Torsion, TQFT, and Seiberg–Witten invariants of 3–manifolds

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Abstract

We prove a conjecture of Hutchings and Lee relating the Seiberg–Witten invariants of a closed 3–manifold $X$ with $b_1 \geq 1$ to an invariant that “counts” gradient flow lines—including closed orbits—of a circle-valued Morse function on the manifold. The proof is based on a method described by Donaldson for computing the Seiberg–Witten invariants of 3–manifolds by making use of a “topological quantum field theory,” which makes the calculation completely explicit. We also realize a version of the Seiberg–Witten invariant of $X$ as the intersection number of a pair of totally real submanifolds of a product of vortex moduli spaces on a Riemann surface constructed from geometric data on $X$. The analogy with recent work of Ozsváth and Szabó suggests a generalization of a conjecture of Salamon, who has proposed a model for the Seiberg–Witten–Floer homology of $X$ in the case that $X$ is a mapping torus.

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1 Introduction

In [5] and [6], Hutchings and Lee investigate circle-valued Morse theory for Riemannian manifolds $X$ with first Betti number $b_1 \geq 1$. Given a generic Morse function $\phi : X \to S^1$ representing an element of infinite order in $H^1(X; \mathbb{Z})$ and having no extrema, they determine a relationship between the Reidemeister torsion $\tau(X, \phi)$ associated to $\phi$, which is in general an element of the field $\mathbb{Q}(t)$, and the torsion of a “Morse complex” $M^*$ defined over the ring $L_{\mathbb{Z}}$ of integer-coefficient Laurent series in a single variable $t$. If $S$ is the inverse image of a regular value of $\phi$ then upward gradient flow of $\phi$ induces a return map $F : S \to S$ that is defined away from the descending manifolds of the critical points of $\phi$. The two torsions $\tau(X, \phi)$ and $\tau(M^*)$ then differ by multiplication by the zeta function $\zeta(F)$. In the case that $X$ has dimension three, which will be our exclusive concern in this paper, the statement reads

$$\tau(M^*)\zeta(F) = \tau(X, \phi),$$

up to multiplication by $\pm t^k$. One should think of the left-hand side as “counting” gradient flows of $\phi$; $\tau(M^*)$ is concerned with gradient flows between critical points of $\phi$, while $\zeta(F)$, defined in terms of fixed points of the return map, describes the closed orbits of $\phi$. It should be remarked that $\tau(X, \phi) \in \mathbb{Q}(t)$ is in fact a polynomial if $b_1(X) > 1$, and “nearly” so if $b_1(X) = 1$; see [10] or [17] for details.

If the three–manifold $X$ is zero-surgery on a knot $K \subset S^3$ and $\phi$ represents a generator in $H^1(X; \mathbb{Z})$, the Reidemeister torsion $\tau(X, \phi)$ is essentially (up to a standard factor) the Alexander polynomial $\Delta_K$ of the knot. It has been proved by Fintushel and Stern [4] that the Seiberg–Witten invariant of $X \times S^1$, which can be identified with the Seiberg–Witten invariant of $X$, is also given by the Alexander polynomial (up to the same standard factor). More generally, Meng and Taubes [10] show that the Seiberg–Witten invariant of any closed three–manifold with $b_1(X) \geq 1$ can be identified with the Milnor torsion $\tau(X)$ (after summing over the action of the torsion subgroup of $H^2(X; \mathbb{Z})$), from which it follows that if $S$ denotes the collection of spin$^c$ structures on $X$,

$$\sum_{\alpha \in S} SW(\alpha)t^{c_1(\alpha) \cdot S/2} = \tau(X, \phi),$$

up to multiplication by $\pm t^k$ (in [10] the sign is specified). Here $c_1(\alpha)$ denotes the first Chern class of the complex line bundle $\det \alpha$ associated to $\alpha$.

These results point to the natural conjecture, made in [6], that the left-hand side of (1) is equal to the Seiberg–Witten invariant of $X$—or more precisely
to a combination of invariants as in (2)—independently of the results of Meng and Taubes. We remark that the theorem of Meng and Taubes announced in [10] depends on surgery formulae for Seiberg–Witten invariants, and a complete proof of these results has not yet appeared in the literature. The conjecture of Hutchings and Lee gives a direct interpretation of the Seiberg–Witten invariants in terms of geometric information, reminiscent of Taubes’s work relating Seiberg–Witten invariants and holomorphic curves on symplectic 4–manifolds. The proof of this conjecture is the aim of this paper; combined with the work in [6] and [5] it establishes an alternate proof of the Meng–Taubes result (for closed manifolds) that does not depend on the surgery formulae for Seiberg–Witten invariants used in [10] and [4].

Remark 1.1 In fact, the conjecture in [6] is more general, as follows: Hutchings and Lee define an invariant $I : S \to \mathbb{Z}$ of spin$^c$ structures based on the counting of gradient flows, which is conjectured to agree with the Seiberg–Witten invariant. The proof presented in this paper gives only an “averaged” version of this statement, i.e., that the left hand side of (1) is equal to the left hand side of (2). It can be seen from the results of [6] that this averaged statement is in fact enough to recover the full Meng–Taubes theorem: see in particular [6], Lemma 4.5. It may also be possible to extend the methods of this paper to distinguish the Seiberg–Witten invariants of spin$^c$ structures whose determinant lines differ by a non-torsion element $a \in H^2(X; \mathbb{Z})$ with $a \cdot S = 0$.

We also show that the “averaged” Seiberg–Witten invariant is equal to the intersection number of a pair of totally real submanifolds in a product of symmetric powers of a slice for $\phi$. This is a situation strongly analogous to that considered by Ozsváth and Szabó in [14] and [15], and one might hope to define a Floer-type homology theory along the lines of that work. Such a construction would suggest a generalization of a conjecture of Salamon, namely that the Seiberg–Witten–Floer homology of $X$ agrees with this new homology (which is a “classical” Floer homology in the case that $X$ is a mapping torus—see [16]).

2 Statement of results

Before stating our main theorems, we need to recall a few definitions and introduce some notation. First is the notion of the torsion of an acyclic chain complex; basic references for this material include [11] and [17].
2.1 Torsion

By a volume $\omega$ for a vector space $W$ of dimension $n$ we mean a choice of nonzero element $\omega \in \Lambda^n W$. Let $0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0$ be an exact sequence of finite-dimensional vector spaces over a field $k$. For volumes $\omega'$ on $V'$ and $\omega''$ on $V''$, the induced volume on $V$ will be written $\omega' \omega''$; if $\omega_1$, $\omega_2$ are two volume elements for $V$, then we can write $\omega_1 = c \omega_2$ for some nonzero element $c \in k$ and by way of shorthand, write $c = \omega_1 / \omega_2$. More generally, let $\{C_i\}_{i=0}^n$ be a complex of vector spaces with differential $\partial : C_i \rightarrow C_{i-1}$, and let us assume that $C_n$ is acyclic, ie, $H_n(C_n) = 0$. Suppose that each $C_i$ comes equipped with a volume element $\omega_i$, and choose volumes $\nu_i$ arbitrarily on each image $\partial C_i$, $i = 2, \ldots, n-1$. From the exact sequence

$$0 \rightarrow C_n \rightarrow C_{n-1} \rightarrow \partial C_{n-1} \rightarrow 0$$

define $\tau_{n-1} = \omega_n \nu_{n-1} / \omega_{n-1}$. For $i = 2, \ldots, n-2$ use the exact sequence

$$0 \rightarrow \partial C_{i+1} \rightarrow C_i \rightarrow \partial C_i \rightarrow 0$$

to define $\tau_i = \nu_{i+1} \nu_i / \omega_i$. Finally, from

$$0 \rightarrow \partial C_2 \rightarrow C_1 \rightarrow C_0 \rightarrow 0$$

define $\tau_1 = \nu_2 \omega_0 / \omega_1$. We then define the torsion $\tau(C, \{\omega_i\}) \in k \setminus \{0\}$ of the (volumed) complex $C$ to be:

$$\tau(C) = \prod_{i=1}^{n-1} \tau_i^{(-1)^{i+1}}$$

(3)

It can be seen that this definition does not depend on the choice of $\nu_i$. Note that in the case that our complex consists of just two vector spaces,

$$C = 0 \rightarrow C \xrightarrow{\partial} C_{i-1} \rightarrow 0,$$

we have that $\tau(C) = \text{det}(\partial)^{(-1)^i}$. We extend the definition of $\tau(C)$ to non-acyclic complexes by setting $\tau(C) = 0$ in this case.

