Burnside obstructions to the Montesinos-Nakanishi 3-move conjecture

Mieczyslaw K. Dabkowski
Jozef H. Przytycki

Department of Mathematics, The George Washington University
Washington, DC 20052, USA

Email: mdab@gwu.edu, przytyck@gwu.edu

Abstract

Yasutaka Nakanishi asked in 1981 whether a 3-move is an unknotting operation. In Kirby’s problem list, this question is called The Montesinos-Nakanishi 3-move conjecture. We define the nth Burnside group of a link and use the 3rd Burnside group to answer Nakanishi’s question; i.e., we show that some links cannot be reduced to trivial links by 3-moves.

AMS Classification numbers  
Primary: 57M27  
Secondary: 20D99

Keywords: Knot, link, tangle, 3-move, rational move, braid, Fox coloring, Burnside group, Borromean rings, Montesinos-Nakanishi conjecture, branched cover, core group, lower central series, associated graded Lie ring, skein module

Proposed: Robion Kirby  
Received: 5 May 2002
Seconded: Walter Neumann, Vaughan Jones  
Revised: 19 June 2002
One of the oldest elementary formulated problems in classical Knot Theory is the 3-move conjecture of Nakanishi. A 3-move on a link is a local change that involves replacing parallel lines by 3 half-twists (Figure 1).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Figure 1}
\end{figure}

**Conjecture 1** (Montesinos-Nakanishi, Kirby’s problem list; Problem 1.59(1), [4]) Any link can be reduced to a trivial link by a sequence of 3-moves.

The conjecture has been proved to be valid for several classes of links by Chen, Nakanishi, Przytycki and Tsukamoto (eg, closed 4-braids and 4-bridge links).

Nakanishi, in 1994, and Chen, in 1999, have presented examples of links which they were not able to reduce: \( L_{2BR} \), the 2{parallel of the Borromean rings, and \( \mathcal{\gamma} \), the closure of the square of the center of the \( n \)th braid group, ie, \( \mathcal{\gamma} = (1 2 3 4)^{10} \).

**Remark 2** In [6] it was noted that 3-moves preserve the first homology of the double branched cover of a link \( L \) with \( Z_3 \) coefficients \( (H_1(M_{L}^{(2)}; Z_3)) \). Suppose that \( \mathcal{\gamma} \) (respectively \( L_{2BR} \)) can be reduced by 3-moves to the trivial link \( T_n \). Since \( H_1(M_{\mathcal{\gamma}}^{(2)}; Z_3) = Z_3^4 \), \( H_1(M_{L_{2BR}}^{(2)}; Z_3) = Z_3^5 \) and \( H_1(M_{T_n}^{(2)}; Z_3) = Z_3^{n-1} \) where \( T_n \) is a trivial link of \( n \) components, it follows that \( n = 5 \) (respectively \( n=6 \)).

We show below that neither \( \mathcal{\gamma} \) nor \( L_{2BR} \) can be reduced by 3-moves to trivial links.

The tool we use is a non-abelian version of Fox n-colorings, which we shall call the \( n \)th Burnside group of a link, \( B_L(n) \).

**Definition 3** The \( n \)th Burnside group of a link is the quotient of the fundamental group of the double branched cover of \( S^3 \) with the link as the branch set divided by all relations of the form \( a^n = 1 \). Succinctly: \( B_L(n) = 1(M_L^{(2)}) = (a^n) \).

**Proposition 4** \( B_L(3) \) is preserved by 3-moves.
**Proof** In the proof we use the core group interpretation of \( 1(M^{(2)}_L) \). Let \( D \) be a diagram of a link \( L \). We define (after [3, 2]) the associated core group \( D^{(2)} \) as follows: generators of \( D^{(2)} \) correspond to arcs of the diagram. Any crossing \( v_s \) yields the relation \( r_s = y_1^{-1}y_1y_k^{-1} \) where \( y_i \) corresponds to the overcrossing and \( y_j; y_k \) correspond to the undercrossings at \( v_s \) (see Figure 2).

