7(P)$; i.e., $C_{\text{Spin}_7(P)}(P) = Z(P)$.

**Proof** Let $z$ be the central involution in $\text{Spin}_7(q)$. By Lemma A.11(e), $zPz = P$, and $\bar{P} \overset{\text{def}}{=} Pz$ is 2-centric in $\Omega_7(q)$ and is $F_2(\Omega_7(q))$-radical. So by Lemma A.11(d), there is a 2-subgroup $\mathfrak{p} \in \Omega_7(q)$ such that $\mathfrak{p} \setminus \Omega_7(q) = \bar{P}$, and such that $\mathfrak{p}$ is 2-centric in $\Omega_7(q)$ and is $F_2(\Omega_7(q))$-radical.

Let $V = \bigoplus_{i=1}^m V_i$ be a maximal decomposition of $V$ as an orthogonal direct sum of $\mathfrak{p}$-representations, and set $b_i = b_iV_i$. We assume these are arranged so that for some $k$, $\dim(V_i) > 1$ when $i < k$ and $\dim(V_i) = 1$ when $i > k$. Let $V_+$ be the sum of those 1-dimensional components $V_i$ with square discriminant, and let $V_-$ be the sum of those 1-dimensional components $V_i$ with nonsquare discriminant. We will be referring to the two decompositions

$$M^+ = \bigoplus_{i=1}^k (V_i; b_i), \quad M^- = \bigoplus_{i=1}^m (V_i; b_i) \quad (V_+; b_+) \quad (V_-; b_-);$$

both of which are orthogonal direct sums. We also write

$$V^{(1)} = \bar{P} \otimes_{\mathbf{F}_q} V \quad \text{and} \quad V_1^{(1)} = \bar{P} \otimes_{\mathbf{F}_q} V_i;$$

and let $b^{(1)}_1$ and $b^{(1)}_1$ be the induced quadratic forms.
Step 1. For each \( i \), set
\[
D_i = f_{b_i} \Omega(V_i; b_i); 
\]
a subgroup of order 2; and write
\[
D = \bigoplus_{i=1}^{\eta} D_i \text{ O(V; b); and } D = \bigoplus_{\nu \in V} Y D_i \text{ O(V; b);}. 
\]
Thus \( D \) and \( D \) are elementary abelian 2-groups of rank \( m \) and \( \text{dim}(V) \), respectively. We first claim that
\[
\widehat{D} \text{ D; } (1) 
\]
and that
\[
\widehat{D} = \bigoplus_{i=1}^{\eta} P_i \text{ where } 8 \in P_i \text{ is 2-centric in } O(V_i; b_i) \text{ and } F_2(O(V_i; b_i)) \{ \text{radical}.} 
\]
Clearly, \([D; \widehat{D}] = 1 \) (and \( D \) is a 2-group), so \( D \) is 2-centric. This proves (1). The \( V_i \) are thus distinct (pairwise nonisomorphic) as \( D \)-representations, since they are pairwise nonisomorphic as \( D \)-representations. The decomposition as a sum of \( V_i \)'s is thus unique (not only up to isomorphism), since \( \text{Hom}_b(V; V_j) = 0 \) for \( i \neq j \).

Let \( \Theta \) be the group of elements of \( O(V; b) \) which send each \( V_i \) to itself, and let \( \Phi \) be the group of elements which permute the \( V_i \). By the uniqueness of the decomposition of \( V \),
\[
\Phi \Theta_{\text{O(V; b)}}(\widehat{D}) = \bigoplus_{i=1}^{\eta} O(V_i; b_i) \text{ and } N_{\text{O(V; b)}}(\Phi) \Phi: 
\]
Since \( \Phi \) is 2-centric in \( O(V; b) \) and \( F_2(O(V; b)) \{ \text{radical, it is also 2-centric in } \Phi \text{ and } F_2(\Phi) \{ \text{radical (this holds for any subgroup which contains } N_{\text{O(V; b)}}(\Phi)).} 
\]
So by Lemma A.11(b) (and since \( \Theta \triangleleft \Phi \)), \( \Phi \) is 2-centric in \( \Theta \) and \( F_2(\Theta) \{ \text{radical. Point (2) now follows from Lemma A.11(a).}