As a slight generalization, we can allow the chain groups $C_i$ to be finitely generated free modules over an integral domain $K$ with fixed ordered bases rather than vector spaces with fixed volume elements, as follows. Write $Q(K)$ for the field of fractions of $K$, then form the complex of vector spaces $Q(K) \otimes_K C_i$. The bases for the $C_i$ naturally give rise to bases, and hence volumes, for $Q(K) \otimes_K C_i$. We understand the torsion of the complex of $K$–modules $C_i$ to be the torsion of this latter complex, and it is therefore a nonzero element of the field $Q(K)$.
Let X be a connected, compact, oriented smooth manifold with a given CW decomposition. Following [17], suppose \( \varphi : \mathbb{Z}[H_1(X;\mathbb{Z})] \to K \) is a ring homomorphism into an integral domain K. The universal abelian cover \( \tilde{X} \) has a natural CW decomposition lifting the given one on X, and the action of the deck transformation group \( H_1(X;\mathbb{Z}) \) naturally gives the cell chain complex \( C_*(\tilde{X}) \) the structure of a \( \mathbb{Z}[H_1(X;\mathbb{Z})] \)-module. As such, \( C_*(\tilde{X}) \) is free of rank equal to the number of \( i \)-cells of \( X \). We can then form the twisted complex \( C_\varphi(\tilde{X}) = K \otimes_\varphi C_*(\tilde{X}) \) of \( K \)-modules. We choose a sequence \( \epsilon \) of cells of \( \tilde{X} \) such that over each cell of \( X \) there is exactly one element of \( \epsilon \), called a base sequence; this gives a basis of \( C_\varphi(\tilde{X}) \) over \( K \) and allows us to form the torsion \( \tau_\varphi(X,\epsilon) \in K(\mathbb{Z}) \) relative to this basis. Note that the torsion \( \tau_\varphi(X,\epsilon') \) arising from a different choice \( \epsilon' \) of base sequence stands in the relationship \( \tau_\varphi(X,\epsilon)=\pm\varphi(h)\tau_\varphi(X,\epsilon') \) for some \( h \in H_1(X;\mathbb{Z}) \) (here, as is standard practice, we write the group operation in \( H_1(X;\mathbb{Z}) \) multiplicatively when dealing with elements of \( \mathbb{Z}[H_1(X;\mathbb{Z})] \)). The set of all torsions arising from all such choices of \( \epsilon \) is “the” torsion of \( X \) associated to \( \varphi \) and is denoted \( \tau_\varphi(X) \).

We are now in a position to define the torsions we will need.

**Definition 2.1** (1) For \( X \) a smooth manifold as above with \( b_1(X) \geq 1 \), let \( \phi : X \to S^1 \) be a map representing an element \([\phi]\) of infinite order in \( H^1(X;\mathbb{Z}) \). Let \( C \) be the infinite cyclic group generated by the formal variable \( t \), and let \( \varphi_1 : \mathbb{Z}[H_1(X;\mathbb{Z})] \to \mathbb{Z}[C] \) be the map induced by the homomorphism \( H_1(X;\mathbb{Z}) \to C, a \mapsto t([\phi],a) \). Then the Reidemeister torsion \( \tau(X,\phi) \) of \( X \) associated to \( \phi \) is defined to be the torsion \( \tau_{\varphi_1}(X) \).

(2) Write \( H \) for the quotient of \( H_1(X;\mathbb{Z}) \) by its torsion subgroup, and let \( \varphi_2 : \mathbb{Z}[H_1(X;\mathbb{Z})] \to \mathbb{Z}[H] \) be the map induced by the projection \( H_1(X;\mathbb{Z}) \to H \). The Milnor torsion \( \tau(X) \) is defined to be \( \tau_{\varphi_2}(X) \).

**Remark 2.2** (1) Some authors use the term Reidemeister torsion to refer to the torsion \( \tau_\varphi(X) \) for arbitrary \( \varphi \); and other terms, eg, Reidemeister–Franz–DeRham torsion, are also in use.

(2) The torsions in Definition 2.1 are defined for manifolds \( X \) of arbitrary dimension, with or without boundary. We will be concerned only with the case that \( X \) is a closed manifold of dimension 3 with \( b_1(X) \geq 1 \). In the case \( b_1(X) > 1 \), work of Turaev [17] shows that \( \tau(X) \) and \( \tau(X,\phi) \), naturally subsets of \( \mathbb{Q}(H) \) and \( \mathbb{Q}(t) \), are actually subsets of \( \mathbb{Z}[H] \) and \( \mathbb{Z}[t,t^{-1}] \). Furthermore, if \( b_1(X) = 1 \) and \([\phi] \in H^1(X;\mathbb{Z}) \) is a generator, then \( \tau(X) = \tau(X,\phi) \) and \((t-1)^2\tau(X) \in \mathbb{Z}[t,t^{-1}] \). Rather than thinking of torsion as a set of elements in a field we normally identify it with a representative “defined up to multiplication
by $\pm t^k$ or similar, since by the description above any two representatives of the torsion differ by some element of the group ($C$ or $H$) under consideration.

### 2.2 $S^1$–Valued Morse Theory

We review the results of Hutchings and Lee that motivate our theorems. As in the introduction, let $X$ be a smooth closed oriented 3–manifold having $b_1(X) \geq 1$ and let $\phi : X \to S^1$ be a smooth Morse function. We assume (1) $\phi$ represents an indivisible element of infinite order in $H^1(X, \mathbb{Z})$; (2) $\phi$ has no critical points of index 0 or 3; and (3) the gradient flow of $\phi$ with respect to a Riemannian metric on $X$ is Morse–Smale. Such functions always exist given our assumptions on $X$.

Given such a Morse function $\phi$, fix a smooth level set $S$ for $\phi$. Upward gradient flow defines a return map $F : S \to S$ away from the descending manifolds of the critical points of $\phi$. The zeta function of $F$ is defined by the series

$$\zeta(F) = \exp \left( \sum_{k \geq 1} \text{Fix}(F^k) \frac{t^k}{k} \right)$$

where $\text{Fix}(F^k)$ denotes the number of fixed points (counted with sign in the usual way) of the $k$-th iterate of $F$. One should think of $\zeta(F)$ as keeping track of the number of closed orbits of $\phi$ as well as the “degree” of those orbits. For future reference we note that if $h : S \to S$ is a diffeomorphism of a surface $S$ then

$$\zeta(h) = \sum_k L(h^{(k)}) t^k$$

where $L(h^{(k)})$ is the Lefschetz number of the induced map on the $k$-th symmetric power of $S$ (see [16], [7]).

We now introduce a Morse complex that can be used to keep track of gradient flow lines between critical points of $\phi$. Write $L_\mathbb{Z}$ for the ring of Laurent series in the variable $t$, and let $M^i$ denote the free $L_\mathbb{Z}$–module generated by the index-$i$ critical points of $\phi$. The differential $d_M : M^i \to M^{i+1}$ is defined to be

$$d_M x_\mu = \sum_{\nu} a_{\mu\nu}(t) y_\nu$$

where $x_\mu$ is an index-$i$ critical point, $\{y_\nu\}$ is the set of index-$(i + 1)$ critical points, and $a_{\mu\nu}(t)$ is a series in $t$ whose coefficient of $t^n$ is defined to be the number of gradient flow lines of $\phi$ connecting $x_\mu$ with $y_\nu$ that cross $S$ n
times. Here we count the gradient flows with sign determined by orientations on the ascending and descending manifolds of the critical points; see [6] for more details.

**Theorem 2.3** (Hutchings–Lee) In this situation, the relation (1) holds up to multiplication by $\pm t^k$.

### 2.3 Results

The main result of this work is that the left hand side of (1) is equal to the left hand side of (2), without using the results of [10]. Hence the current work, together with that of Hutchings and Lee, gives an alternative proof of the theorem of Meng and Taubes in [10].

Our proof of this fact is based on ideas of Donaldson for computing the Seiberg–Witten invariants of 3–manifolds. We outline Donaldson’s construction here; see Section 4 below for more details. Given $\phi : X \to S^1$ a generic Morse function as above and $S$ the inverse image of a regular value, let $W = X \setminus \text{nbhd}(S)$ be the complement of a small neighborhood of $S$. Then $W$ is a cobordism between two copies of $S$ (since we assumed $\phi$ has no extrema—note we may also assume $S$ is connected). Note that two spin$^c$ structures on $X$ that differ by an element $a \in H^2(X; \mathbb{Z})$ with $a([S]) = 0$ restrict to the same spin$^c$ structure on $W$, in particular, spin$^c$ structures $\sigma$ on $W$ are determined by their degree $m = \langle c_1(\sigma), S \rangle$. Note that the degree of a spin$^c$ structure is always even.

Now, a solution of the Seiberg–Witten equations on $W$ restricts to a solution of the vortex equations on $S$ at each end of $W$ (more accurately, we should complete $W$ by adding infinite tubes $S \times (-\infty, 0], S \times [0, \infty)$ to each end, and consider the limit of a finite-energy solution on this completed space)—see [3], [13] for example. These equations have been extensively studied, and it is known that the moduli space of solutions to the vortex equations on $S$ can be identified with a symmetric power $\text{Sym}^n S$ of $S$ itself: see [1], [8]. Donaldson uses the restriction maps on the Seiberg–Witten moduli space of $W$ to obtain a self-map $\kappa_n$ of the cohomology of $\text{Sym}^n S$, where $n$ is defined by $n = g(S) - 1 - \frac{1}{2}|m|$ if $b_1(X) > 1$ and $n = g(S) - 1 + \frac{1}{2}m$ if $b_1(X) = 1$ (here $g(S)$ is the genus of the orientable surface $S$). The alternating trace $\text{Tr} \kappa_n$ is identified as the sum of Seiberg–Witten invariants of spin$^c$ structures on $X$ that restrict to the given spin$^c$ structure on $W$—that is, the coefficient of $t^n$ on the left hand side of (2). For a precise statement, see Theorem 4.1.

Our main result is the following.
Theorem 2.4  Let $X$ be a Riemannian 3–manifold with $b_1(X) \geq 1$, and fix an integer $n \geq 0$ as above. Then we have

$$\text{Tr} \kappa_n = [\tau(M^*) \zeta(F)]_n,$$

(6)

where $\tau(M^*)$ is represented by $t^N \det(d_M)$, and $N$ is the number of index 1 critical points of $\phi$. Here $\text{Tr}$ denotes the alternating trace and $[\cdot]_n$ denotes the coefficient of $t^n$ of the polynomial enclosed in brackets.