In this presentation of \( D^{(2)} \) one relation can be dropped since it is a consequence of others. Wada proved that \( D^{(2)} = 1(M^{(2)}_L) \cong \mathbb{Z} \), [10] (see [7] for an elementary proof using only Wirtinger presentation). Furthermore, if we put \( y_i = 1 \) for any fixed generator, then \( D^{(2)} \) reduces to \( 1(M^{(2)}_L) \). The last part of our proof is illustrated in Figure 2.

\[\begin{align*}
\gamma &\xrightarrow{y_1} \gamma^{-1}y_1 \quad \gamma^{-1}y_2^{-1}y_1 \\
\gamma &\xrightarrow{y_2} \gamma^{-1}y_2^{-1}y_1 \quad \gamma^{-1}y_2^{-1}y_1 \\
\end{align*}\]

Figure 2

**Lemma 5** \( B_\varphi(3) = f x_1; x_2; x_3; x_4 \ j a^3 \) for any word \( a; P_1; P_2; P_3; P_4 \), where

\[ P_1 = x_1^{-1}x_3^{-1}x_4^{-1}x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1}x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1}x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1}x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1}x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1}x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1}x_1^{-1}. \]

**Proof** Consider the 5(braid) \( \gamma = (1 \ 2 \ 3 \ 4)^{10} \) (Figure 3). If we label initial arcs of the braid by \( x_1; x_2; x_3; x_4 \) and \( x_5 \), and use core relations (progressing from left to right) we obtain labels \( Q_1; Q_2; Q_3; Q_4 \) and \( Q_5 \) on the final arcs of the braid where

\[ Q_1 = x_1^{-1}x_3^{-1}x_4^{-1}x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1}x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1}x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1}x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1}x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1}x_1^{-1}. \]

For a group \( \varphi^{(2)} \) of the closed braid \( \varphi \), we have relations \( Q_i = x_i \). To obtain \( 1(M^{(2)}_\varphi) \) we can put \( x_5 = 1 \), and delete one relation, say \( Q_5x_5^{-1} \). These lead to the presentation of \( B_\varphi(3) \) described in the lemma.

**Theorem 6** The links \( \varphi \) and \( L_{2BR} \) are not 3(move reducible to trivial links.)
Proof  Let $B(n; 3)$ denote the classical free $n$ generator Burnside group of exponent $3$. As shown by Burnside [1], $B(n; 3)$ is a finite group. Its order, $|B(n; 3)|$, is equal to $3^{n^2}$. For a trivial link: $B_{T_i}(3) = B(k - 1; 3)$. In order to prove that $\gamma$ and $L_{2B_R}$ are not movable reducible to trivial links, it suffices to show that $B_\gamma(3) \cong B(4; 3)$ and $B_{L_{2B_R}}(3) \cong B(5; 3)$ (see Remark 2). We have demonstrated these to be true both by manual computation, and by using the programs GAP, Magnus and Magma. More details in the case of $\gamma$ are provided below.

For the manual calculations, one first observes that for any $i$, $P_i$ is in the third term of the lower central series of $B(4; 3)$. In particular, for $u = x_1x_2^{-1}x_3x_4^{-1}$ and $u = x_1^{-1}x_2^{-1}x_3x_4$, one has $uu = [B(4; 3); B(4; 3)]$ and $P_i = [uu; x_iu]$. It is known ([9]), that $B(4; 3)$ is of class $3$ (the lower central series has $3$ terms), and that the third term is isomorphic to $Z_3^3$ with basis: $e_1 = [[x_2; x_3]; x_4], e_2 = [[x_1; x_3]; x_4], e_3 = [[x_1; x_2]; x_4]$ and $e_4 = [[x_1; x_2]; x_3]$. It now takes an elementary linear algebra calculation (see Lemma 7 below) to show that $P_1; P_2; P_3; P_4$ form a basis of the third term of the lower central series of $B(4; 3)$. Thus $|B_\gamma(3)| = 3^{10}$. \hfill $\Box$

Lemma 7  $P_1; P_2; P_3$, and $P_4$ form a basis of the third term of the lower central series of $B(4; 3)$.