Step 2. Whenever \( \text{dim}(V_i) > 1 \) (i.e., \( 1 \leq i \leq k \)), then by Lemma A.6, \( \text{dim}(V_i) \) is even, and \( b_i \) has square discriminant. So by Lemma A.4(a), \( -f_{b_i} \Omega(V_i; b_i) \) for such \( i \). Together with (1), this shows that
\[
\Phi = \Phi \backslash \Omega_2(q) \bigoplus_{i=1}^{\nu} D_i \backslash \Omega(V_i; b_i) \ D_+ \backslash \Omega(V_; b_-) \ D_-: (3)
\]
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Also, by Lemma A.4(a) again,
\[ \Omega(V; b) \setminus D = \text{SO}(V; b) \setminus D \]
\[ = -\text{Id}_{V_i} V_i \quad k+1 \quad i < j \quad m; \quad V_i; V_j \quad V. \]  \hspace{1cm} (4)

**Step 3** By (3) and (4), the \( V_i \) are distinct as \( P \) {representations (not only as \( \overline{P} \) {representations), except possibly when \( \dim(V) = 2 \). We first check that this exceptional case cannot occur. If \( \dim(V_+) = 2 \) and its two irreducible summands are isomorphic as \( P \) {representations, then the image of \( P \) under projection to \( \text{O}(V; b) \) is just \( \text{Id}_V, g \). Hence we can write \( V_+ = W \) \( W^0 \), where \( W, W^0 \) are \( 1 \) {dimensional, \( P \) {invariant, and have nonsquare discriminant. Also, \( \dim(V_-) \) is odd, since \( V_+ \) and the \( V_i \) for \( i = k \) are all even dimensional. So \( -\text{Id}_{V_i} \) lies in \( C_{\Omega(q)}(\overline{P}) \) but not in \( \overline{P} \). But this is impossible, since \( \overline{P} \) is \( 2 \) {centric in \( \Omega(q) \). The argument when \( \dim(V_-) = 2 \) is similar.

The \( V_i \) are thus distinct as \( \overline{P} \) {representations. So for all \( i \neq j \), \( \text{Hom}_P(V_i; V_j) = 0 \), and hence
\[ \text{Hom}_{\overline{P}}(V_i^{(1)}; V_j^{(1)}) = \overline{F}_q \otimes \overline{F}_q \text{Hom}_{\overline{P}}(V_i; V_j) = 0. \]
Thus any element of \( \text{O}(V^{(1)}; b^{(1)}) \) which centralizes \( \overline{P} \) sends each \( V_i^{(1)} \) to itself. In other words,
\[ C_{\text{Spin}_7(\overline{F}_q)}(\overline{P}) \cong_{i=1}^{m} C_{\Omega(q)}(\overline{P}) \]
\[ = \text{O}(V_i^{(1)}; b_i^{(1)}): \]
If \( \dim(V) = 2 \), then since \( \overline{P} \) contains all involutions in \( \text{O}(V; b) \) which are \( P \) {invariant and have even dimensional \( (-1) \) {eigenspace (see (3)), Lemma A.4(c) shows that each element of \( \text{Spin}_7(\overline{F}_q) \) which commutes with \( P \) must act on \( V \) via \( \text{Id} \). Also, for \( 1 \leq i \leq k \), since \( -\text{Id}_{V_i} 2 \overline{P} \) by (3), each element in the centralizer of \( P \) acts on \( V_i \) with determinant 1 (Lemma A.4(c) again). We thus conclude that
\[ C_{\text{Spin}_7(\overline{F}_q)}(\overline{P}) \cong_{i=1}^{k} \text{SO}(V_i^{(1)}; b_i^{(1)}) \quad \text{Id}_{V_i} g \quad \text{Id}_{V_i} g. \]  \hspace{1cm} (5)

**Step 4** We next show that
\[ C_{\text{Spin}_7(\overline{F}_q)}(\overline{P}) \cong_{i=1}^{k} \text{Id}_{V_i} g \quad \text{Id}_{V_i} g. \]  \hspace{1cm} (6)
Using (5), this means showing, for each \( 1 \leq k \leq l \), that

\[
pr_i \ C_{\text{Spin}^r(\mathbb{F}_q_i)}(P) = \text{Id}_{V_i} \quad (7)
\]

where \( pr_i \) denotes the projection of \( O_\gamma(\mathbb{F}_q) = O(V_i^{(1)}; b^{(1)}) \) to \( O(V_i^{(1)}; b_i^{(1)}) \).

By Lemma A.6, \( \dim(V_i) = 2 \) or 4. We consider these two cases separately.