This fact immediately implies the conjecture of Hutchings and Lee. Furthermore, we will make the following observation:

Theorem 2.5  There is a smooth connected representative $S$ for the Poincaré dual of $[\phi] \in H^1(X; \mathbb{Z})$ such that $\text{Tr} \kappa_n$ is given by the intersection number of a pair of totally real embedded submanifolds in $\text{Sym}^{n+N}S \times \text{Sym}^{n+N}S$.

This may be the first step in defining a Lagrangian-type Floer homology theory parallel to that of Ozsváth and Szabó, one whose Euler characteristic is a priori a combination of Seiberg–Witten invariants. In the case that $X$ is a mapping torus, a program along these lines has been initiated by Salamon [16]. In this case the two totally real submanifolds in Theorem 2.5 reduce to the diagonal and the graph of a symplectomorphism of $\text{Sym}^nS$ determined by the monodromy of the mapping torus, both of which are in fact Lagrangian.

The remainder of the paper is organized as follows: Section 3 gives a brief overview of some elements of Seiberg–Witten theory and the dimensional reduction we will make use of, and Section 4 gives a few more details on this reduction and describes the TQFT we use to compute Seiberg–Witten invariants. Section 5 proves a theorem that gives a means of calculating as though a general cobordism coming from an $S^1$–valued Morse function of the kind we are considering possessed a naturally-defined monodromy map; Section 6 collects a few other technical results of a calculational nature, the proof of one of which is the content of Section 9. In Section 7 we prove Theorem 2.4 by a calculation that is fairly involved but is not essentially difficult, thanks to the tools provided by the TQFT. Section 8 proves Theorem 2.5.

3 Review of Seiberg–Witten theory

We begin with an outline of some aspects of Seiberg–Witten theory for a 3–manifolds. Recall that a spin$^c$ structure on a 3–manifold $X$ is a lift of the
oriented orthogonal frame bundle of \( X \) to a principal spin\(^c\)(3)–bundle \( \sigma \). There are two representations of spin\(^c\)(3) = Spin(3) \times U(1)/\pm 1 = SU(2) \times U(1)/\pm 1 \) that will interest us, namely the spin representation spin\(^c\)(3) \to SU(2) \) and also the projection spin\(^c\)(3) \to U(1) \) given by \([g, e^{i\theta}] \mapsto e^{2i\theta}\). For a spin\(^c\) structure \( \sigma \) the first of these gives rise to the associated spinor bundle \( \mathcal{W} \) which is a hermitian 2–plane bundle, and the second to the determinant line bundle \( \mathcal{L} = \wedge^2 \mathcal{W} \). We define \( c_1(\sigma) := c_1(\mathcal{L}) \). The Levi–Civita connection on \( X \) together with a choice of hermitian connection \( A \) on \( \mathcal{L}^{1/2} \) gives rise to a hermitian connection on \( \mathcal{W} \) that is compatible with the action of Clifford multiplication \( c : T^*_c X \to \text{End}_0 \mathcal{W} = \{\text{traceless endomorphisms of } \mathcal{W}\} \), and thence to a Dirac operator \( D_A : \Gamma(\mathcal{W}) \to \Gamma(\mathcal{W}) \).

The Seiberg–Witten equations are equations for a pair \((A, \psi) \in \mathcal{A}(\mathcal{L}) \times \Gamma(\mathcal{W})\) where \( \mathcal{A}(\mathcal{L}) \) denotes the space of hermitian connections on \( \mathcal{L}^{1/2} \), and read:

\[
\begin{align*}
D_A \psi &= 0 \\
c(\ast F_A + i \ast \mu) &= \psi \otimes \psi^* - \frac{1}{2} |\psi|^2
\end{align*}
\]

(7)

Here \( \mu \in \Omega^2(X) \) is a closed form used as a perturbation; if \( b_1(X) > 1 \) we may choose \( \mu \) as small as we like.

On a closed oriented 3–manifold the Seiberg–Witten moduli space is the set of \( L^{2,2} \) solutions to the above equations modulo the action of the gauge group \( \mathcal{G} = L^{2,3}(X; S^1) \), which acts on connections by conjugation and on spinors by complex multiplication. For generic choice of perturbation \( \mu \) the moduli space \( \mathcal{M}_\sigma \) is a compact zero–dimensional manifold that is smoothly cut out by its defining equations (if \( b_1(X) > 0 \)). There is a way to orient \( \mathcal{M}_\sigma \) using a so–called homology orientation of \( X \), and the Seiberg–Witten invariant of \( X \) in the spin\(^c\) structure \( \sigma \) is defined to be the signed count of points of \( \mathcal{M}_\sigma \). One can show that if \( b_1(X) > 1 \) then the resulting number is independent of all choices involved and depends only on \( X \) (with its orientation); while if \( b_1(X) = 1 \) there is a slight complication: in this case we need to make a choice of generator \( o \) for the free part of \( H^1(X; \mathbb{Z}) \) and require that \( \langle [\mu] \cup o, [X] \rangle > \pi(c_1(\sigma) \cup o, [X]) \).

Suppose now that rather than a closed manifold, \( X \) is isometric to a product \( \Sigma \times \mathbb{R} \) for some Riemann surface \( \Sigma \). If \( t \) is the coordinate in the \( \mathbb{R} \) direction, then Clifford multiplication by \( dt \) is an automorphism of square \(-1\) of \( W \) and therefore splits \( W \) into eigen-bundles \( E \) and \( F \) on which \( dt \) acts as multiplication by \(-i\) and \( i \), respectively. In fact \( F = K^{-1}E \) where \( K \) is the canonical bundle of \( \Sigma \), and \( 2E - K = L \), the determinant line of \( \sigma \). Writing a section \( \psi \) of \( W \) as \( (\alpha, \beta) \in \Gamma(E \oplus K^{-1}E) \), we can express the Dirac operator in this
decomposition as:

$$D_A\psi = \begin{pmatrix} -i\partial_t & \bar{\partial}_{B,J} \alpha \\ i\bar{\partial}_{B,J} & i\partial_t \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Here we have fixed a spin structure (with connection) $K^{1/2}$ on $\Sigma$ and noted that the choice of a connection $A$ on $L^{1/2} = E - K^{1/2}$ is equivalent to a choice of connection $B$ on $E$. The metric on $\Sigma \times \mathbb{R}$ induces a complex structure $J$ and area form $\omega_{\Sigma}$ on $\Sigma$. Then $\bar{\partial}_{B,J}$ is the associated $\partial$ operator on sections of $E$ with adjoint operator $\bar{\partial}_{B,J}^\ast$.

The 2-forms $\Omega^2_\mathbb{C}(\Sigma \times \mathbb{R})$ split as $\Omega^{1,1}(\Sigma) \oplus [(\Omega^{1,0}(\Sigma) \oplus \Omega^{0,1}(\Sigma)) \otimes \Omega^1_\mathbb{C}(\mathbb{R})]$, and we will write a form $\nu$ as $\Lambda\nu \cdot \omega_{\Sigma} + \nu^{1,0}dt + \nu^{0,1}dt$ in this splitting. Thus $\Lambda\nu$ is a complex function on $\Sigma \times \mathbb{R}$, while $\nu^{1,0}$ and $\nu^{0,1}$ are 1-forms on $\Sigma$. With these conventions, the Seiberg–Witten equations become

$$i\alpha = \bar{\partial}_{B,J}\beta$$

$$i\beta = -\bar{\partial}_{B,J}\alpha$$

$$2\Lambda F_B - \Lambda F_K + 2i\Lambda \mu = i(|\alpha|^2 - |\beta|^2)$$

$$2(F_B - F_K)^{1,0} + 2i\mu^{1,0} = \alpha \otimes \beta$$

One can show that for a finite-energy solution either $\alpha$ or $\beta$ must identically vanish; apparently this implies any such solution is constant, and the above system of equations descends to $\Sigma$ when written in temporal gauge (ie, so the connection has no $dt$ component). The above equations (with $\beta = 0$) therefore reduce to the vortex equations in $E$, which are for a pair $(B,\alpha) \in \mathcal{A}(E) \times \Gamma(E)$ and read

$$\bar{\partial}_{B,J}\alpha = 0$$

$$i \star F_B + \frac{1}{2}|\alpha|^2 = \tau$$

where $\tau$ is a function on $\Sigma$ satisfying $\int \tau > 2\pi \deg(E)$ and incorporates the curvature $F_K$ and perturbation above. These equations are well-understood, and it is known that the space of solutions to the vortex equations modulo $\text{Map}(\Sigma, S^1)$ is isomorphic to the space of solutions $(B,\alpha)$ of the single equation

$$\bar{\partial}_{B,J}\alpha = 0$$

modulo the action of $\text{Map}(\Sigma, \mathbb{C}^*)$. The latter is naturally identified with the space of divisors of degree $d = \deg(E)$ on $\Sigma$ via the zeros of $\alpha$, and forms a Kähler manifold isomorphic to the $d$-th symmetric power $\text{Sym}^d \Sigma$, which for brevity we will abbreviate as $\Sigma^{(d)}$ from now on. We write $\mathcal{M}_d(\Sigma, J)$ (or simply $\mathcal{M}(\Sigma)$) for the moduli space of vortices in a bundle $E$ of degree $d$ on $\Sigma$. 

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The situation for $\alpha \equiv 0$ is analogous to the above: in this case $\beta$ satisfies $\partial_{B,J}^* \beta = 0$ so that $*_{2}\beta$ is a holomorphic section of $K \otimes E^*$. Replacing $\beta$ by $*_{2}\beta$ shows that the Seiberg–Witten equations reduce to the vortex equations in the bundle $K \otimes E^*$, giving a moduli space isomorphic to $\Sigma^{(2g-2-d)}$.