Proof  In the associated graded Lie ring $L(4; 3)$ of $B(4; 3)$ ([9]), the third term (denoted $L_3$) is isomorphic to $Z_3^3$ with basis $e_1; e_2; e_3$. In $L(4; 3)$, which is a linear space over $Z_3$, one uses an additive notation and the bracket in the group becomes a (non-associative) product ([9]). In this notation $e_1 = x_2x_3x_4$, $e_2 = x_1x_3x_4$, $e_3 = x_1x_2x_4$, and $e_4 = x_1x_2x_3$. In the calculation expressing $P_i$ in the basis we use the following identities in $L_3$ ([9], page 89).

\[
\text{xyz} = 0; \text{xyz} = \text{yxz} = \text{zxy} = -\text{zxy} = -\text{yxz} = -\text{xyz}; \text{xzy} = 0;
\]

Now we have: $P_1 = (uu)(x_1u)(uu)^{-1}(x_1u)^{-1} = [(uu)^{-1}(x_1u)^{-1}] = [uu; x_1u]$ as the last term of the lower central series is in the center of $B(4; 3)$. Furthermore, we have $uu = x_1x_2^{-1}x_3x_4^{-1}x_1^{-1}x_2^{-1}x_3^{-1}x_4 = [x_2^{-1}x_3x_4^{-1}, x_1^{-1}]x_3x_4^{-1}, x_2][x_4^{-1}, x_3^{-1}]$.

Geometry & Topology, Volume 6 (2002)
Writing $P_i$ additively in $L_3$ one obtains:

$$P_1 = ((-x_2 + x_3 - x_4)(-x_1) + (x_3 - x_4)x_2 + x_4x_3)(x_i - x_1 + x_2 - x_3 + x_4):$$

After simplifications one gets:

$$P_1 = -e_1; P_2 = e_1 + e_2; P_3 = e_1 - e_2 - e_3; \text{ and } P_4 = e_1 - e_2 + e_3 + e_4;$$

The matrix expressing $P_i$ in terms of $e_i$'s is the upper triangular matrix with the determinant equal to 1. Therefore the lemma follows.

A similar calculation establishes that $|B_{L_{2BR}}(3)| < |B(5; 3)|$. $B(5; 3)$ is of class 3 and has $3^{25}$ elements. Considering $L_{2BR}$ as a closed 6-braid we note that $B_{L_{2BR}}(3)$ is obtained from $B(5; 3)$ by adding 5 relations $R_1; \ldots; R_5$. Relations $f_{R_1;g}$ are in the last term of the lower central series of $B(5; 3)$ (and of the associated graded algebra $L(5; 3)$). Relations form a 4-dimensional subspace in $L_3 = \mathbb{Z}_3^{10}$. Thus $|B_{L_{2BR}}(3)| = 3^{21}$.

For a computer verification showing that $B_{L_{2BR}}(3) \not= B(4; 3)$ consider any presentation of $B(4; 3)$ (eg, Magma solution by Mike Newman [5]) and add the relations $P_i$ to obtain a presentation of $B_{L_{2BR}}(3)$. Using any of the algebra programs mentioned above, one verifies that $|B_{L_{2BR}}(3)| = 3^{10}$ while $|B(4; 3)| = 3^{14}$.

The solution of the Nakanishi-Montesinos 3-move conjecture, presented above, is the first instance of application of Burnside groups of links. It was motivated by the analysis of cubic skein modules of 3-manifolds. The next step is the application of Burnside groups to rational moves on links. This, in turn, should have deep implications to the theory of skein modules [7].

References


Geometry & Topology, Volume 6 (2002)