**Case 4A** If \( \dim(V_i) = 4 \), then by (2) and Lemma A.11(c), \( P_i^{0} \) is 2-central in \( \Omega(V_i; b_i) \) and is \( F_2(\Omega(V_i; b_i)) \) radical. Also, by Proposition A.5,

\[
\Omega(V_i; b_i) = \Omega^+_4(q) = SL_2(q) \quad c_2 SL_2(q):
\]

By Lemma A.11(a,f), under this identification, we have \( P_i^{0} = Q \subset Q^0 \), where \( Q \) and \( Q^0 \) are 2-central in \( SL_2(q) \) and \( F_2(SL_2(q)) \) radical. The Sylow 2-subgroups of \( SL_2(q) \) are quaternion groups of order 8, all subgroups of a quaternion 2-group are quaternion or cyclic, and cyclic 2-subgroups of \( SL_2(q) \) cannot be both 2-central and \( F_2(SL_2(q)) \) radical. So \( Q \) and \( Q^0 \) must be quaternion of order 8. By [23, 3.6.3], any cyclic 2-subgroup of \( SL_2(\mathbb{F}_q) \) of order 4 is conjugate to a subgroup of diagonal matrices, whose centralizer is the group of all diagonal matrices in \( SL_2(\mathbb{F}_q) \). Knowing this, one easily checks that all nonabelian quaternion 2-subgroups of \( SL_2(\mathbb{F}_q) \) are central in \( SL_2(\mathbb{F}_q) \). It follows that \( P_i^{0} \) is central in

\[
SO(V_i^{(1)}; b_i^{(1)}) = SL_2(\mathbb{F}_q) \quad c_2 SL_2(\mathbb{F}_q);
\]

and hence that

\[
pr_i \ C_{\text{Spin}^r(\mathbb{F}_q_i)}(P) = \text{Id}_{V_i} \quad C_{SO(V_i^{(1)}; b_i^{(1)})}(P_i^{0}) = Z(P_i^{0}) = \text{Id}_{V_i}.
\]

Thus (7) holds in this case.

**Case 4B** If \( \dim(V_i) = 2 \), then \( O(V_i; b_i) = O_2(q) \) is a dihedral group of order 2(q−1) [24, Theorem 11.4]. Hence \( P_i \) is a Sylow \( 2 \) subgroup of \( O(V_i; b_i) \), since the Sylow subgroups of orders the only radical \( 2 \) subgroups of a dihedral group. Fix \( V_i \) for any \( k < j \) and choose \( 2 \) \( O(V_i; b_i) \) of determinant \(-1\) whose \(-1\) eigenspace has the same discriminant as \( V_j \). Since \( P_i \) is a Sylow \( 2 \) subgroup of \( O(V_i; b_i) \), we can assume (after conjugating if necessary) that \( P_i \) is a radical 2-subgroup. Then \(-\text{Id}_{V_i}\) lies in \( P_i \). Hence for any \( g \in P_i \), \( g \) leaves both eigenspaces of \( V_i \) invariant, and has determinant 1 by (5). Thus \( pr_i(g) = \text{Id}_{V_i} \) and so (7) holds in this case.

**Step 5** Clearly, \(-\text{Id}_{V_i}\) lies in \( SO(V_i; b_i) \) if and only if \( \dim(V_i) \) is even (which is the case for exactly one of the two spaces \( V_i \)), and this holds if and
only if \(-\text{Id}_V \equiv 2 \Omega(V; b)\). Also, since each \(V_i\) for \(1 \leq i \leq k\) has square discriminant (Lemma A.6 again), \(-\text{Id}_V \equiv 2 \Omega(V; b)\) for all such \(i\). Thus (6) and (1) imply that

\[ C_{\text{Spin}_7(F_q)}(P) \cong \Phi \setminus \Omega_7(q) = \mathcal{P}; \]

and hence that \(P\) is centric in \(\text{Spin}_7(F_q)\).

Proposition A.12 does not hold in general if \(\text{Spin}_7(-)\) is replaced by an arbitrary algebraic group. For example, assume \(q\) is an odd prime power, and let \(P = \text{SL}_5(F_q)\) be the group of diagonal matrices of \(2\)-power order. Then \(P\) is \(2\)-centric in \(\text{SL}_5(F_q)\) and \(F_2(\text{SL}_5(F_q))\) radical, but is definitely not \(2\)-centric in \(\text{SL}_5(F_q)\).

References


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