4 A TQFT for Seiberg–Witten invariants

In this section we describe Donaldson’s “topological quantum field theory” for computing the Seiberg–Witten invariants. Suppose $W$ is a cobordism between two Riemann surfaces $S_-$ and $S_+$. We complete $W$ by adding tubes $S_\pm \times [0, \infty)$ to the boundaries and endow the completed manifold $\hat{W}$ with a Riemannian metric that is a product on the ends. By considering finite-energy solutions to the Seiberg–Witten equations on $\hat{W}$ in some spin$^c$ structure $\sigma$, we can produce a Fredholm problem and show that such solutions must approach solutions to the vortex equations on $S_\pm$. Following a solution to its limiting values, we obtain smooth maps between moduli spaces, $\rho_\pm : \mathcal{M}(W) \to \mathcal{M}(S_\pm)$. Thus we can form

$$\kappa_\sigma = (\rho_- \otimes \rho_+)*[\mathcal{M}(\hat{W})] \in H_*(\mathcal{M}(S_-)) \otimes H_*(\mathcal{M}(S_+)) \cong \text{hom}(H^*(\mathcal{M}(S_-)), H^*(\mathcal{M}(S_+))).$$

Here we use Poincaré duality and work with rational coefficients.

This is the basis for our “TQFT.” To a surface $S$ we associate the cohomology of the moduli space $\mathcal{M}(S)$, and to a cobordism $W$ between $S_-$ and $S_+$ we assign the homomorphism $\kappa_\sigma$:

$$S \mapsto V_S = H^*(\mathcal{M}(S)) \quad \text{and} \quad W \mapsto \kappa_\sigma : V_{S_-} \to V_{S_+}.$$

In the sequel we will be interested only in cobordisms $W$ that satisfy the topological assumption $H_1(W, \partial W) = \mathbb{Z}$. Under this assumption, gluing theory for Seiberg–Witten solutions provides a proof of the central property of TQFTs, namely that if $W_1$ and $W_2$ are composable cobordisms then $\kappa_{W_1 \cup W_2} = \kappa_{W_2} \circ \kappa_{W_1}$.

If $X$ is a closed oriented 3–manifold with $b_1(X) > 0$ then the above constructions can be used to calculate the Seiberg–Witten invariants of $X$, as seen in [2]. We now describe the procedure involved. Begin with a Morse function $\phi : X \to S^1$ as in the introduction, and cut $X$ along the level set $S$ to produce a cobordism $W$ between two copies of $S$, which come with an identification or
“gluing map” \( \partial_- W \to \partial_+ W \). Write \( g \) for the genus of \( S \). The cases \( b_1(X) > 1 \) and \( b_1(X) = 1 \) are slightly different and we consider them separately.

Suppose \( b_1(X) > 1 \), so the perturbation \( \mu \) in (7) can be taken to be small. Consider the constant solutions to the equations (8) on the ends of \( \tilde{W} \), or equivalently the possible values of \( \rho_{\pm} \). If \( \beta \equiv 0 \) then \( \alpha \) is a holomorphic section of \( E \) and so the existence of a nonvanishing solution requires \( \deg(E) \geq 0 \). Since \( \mu \) is small, integrating the third equation in (8) tells us that \( 2E - K \) is nonpositive. Hence existence of nonvanishing solutions requires \( 0 \leq \deg(E) \leq \frac{1}{2} \deg(K) = g - 1 \). If \( \alpha \equiv 0 \), then \( +_2 \beta \) is a holomorphic section of \( K - E \) so to avoid triviality we must have \( 0 \leq \deg(K) - \deg(E) \), i.e. \( \deg(E) \leq 2g - 2 \).

On the other hand, integrating the third Seiberg–Witten equation tells us that \( 2E - K \) is nonnegative, so that \( \deg(E) \geq g - 1 \). To summarize we have shown that constant solutions to the Seiberg–Witten equations on the ends of \( \tilde{W} \) in a spin\(^c\) structure \( \sigma \) are just the vortices on \( S \) (with the finite-energy hypothesis). If \( \det(\sigma) = L \) a necessary condition for the existence of such solutions is \( -2g + 2 \leq \deg(L) \leq 2g - 2 \) (recall \( L = 2E - K \) so in particular \( L \) is even). If this condition is satisfied then the moduli space on each end is isomorphic to \( \mathcal{M}_n(S) \cong S^{(n)} \) where \( n = g - 1 - \frac{1}{2} \deg(L) \). Note that by suitable choice of perturbation \( \mu \) we can eliminate the “reducible” solutions, i.e., those with \( \alpha \equiv 0 \equiv \beta \), which otherwise may occur at the extremes of our range of values for \( \deg(L) \).

Now assume \( b_1(X) = 1 \). Integrating the third equation in (8) shows

\[
\langle c_1(\sigma), S \rangle - \frac{1}{\pi} \langle [\mu], S \rangle = \frac{1}{2\pi} \int_S |\beta|^2 - |\alpha|^2.
\]

The left hand side of this is negative by our assumption on \( \mu \), and we know that either \( \alpha \equiv 0 \) or \( \beta \equiv 0 \). The first of these possibilities gives a contradiction; hence \( \beta \equiv 0 \) and the system (8) reduces to the vortex equations in \( E \) over \( S \). Existence of nontrivial solutions therefore requires \( \deg(E) \geq 0 \), i.e., \( \deg(L) \geq 2 - 2g(S) \). Thus the moduli space on each end of \( \tilde{W} \) is isomorphic to \( \mathcal{M}_n(S) \cong S^{(n)} \), where \( n = \deg(E) = g - 1 + \frac{1}{2} \deg(L) \) and \( \deg(L) \) is any even integer at least \( 2 - 2g(S) \).

**Theorem 4.1** (Donaldson) Let \( X, \sigma, \phi, S, \) and \( W \) be as above. Write \( \langle c_1(\sigma), [S] \rangle = m \) and define either \( n = g(S) - 1 - \frac{1}{2} |m| \) or \( n = g(S) - 1 + \frac{1}{2} m \) depending whether \( b_1(X) > 1 \) or \( b_1(X) = 1 \). Then if \( n \geq 0 \),

\[
\text{Tr} \kappa_\sigma = \sum_{\tilde{\sigma} \in \mathcal{S}_m} SW(\tilde{\sigma}) \tag{11}
\]
where \( S_m \) denotes the set of spin\(^c\) structures \( \tilde{\sigma} \) on \( X \) such that \( \langle c_1(\tilde{\sigma}), [S] \rangle = m \). If \( n < 0 \) then the right hand side of (11) vanishes. Here \( \text{Tr} \) denotes the graded trace.

Note that with \( n \) as in the theorem, \( \kappa_\sigma \) is a linear map 
\[
\kappa_\sigma : H^*(S^{(n)}) \to H^*(S^{(n)}),
\]
as the trace of \( \kappa_\sigma \) computes a sum of Seiberg–Witten invariants rather than just \( SW(\sigma) \), we use the notation \( \kappa_n \) rather than \( \kappa_\sigma \).

Since \( \kappa_n \) obeys the composition law, in order to determine the map corresponding to \( W \) we need only determine the map generated by elementary cobordisms, i.e, those consisting of a single 1–2–handle addition (we need not consider 0– or 3–handles by our assumption on \( \phi \)). In [2], Donaldson uses an elegant algebraic argument to determine these elementary homomorphisms. To state the result, recall that the cohomology of the \( n \)-th symmetric power \( S^{(n)} \) of a Riemann surface \( S \) is given over \( \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \) or \( \mathbb{C} \) by
\[
H^*(S^{(n)}) = \bigoplus_{i=0}^n \Lambda^i H^1(S) \otimes \text{Sym}^{n-i}(H^0(S) \oplus H^2(S)).
\tag{12}
\]

Suppose that \( W \) is an elementary cobordism connecting two surfaces \( \Sigma_g \) and \( \Sigma_{g+1} \). Thus there is a unique critical point (of index 1) of the height function \( h : W \to \mathbb{R} \), and the ascending manifold of this critical point intersects \( \Sigma_{g+1} \) in an essential simple closed curve that we will denote by \( c \).

Now, \( c \) obviously bounds a disk \( D \subset W \); the Poincaré–Lefschetz dual of \([D] \in H_2(W, \partial W)\) is a 1–cocycle that we will denote \( \eta_0 \in H^1(W) \). It is easy to check that \( \eta_0 \) is in the kernel of the restriction \( r_1 : H^1(W) \to H^1(\Sigma_g) \), so we may complete \( \eta_0 \) to a basis \( \eta_0, \eta_1, \ldots, \eta_{2g} \) of \( H^1(W) \) with the property that \( \xi_1 := r_1(\eta_1), \ldots, \xi_{2g} := r_1(\eta_{2g}) \) form a basis for \( H^1(\Sigma_g) \). Since the restriction \( r_2 : H^1(W) \to H^1(\Sigma_{g+1}) \) is injective, we know \( \xi_0 := r_2(\eta_0), \ldots, \xi_{2g} := r_2(\eta_{2g}) \) are linearly independent; note that \( r_2(\eta_0) \) is just \( c^* \), the Poincaré dual of \( c \).

The choice of basis \( \eta_j \) with its restrictions \( \xi_j \), \( \tilde{\xi}_j \) gives rise to an inclusion \( i : H^1(\Sigma_g) \to H^1(\Sigma_{g+1}) \) in the obvious way, namely \( i(\xi_j) = \tilde{\xi}_j \). One may check that this map is independent of the choice of basis \( \{\eta_j\} \) for \( H^1(W) \) having \( \eta_0 \) as above. From the decomposition (12), we can extend \( i \) to an inclusion \( i : H^*(S^{(n)}) \to H^*(S^{(n)}) \). Having produced this inclusion, we now proceed to suppress it from the notation, in particular in the following theorem.

**Theorem 4.2** (Donaldson) In this situation, and with \( \sigma \) and \( n \) as previously, the map \( \kappa_n \) corresponding to the elementary cobordism \( W \) is given by 
\[
\kappa_n(\alpha) = c^* \wedge \alpha.
\]
If $\bar{W}$ is the “opposite” cobordism between $\Sigma_{g+1}$ and $\Sigma_g$, the corresponding $\kappa_n$ is given by the contraction

$$\kappa_n(\beta) = \iota_c \beta,$$

where contraction is defined using the intersection pairing on $H^1(\Sigma_{g+1})$.

This result makes the calculation of Seiberg–Witten invariants completely explicit, as we see in the next few sections.

5 Standardization of $X$

We now return to the situation of the introduction: namely, we consider a closed 3–manifold $X$ having $b_1(X) \geq 1$, with its circle-valued Morse function $\phi : X \to S^1$ having no critical points of index 0 or 3, and $N$ critical points of each index 1 and 2. We want to show how to identify $X$ with a “standard” manifold $M(g, N, h)$ that depends only on $N$ and a diffeomorphism $h$ of a Riemann surface of genus $g + N$. This standard manifold will be obtained from two “compression bodies,” ie, cobordisms between surfaces incorporating handle additions of all the same index. Two copies of the same compression body can be glued together along their smaller-genus boundary by the identity map, then by a “monodromy” diffeomorphism of the other boundary component to produce a more interesting 3–manifold. Such a manifold lends itself well to analysis using the TQFT from the previous section, as the interaction between the curves $c$ corresponding to each handle is completely controlled by the monodromy. We now will show that every closed oriented 3–manifold $X$ having $b_1(X) > 0$ can be realized as such a glued-up union of compression bodies.

To begin with, we fix a closed oriented genus 0 surface $\Sigma_0$ (that is, a standard 2–sphere) with an orientation-preserving embedding $\psi_{0,0} : S^0 \times D^2 \to \Sigma_0$. Here we write $D^n = \{x \in \mathbb{R}^n | |x| < 1\}$ for the unit disk in $\mathbb{R}^n$. There is a standard way to perform surgery on the image of $\psi_{0,0}$ (see [12]) to obtain a new surface $\Sigma_1$ of genus 1 and an orientation-preserving embedding $\psi_{1,1} : S^1 \times D^1 \to \Sigma_1$. In fact we can get a cobordism $(W_{0,1}, \Sigma_0, \Sigma_1)$ with a “gradient-like vector field” $\xi$ for a Morse function $f : W_{0,1} \to [0, 1]$. Here $f^{-1}(0) = \Sigma_0$, $f^{-1}(1) = \Sigma_1$, and $f$ has a single critical point $p$ of index 1 with $f(p) = \frac{1}{2}$. We have that $\xi[f] > 0$ away from $p$ and that in local coordinates near $p$, $f = \frac{1}{2} - x_1^2 + x_2^2 + x_3^2$ and $\xi = -x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}$. The downward flow of $\xi$ from $p$ intersects $\Sigma_0$ in $\psi_{0,0}(S^0 \times 0)$ and the upward flow intersects $\Sigma_1$ in $\psi_{1,1}(S^1 \times 0)$. 

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Choose an embedding $\psi_{0,1}: S^0 \times D^2 \to \Sigma_1$ whose image is disjoint from $\psi_{1,1}(S^1 \times D^3)$. Then we can repeat the process above to get another cobordism $(W_{1,2}, \Sigma_1, \Sigma_2)$ with Morse function $f: W_{1,2} \to [1,2]$ having a single critical point of index 1 at level $\frac{1}{2}$, and gradient-like vector field $\xi$ as before.

Continuing in this way, we get a sequence of cobordisms $(W_{g,g+1}, \Sigma_g, \Sigma_{g+1})$ between surfaces of genus difference 1, with Morse functions $f: W_{g,g+1} \to [g,g+1]$ and gradient-like vector fields $\xi$. To each $\Sigma_g$, $g \geq 1$, is also associated a pair of embeddings $\psi_{1,g}: S^i \times D^{2-i} \to \Sigma_g$, $i = 0,1$. These embeddings have disjoint images, and are orientation-preserving with respect to the given, fixed orientations on the $\Sigma_g$. Note that the orientation on $\Sigma_g$ induced by $W_{g,g+1}$ is opposite to the given one, so the map $\psi_{0,g}: S^0 \times D^2 \to -\Sigma_g = \partial_- W_{g,g+1}$ is orientation-reversing.

Since the surfaces $\Sigma_g$ are all standard, we have a natural way to compose $W_{g-1,g}$ and $W_{g,g+1}$ to produce a cobordism $W_{g-1,g+1} = W_{g-1,g} + W_{g,g+1}$ with a Morse function to $[g-1,g+1]$ having two index-1 critical points. Furthermore, by replacing $f$ by $-f$ we can obtain cobordisms $(W_{g+1,g}, \Sigma_{g+1}, \Sigma_g)$ with Morse functions having a single critical point of index 2, and these cobordisms may be naturally composed with each other or with the original index-1 cobordisms obtained before (after appropriately adjusting the values of the corresponding Morse functions), whenever such composition makes sense. We may think of $W_{g+1,g}$ as being simply $W_{g,g+1}$ with the opposite orientation.

In particular, we can fix integers $g,N \geq 0$ and proceed as follows. Beginning with $\Sigma_{g+N}$, compose the cobordisms $W_{g+1,g+N-1}, \ldots, W_{g+1,g}$ to form a “standard” compression body, and glue this with the composition $W_{g+1,g} + \cdots + W_{g+N-1,g+N}$ using the identity map on $\Sigma_g$. The result is a cobordism $(W, \Sigma_{g+N}, \Sigma_{g+N})$ and a Morse function $f: W \to \mathbb{R}$ that we may rescale to have range $[-N,N]$, having $N$ critical points each of index 1 and 2. By our construction, the first half of this cobordism, $W_{g+1,g}$, is identical with the second half, $W_{g,N+1}$: they differ only in their choice of Morse function and associated gradient-like vector field.

Now, by our construction the circles $\psi_{1,g+k}: S^1 \times 0 \to f^{-1}(-k) = \Sigma_{g+k} \subset W$, $1 \leq k \leq N$, all survive to $\Sigma_{g+N}$ under downward flow of $\xi$. This is because the images of $\psi_{1,g}$ and $\psi_{0,g}$ are disjoint for all $g$. Thus on the “lower” copy of $\Sigma_{g+N}$ we have $N$ disjoint primitive circles $c_1, \ldots, c_N$ that, under upward flow of $\xi$, each converge to an index 2 critical point. Similarly, (since $W_{g+1,N} = W_{g+1,N}$) the circles $\psi_{1,N+k}: S^1 \times 0 \to f^{-1}(k) = \Sigma_{g+k} \subset W$, $1 \leq k \leq N$, survive to $\Sigma_{g+N}$ under upward flow of $\xi$, and intersect the “upper” copy of $\Sigma_{g+N}$ in the circles $c_1, \ldots, c_N$.
Now suppose \( h: \partial^+W = \Sigma_{g+N} \to \Sigma_{g+N} = -\partial^-W \) is a diffeomorphism; then we can use \( h \) to identify the boundaries \( f^{-1}(-N), f^{-1}(N) \) of \( W \), and produce a manifold that we will denote by \( M(g,N,h) \). Note that this manifold is entirely determined by the isotopy class of the map \( h \), and that if \( h \) preserves orientation then \( M(g,N,h) \) is an orientable manifold having \( b_1 \geq 1 \).

**Theorem 5.1** Let \( X \) be a closed oriented 3-manifold and \( \phi: X \to S^1 \) a circle-valued Morse function with no critical points of index 0 or 3, and with \( N \) critical points each of index 1 and 2. Assume that \( [\phi] \in H^1(X;\mathbb{Z}) \) is of infinite order and indivisible. Arrange that \( 0 < \arg\phi(p) < \pi \) for an index 1 critical point and \( \pi < \arg\phi(q) < 2\pi \) for an index 2 critical point, and let \( S_g = \phi^{-1}(1) \), where \( S_g \) has genus \( g \). Then \( X \) is diffeomorphic to \( M(g,N,h) \) for some \( h: \Sigma_{g+N} \to \Sigma_{g+N} \) as above.

Note that \( S_g \) has by construction the smallest genus among smooth slices for \( f \).

**Proof** By assumption \(-1\) is a regular value of \( \phi \), so \( S_{g+N} = \phi^{-1}(-1) \) is a smooth orientable submanifold of \( X \); it is easy to see that \( S_{g+N} \) is a closed surface of genus \( g + N \). Cut \( X \) along \( S_{g+N} \); then we obtain a cobordism \((W_\phi,S_-,S_+)\) between two copies \( S_\pm \) of \( S_{g+N} \), and a Morse function \( f: W_\phi \to [-\pi,\pi] \) induced by \( \arg\phi \). The critical points of \( f \) are exactly those of \( \phi \) (with the same index), and by our arrangement of critical points we have that \( f(q) < 0 \) for any index 2 critical point \( q \) and \( f(p) > 0 \) for any index 1 critical point \( p \).

It is well-known that we can arrange for the critical points of \( f \) to have distinct values, and that in this case \( W_\phi \) is diffeomorphic to a composition of elementary cobordisms, each containing a single critical point of \( f \). For convenience we rescale \( f \) so that its image is the interval \([-N,N]\) and the critical values of \( f \) are the half-integers between \(-N\) and \( N \). Orient each smooth level set \( f^{-1}(x) \) by declaring that a basis for the tangent space of \( f^{-1}(x) \) is positively oriented if a gradient-like vector field for \( f \) followed by that basis is a positive basis for the tangent space of \( W_\phi \).

We will show that \( W_\phi \) can be standardized by working “from the middle out.” Choose a gradient-like vector field \( \xi_f \) for \( f \), and consider \( S_g = f^{-1}(0) \)—the “middle level” of \( W_\phi \), corresponding to \( \phi^{-1}(1) \). There is exactly one critical point of \( f \) in the region \( f^{-1}([0,1]) \), of index 1, and as above \( \xi_f \) determines a “characteristic embedding” \( \theta_{0,g}: S^0 \times D^2 \to S_g \). Choose a diffeomorphism \( \Theta_0: S_g \to \Sigma_g \) such that \( \Theta_0 \circ \theta_{0,g} = \psi_{0,g} \); then it follows from [12], Theorem 3.13, that \( f^{-1}([0,1]) \) is diffeomorphic to \( W_{g,g+1} \) by some diffeomorphism \( \Theta \) sending \( \xi_f \) to \( \xi \). (Recall that \( \xi \) is the gradient-like vector field fixed on \( W_{g,g+1} \).)
Let $\Theta_1: S_{g+1} \to \Sigma_{g+1}$ be the restriction of $\Theta$ to $S_{g+1} = f^{-1}(1)$, and let $\mu_{0,g+1} = \Theta_1^{-1} \circ \psi_{0,g+1}: S^0 \times D^2 \to S_{g+1}$. Now $\xi_f$ induces an embedding $\theta_{0,g+1}: S^0 \times D^2 \to S_{g+1}$, by considering downward flow from the critical point in $f^{-1}([1,2])$. Since any two orientation-preserving diffeomorphisms $D^2 \to D^2$ are isotopic and $S_{g+1}$ is connected, we have that $\mu_{0,g+1}$ and $\theta_{0,g+1}$ are isotopic. It is now a simple matter to modify $\xi_f$ in the region $f^{-1}([1,1+\epsilon])$ using the isotopy, and arrange that $\theta_{0,g+1} = \mu_{0,g+1}$. Equivalently, $\Theta \circ \theta_{0,g+1} = \psi_{0,g+1}$, so the theorem quoted above shows that $f^{-1}([1,2])$ is diffeomorphic to $W_{g+1,g+2}$. 

In fact, since the diffeomorphism sends $\xi_f$ to $\xi$, we get that $\Theta$ extends smoothly to a diffeomorphism $f^{-1}([0,2]) \to W_{g,g+2}$.

Continuing in this way, we see that after successive modifications of $\xi_f$ in small neighborhoods of the levels $f^{-1}(k)$, $k = 1, \ldots, N-1$, we obtain a diffeomorphism $\Theta: f^{-1}([0,N]) \to W_{g,g+N}$ with $\Theta \circ \xi_f = \xi$.

The procedure is entirely analogous when we turn to the “lower half” of $W_\phi$, but the picture is upside-down. We have the diffeomorphism $\Theta_0: S_g \to \Sigma_g$, but before we can extend it to a diffeomorphism $\Theta: f^{-1}([-1,0]) \to W_{g+1,g}$ we must again make sure the characteristic embeddings match. That is, consider the map $\theta_{0,g}^1: S^0 \times D^2 \to S_g$ induced by upward flow from the critical point, and compare it to $\Psi_0^1 \circ \psi_{0,g}$. As before we can isotope $\xi_f$ in (an open subset whose closure is contained in) the region $f^{-1}([-\epsilon,0])$ so that these embeddings agree, and we then get the desired extension of $\Theta$ to $f^{-1}([-1,N])$. Then the procedure is just as before: alter $\xi_f$ at each step to make the characteristic embeddings agree, and extend $\Theta$ one critical point at a time.

Thus $\Theta: W_\phi \cong W = W_{g+N,g+N-1} + \cdots + W_{g+1,g} + W_{g,g+1} + \cdots + W_{g,N-1,g+N}$. Since $W_\phi$ was obtained by cutting $X$, it comes with an identification $\iota: \Sigma_{+} \to \Sigma_{-}$. Hence $X \cong M(g,N,h)$ where $h = \Theta \circ \iota \circ \Theta^{-1}: \Sigma_{g+N} \to \Sigma_{g+N}$.

**Remark 5.2** The identification $X \cong M(g,N,h)$ is not canonical, as it depends on the initial choice of diffeomorphism $\phi^{-1}(1) \cong \Sigma_g$, the final gradient-like vector field on $W_\phi$ used to produce $\Theta$, as well as the function $\phi$. As with a Heegaard decomposition, however, it is the existence of such a structure that is important.

### 6 Preliminary calculations

This section collects a few lemmata that we will use in the proof of Theorem 2.4. Our main object here is to make the quantity $|\zeta(F)\det(d)|_n$ a bit more explicit.
We work in the standardized setup of the previous section, identifying \( X \) with \( M(g, N, h) \). The motivation for doing so is mainly that our invariants are purely algebraic—i.e., homological—and the standardized situation is very easy to deal with on this level.

Choose a metric \( k \) on \( X = M(g, N, h) \); then gradient flow with respect to \( k \) on \((W, \Sigma_{g+N}, \Sigma_{g+N})\) determines curves \( \{c_i\}_{i=1}^N \) and \( \{d_j\}_{j=1}^N \) on \( \Sigma_{g+N} \), namely \( c_i \) is the intersection of the descending manifold of the \( i \)th index-2 critical point with the lower copy of \( \Sigma_{g+N} \) and \( d_j \) is the intersection of the ascending manifold of the \( j \)th index-1 critical point with the upper copy of \( \Sigma_{g+N} \).

**Definition 6.1** The pair \((k, \phi)\) consisting of a metric \( k \) on \( X \) together with the Morse function \( \phi : X \to S^1 \) is said to be symmetric if the following conditions are satisfied. Arrange the critical points of \( \phi \) as in Theorem 5.1, so that all critical points have distinct values. Write \( W_\phi \) for the cobordism \( X \setminus \phi^{-1}(-1) \), and \( f : W_\phi \to [-N,N] \) for the (rescaled) Morse function induced by \( \phi \) as in the proof of Theorem 5.1. Write \( I \) for the (orientation-reversing) involution obtained by swapping the factors in the expression \( W_\phi \cong W_{g+N,g} \cup W_{g,g+N} \).

We require:

1. \( I^*f = -f \).
2. For every \( x \in W_{g+N,g} \) we have \( (\nabla f)_I(x) = -I_*(\nabla f)_x \).

Symmetric pairs \((k, \phi)\) always exist: choose any metric on \( X \), and then in the construction used in the proof of Theorem 5.1, take our gradient-like vector field \( \xi_f \) to be a multiple of the gradient of \( f \) with respect to that metric. It is a straightforward exercise to see that the isotopies of \( \xi_f \) needed in that proof may be obtained by modifications of the metric.

We use the term “symmetric” here because the gradient flows of the Morse function \( f \) on the portions \( W_{g+N,g} \) and \( W_{g,g+N} \) are mirror images of each other. We will also say that the flow of \( \nabla f \) or of \( \nabla \phi \) is symmetric in this case.

Suppose \( M(g, N, h) \) is endowed with a symmetric pair, and consider the calculation of \( \zeta(F)_r(M^*) \) in this case. Recall that \( F \) is the return map of the flow of \( \nabla \phi \) from \( \Sigma_g \) to itself (though \( F \) is only partially defined due to the existence of critical points). Because of the symmetry of the flow, it is easy to see that:

1. The fixed points of iterates \( F^k \) are in 1–1 correspondence with fixed points of iterates \( h^k \) of the gluing map in the construction of \( W \), and the Lefschetz signs of the fixed points agree. Indeed, if \( h \) is sufficiently generic, we can assume that the set of fixed points of \( h^k \) for \( 1 \leq k \leq n \) (an arbitrary, but fixed, \( n \)) occur away from the \( d_j \) (which agree with the \( c_i \) under the identification \( I \) by symmetry).
(II) The \((i, j)\)th entry of the matrix of \(d_M : M^1 \to M^2\) in the Morse complex is given by the series

\[
\sum_{k \geq 1} \langle h^k c_i, c_j \rangle t^{k-1},
\]

where \(\langle \cdot, \cdot \rangle\) denotes the cup product pairing on \(H^1(\Sigma_{g+N}, \mathbb{Z})\) and we have identified the curves \(c_i\) with the Poincaré duals of the homology classes they represent.

We should remark that a symmetric pair is not \textit{a priori} suitable for calculating the invariant \((\zeta(F)\tau(M^*))\) of Hutchings and Lee, since it is not generic. Indeed, for a symmetric flow each index-2 critical point has a pair of upward gradient flow lines into an index-1 critical point. However, this is the only reason the flow is not generic: our plan now is to perturb a symmetric metric to one which does not induce the behavior of the flow just mentioned; then suitable genericity of \(h\) guarantees that the flow is Morse–Smale.

\textbf{Lemma 6.2} Assume that there are no “short” gradient flow lines between critical points, that is, every flow line between critical points intersects \(\Sigma_g\) at least once. Given a symmetric pair \((g_0, \phi)\) on \(M(g, N, h)\) and suitable genericity hypotheses on \(h\), there exists a \(C^0\)-small perturbation of \(g_0\) to a metric \(\tilde{g}\) such that for given \(n \geq 0\)

1. The gradient flow of \(\phi\) with respect to \(\tilde{g}\) is Morse–Smale; in particular the hypotheses of Theorem 2.3 are satisfied.
2. The quantity \([\zeta(F)\tau(M^*)]_m, m \leq n\) does not change under this perturbation.

We defer the proof of this result to Section 9.

\textbf{Remark 6.3} We can always arrange that there are no short gradient flow lines, at the expense of increasing \(g = \text{genus}(\Sigma_g)\). To see this, begin with \(X\) and \(\phi : X \to S^1\) as before, with \(\Sigma_g = \phi^{-1}(1)\) and the critical points arranged according to index. Every gradient flow line then intersects \(\Sigma_{g+N}\). Now rearrange the critical points by an isotopy of \(\phi\) that is constant near \(\Sigma_{g+N}\) so that the index-1 points occur in the region \(\phi^{-1}([e^{i\theta}] | \pi < \theta < 2\pi)\) and the index-2 points in the complementary region. This involves moving all \(2N\) of the critical points past \(\Sigma_g\), and therefore increasing the genus of the slice \(\phi^{-1}(1)\) to \(g + 2N\); we still have that every gradient flow line between critical points intersects \(\Sigma_{g+N}\). Cutting \(X\) along this new \(\phi^{-1}(1)\) gives a cobordism \(W\) between two copies of \(\Sigma_{g+2N}\) and thus standardizes \(X\) in the way we need while ensuring that there are no short flows.
Corollary 6.4  The coefficients of the torsion $\tau(X, \phi)$ may be calculated homologically, as the coefficients of the quantity $\zeta(h)\tau(M_0^*)$ where $M_0^*$ is the Morse complex coming from a symmetric flow.

That is, we can use properties I and II of symmetric pairs to calculate each coefficient of the right-hand side of (1).

Lemma 6.5  If the flow of $\nabla\phi$ is symmetric, the torsion $\tau(M^*)$ is represented by a polynomial whose $k$th coefficient is given by

$$[\tau(M^*)]_k = \sum_{\sigma \in \mathcal{S}_N \atop s_1 + \cdots + s_N = k} (-1)^{\text{sgn}(\sigma)} \langle h^{s_1} c_1, c_{\sigma(1)} \rangle \cdots \langle h^{s_N} c_N, c_{\sigma(N)} \rangle.$$  

Proof  Since there are only two nonzero terms in the Morse complex, the torsion is represented by the determinant of the differential $d_M : M^1 \to M^2$. Our task is to calculate a single coefficient of the determinant of this matrix of polynomials. It will be convenient to multiply the matrix of $d_M$ by $t$; this multiplies $\det(d_M)$ by $t^N$, but $t^N \det(d_M)$ is still a representative for $\tau(M^*)$. Multiplying formula (13) by $t$ shows

$$t^N \det(d_M) = \sum_{\sigma \in \mathcal{S}_N} (-1)^{\text{sgn}(\sigma)} \prod_i \left( \sum_k \langle h^{k_s} c_i, c_{\sigma(i)} \rangle t^k \right) \prod_i \left( \sum_k (-1)^{\text{sgn}(\sigma)} \langle h^{s_i} c_i, c_{\sigma(i)} \rangle \right) t^k$$

and the result follows.

7  Proof of Theorem 2.4

We are now in a position to explicitly calculate $\text{Tr} \kappa_n$ using Theorem 4.2 and as a result prove Theorem 2.4, assuming throughout that $X$ is identified with $M(g, N, h)$ and the flow of $\nabla\phi$ is symmetric. Indeed, fix the nonnegative integer $n$ as in Section 4 and consider the cobordism $W_{\phi}$ as above, identified with a composition of standard elementary cobordisms. Using Theorem 4.2 we see that the first half of the cobordism, $W_{g+N,g} = f_-([0, N])$, induces the map:

$$A_1 : H^*(\Sigma_{g+N}^{(n+N)}) \to H^*(\Sigma_g^{(n)}) \alpha \mapsto c_N^{\epsilon_1} \cdots c_1^{\epsilon_1} \alpha$$
The second half, \( f^{-1}([N,2N]) \), induces:

\[
A_2 : H^*(\Sigma^{(n)}_g) \to H^*(\Sigma^{(n+N)}_{g+N}) \\
\beta \mapsto c_1^* \wedge \cdots \wedge c_N^* \wedge \beta
\]

To obtain the map \( \kappa_n \) we compose the above with the gluing map \( h^* \) acting on the symmetric power \( \Sigma^{(n+N)}_{g+N} \). The alternating trace \( \text{Tr} \kappa_n \) is then given by \( \text{Tr}(h^* \circ A_2 \circ A_1) \).

Following MacDonald [9], we can take a monomial basis for \( H^*(\Sigma^{(n)}_g) \). Explicitly, if \( \{x_i\}_{i=1}^{2g} \) is a symplectic basis for \( H^1(\Sigma) \) having \( x_i \cup x_{i+j} = \delta_{ij} \) for \( 1 \leq i, j \leq g \), and \( x_i \cup x_j = 0 \) for other values of \( i \) and \( j \), \( 1 \leq i < j \leq 2g \), and \( y \) denotes the generator of \( H^2(\Sigma_g) \) coming from the orientation class, the expression (12) shows that the set

\[
B^{(n)}_g = \{\alpha\} = \{x_I y^g = x_{i_1} \wedge \cdots \wedge x_{i_k} \cdot y^g | I = \{i_1 < \cdots < i_k\} \subset \{1, \ldots, 2g\}\},
\]

where \( q = 1, \ldots, n \) and \( k = 0, \ldots, n - q \), forms a basis for \( H^*(\Sigma^{(n)}_g) \). We take \( H^*(\Sigma^{(n+k)}_{g+k}) \) to have similar bases \( B^{(n+k)}_{g+k} \), using the images of the \( x_i \) under the inclusion \( i : H^1(\Sigma_{g+k-1}) \to H^1(\Sigma_{g+k}) \) constructed in section 4, the (Poincaré duals of the) curves \( c_1, \ldots, c_k \), and (the Poincaré duals of) some chosen dual curves \( d_i \) to the \( c_i \) as a basis for \( H^1(\Sigma_{g+k}) \). Our convention is that \( c_i \cup d_j = \delta_{ij} \), where we now identify \( c_i, d_j \) with their Poincaré duals.

The dual basis for \( B^{(n+k)}_{g+k} \) under the cup product pairing will be denoted \( B^{\circ}_{n+k} = \{\alpha^\circ\} \). Thus \( \alpha^\circ \cup \beta = \delta_{\alpha \beta} \) for basis elements \( \alpha \) and \( \beta \). By abuse of notation, we will write \( B^{(n)}_g \subset B^{(m)}_h \) for \( g \leq h \) and \( n \leq m \); this makes use of the inclusions on \( H^1(\Sigma_g) \) induced by our standard cobordisms.

With these conventions, we can write:

\[
\text{Tr} \kappa_n = \sum_{\alpha \in B^{(n+N)}_{g+N}} (-1)^{\deg(\alpha)} \alpha^\circ \cup h^* \circ A_2 \circ A_1(\alpha)
\]

\[
= \sum_{\alpha \in B^{(n+N)}_{g+N}} (-1)^{\deg(\alpha)} \alpha^\circ \cup h^*(c_1 \wedge \cdots \wedge c_N \iota_{c_N} \cdots \iota_{c_1} \alpha)
\]

For a term in this sum to be nonzero, \( \alpha \) must be of a particular form. Namely, we must be able to write \( \alpha = d_1 \wedge \cdots \wedge d_N \wedge \beta \) for some \( \beta \in B^{(N)}_g \). The sum then can be written:

\[
= \sum_{\beta \in B^{(N)}_g} (-1)^{\deg(\beta)+N} (d_1 \wedge \cdots \wedge d_N \wedge \beta)^\circ \cup h^*(c_1 \wedge \cdots \wedge c_N \wedge \beta) \quad (14)
\]
In words, this expression asks us to find the coefficient of $d_1 \land \cdots \land d_N \land \beta$ in the basis expression of $h^*(c_1 \land \cdots \land c_N \land \beta)$, and add up the results with particular signs. Our task is to express this coefficient in terms of intersection data among the $c_i$ and the Lefschetz numbers of $h$ acting on the various symmetric powers of $\Sigma_g$.

Consider the term of (14) corresponding to $\beta = x_I y^g$ for $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, 2g\}$ and $x_I = x_{i_1} \land \cdots \land x_{i_k}$. The coefficient of $d_1 \land \cdots \land d_N \land x_I y^g$ in the basis expression of $h^*(c_1 \land \cdots \land c_N \land x_I y^g)$ is computed by pairing each of $\{c_1, \ldots, c_N, x_{i_1}, \ldots, x_{i_k}\}$ with each of $\{d_1, \ldots, d_N, x_{i_1}, \ldots, x_{i_k}\}$ in every possible way, and summing the results with signs corresponding to the permutation involved. To make the notation a bit more compact, for given $I$ let $\tilde{I} = \{1, \ldots, N, i_1, \ldots, i_k\}$ and write the elements of $\tilde{I}$ as $\{\tilde{i}_m\}_{m=1}^{N+k}$. Likewise, set $\tilde{I}' = \{N+1, \ldots, 2N, i_1, \ldots, i_k\} = \{\tilde{i}'_1, \ldots, \tilde{i}'_{N+k}\}$.

Write $\{\xi_i\}_{i=1}^{2N+2g}$ for our basis of $H^1(\Sigma_{g+N})$:

\[
\begin{align*}
\xi_1 &= c_1, & \xi_N &= c_N, & \xi_{N+1} &= d_1, & \xi_{2N} &= d_N \\
\xi_{2N+1} &= x_1, & \xi_{2N+2g} &= x_{2g}
\end{align*}
\]

and let $\{\xi'_i\}$ be the dual basis: $\langle \xi_i, \xi'_j \rangle = \delta_{ij}$. Define $\zeta_i = h^*(\xi_i)$.

Then since $\deg \beta = |I| = k$ modulo 2, the term of (14) corresponding to $\beta = x_I y^g$ is

\[
(-1)^{k+1} \sum_{\sigma \in S_{k+N}} (-1)^{\text{sgn}(\sigma)} \langle \zeta_{i_1}, \xi'_{\sigma(1)} \rangle \cdots \langle \zeta_{i_k+N}, \xi'_{\sigma(k+N)} \rangle, \quad (15)
\]

and (14) becomes

\[
\text{Tr} \kappa_n = \sum_{k=0}^{\min(n, 2g+2N)} (2n-k+1) \sum_{\substack{I \subset \{2N+1, \ldots, 2N+2g\} \mid |I| = k}} \text{[formula (15)]}, \quad (16)
\]

Here we are using the fact that for each $k = 0, \ldots, \min(n, 2g+2N)$ the space $\Lambda^k H^1(\Sigma_{g+N})$ appears in $H^*(\Sigma^{(n)})$ precisely $2(2n-k+1)$ times, each in cohomology groups of all the same parity.

Note that from (14) we can see that the result is unchanged if we allow not just sets $I \subset \{2N+1, \ldots, 2N+2g\}$ in our sum as above, but extend the sum to include sets $I = \{i_1, \ldots, i_k\}$, where $i_1 \leq \cdots \leq i_k$ and each $i_j \in \{1, \ldots, 2N+2g\}$. That is, we can allow $I$ to include indices referring to the $c_i$ or $d_i$, and allow repeats: terms corresponding to such $I$ contribute 0 to the sum. Likewise, we may assume that the sum in (16) is over $k = 0, \ldots, n$ since values of $k$ larger than $2g+2N$ necessarily involve repetitions in $I$.  

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Consider the permutations \( \sigma \in \mathfrak{S}_{k+N} \) used in the above. The fact that the first \( N \) elements of \( \vec{I} \) and \( \vec{I}' \) are distinguished (corresponding to the \( c_{ij} \) and \( d_{ij} \), respectively) gives such permutations an additional structure. Indeed, writing \( A = \{1, \ldots, N\} \subset \{1, \ldots, N+k\} \), let \( A \) denote the orbit of \( A \) under powers of \( \sigma \), and set \( B = \{1, \ldots, N+k\} \setminus A \). Then \( \sigma \) factors into a product \( \sigma = \rho \cdot \tau \) where \( \rho = \sigma|_{\vec{A}} \) and \( \tau = \sigma|_{\vec{B}} \). By construction, \( \rho \) has the property that the orbit of \( A \) under \( \rho \) is all of \( A \). Given any integers \( 0 \leq m \leq M \), we let \( \mathfrak{S}_{M;m} \) denote the collection of permutations \( \alpha \) of \( \{1, \ldots, M\} \) such that the orbit of \( \{1, \ldots, m\} \) under powers of \( \alpha \) is all of \( \{1, \ldots, M\} \). The discussion above can be summarized by saying that if \( \tilde{A} = \{a_1, \ldots, a_N, a_{N+1}, \ldots, a_{N+r}\} \) (where \( a_i = i \) for \( i = 1, \ldots, N \)) and \( B = \{b_1, \ldots, b_t\} \) then \( \sigma \) preserves each of \( \tilde{A} \) and \( B \), and \( \sigma(\tilde{A}) = \{a_{\rho(1)}, \ldots, a_{\rho(N+r)}\} \), \( \sigma(B) = \{b_{\tau(1)}, \ldots, b_{\tau(t)}\} \) for some \( \rho \in \mathfrak{S}_{N+r;N}, \tau \in \mathfrak{S}_t \). Furthermore, \( \text{sgn}(\sigma) = \text{sgn}(\rho) + \text{sgn}(\tau) \text{ mod } 2 \).

Finally, for \( \rho \in \mathfrak{S}_{N+r;N} \) as above, we define

\[
s_i = \min\{m > 0 | \rho^m(i) \in \{1, \ldots, N\} \}.
\]

The definition of \( \mathfrak{S}_{N+r;N} \) implies that \( \sum_{i=1}^N s_i = r + N \).

In (16) we are asked to sum over all sets \( I \) with \( |I| = k \) and all permutations \( \sigma \in \mathfrak{S}_{N+k} \) of the subscripts of \( \vec{I} \) and \( \vec{I}' \). From the preceding remarks, this is equivalent to taking a sum over all sets \( \tilde{A} \supset \{1, \ldots, N\} \) and \( B \) with \( |\tilde{A}| + |B| = N+k \), and all permutations \( \rho \) and \( \tau \), \( \rho \in \mathfrak{S}_{N+r;N}, \tau \in \mathfrak{S}_t \) (where \( |\tilde{A}| = N+r, |B| = t \)). Since we are to sum over all \( I \) and \( k \) and allow repetitions, we may replace \( \vec{I} \) by \( \tilde{A} \cup B \), meaning we take the sum over all \( \tilde{A} \) and \( B \) and all \( \rho \) and \( \tau \) as above, and eliminate reference to \( I \). Thus, we replace \( \xi_{\tilde{a}_j} \) by \( \xi_{a_j} \), and \( \xi_{\tilde{b}_j} \) by \( \xi_{b_j} \) if we define \( \tilde{A}' = \{N+1, \ldots, 2N\} \cup (\tilde{A} \setminus \{1, \ldots, N\}) \). (Put another way, pairs \((\tilde{I}, \sigma)\) are in 1–1 correspondence with 4–tuples \((\tilde{A}, B, \rho, \tau)\).) Then we can write \( \text{Tr} \kappa_n \) as:

\[
\sum_{k=0}^n (2(n-k) + 1) (-1)^{k+N} \sum_{\tilde{A}, B \atop |\tilde{A}|+|B|=k+N} \sum_{\rho \in \mathfrak{S}_{|\tilde{A}|;N}} (-1)^{\text{sgn}(\rho)} \prod_{i=1}^{N} \langle \xi_{a_{\rho^m(i)}}, \xi_{\rho^m+1(a)} \rangle \\
\times (-1)^{\text{sgn}(\tau)} \prod_{r=1}^{t} \langle \xi_{b_r}, \xi_{b_{\tau(r)}'} \rangle
\]

Carrying out the sum over all \( B \) of a given size \( t \) and all permutations \( \tau \), this
becomes:
\[
\sum_{k=0}^{N} \sum_{\tilde{A} \in \mathcal{A} | \tilde{A} = k + N} \sum_{\rho \in \mathcal{S}_{|\tilde{A}|}} (-1)^{\text{sgn}(\rho) + k + N} (2(n - k) + 1) \prod_{m=0}^{N-1} \langle \zeta_{\rho^m(i)}', \xi_{\rho^m+1(i)}' \rangle \times \text{tr}(h^{*}|_{\Lambda^t H^1(\Sigma_g+N)})
\]
Reordering the summations so that the sum over \( \tilde{A} \) is on the outside and the sum on \( t \) is next, we find that \( k = |\tilde{A}| - N + t \) and the expression becomes:
\[
\sum_{\tilde{A} = 0}^{n-|\tilde{A}|} \sum_{t=0}^{n-|\tilde{A}|} (-1)^{|\tilde{A}| + \text{sgn}(\rho)} \sum_{\rho \in \mathcal{S}_{|\tilde{A}|}} \prod_{m=0}^{N-1} \langle \zeta_{\rho^m(i)}', \xi_{\rho^m+1(i)}' \rangle \times (-1)^t (2(n - (|\tilde{A}| - N)) + 1) \text{tr}(h^{*}|_{\Lambda^t H^1(\Sigma_g+N)})
\]
Again using the fact that \( \Lambda^t H^1(\Sigma_g+N) \) appears exactly \( 2(|\tilde{A}| - t) + 1 \) times in \( H^*(\Sigma^{[\tilde{A}]-N}) \) and writing \( |\tilde{A}| = N + t \), we can carry out the sum over \( t \) to get that \( \text{Tr} \kappa_n \) is:
\[
\sum_{r=0}^{n} \left[ \sum_{\tilde{A} = 0}^{n-|\tilde{A}|} \sum_{\rho \in \mathcal{S}_{|\tilde{A}|}} (-1)^{\text{sgn}(\rho) + |\tilde{A}|} \prod_{m=0}^{N-1} \langle \zeta_{\rho^m(i)}', \xi_{\rho^m+1(i)}' \rangle \right] \cdot L(h^{(n-r)})
\]
Here \( L(h^{(n-r)}) \) is the Lefschetz number of \( h \) acting on the \( (n-r) \)th symmetric power of \( \Sigma_{g+N} \) which, as remarked in (4), is the \( (n-r) \)th coefficient of \( \zeta(h) \). In view of Corollary 6.4, we will be done if we show that the quantity in brackets above are products over all \( i \) of formulae that look like
\[
\langle c_i, \xi_{\rho(i)}' \rangle \langle h^{*}(\zeta_{\rho(i)}), \xi_{\rho(i)}' \rangle \cdots \langle h^{*}(\zeta_{\rho(N-1)}, c_{\tilde{\rho}(i)}) \rangle
\]
where \( \tilde{\rho}(i) \in \{1, \ldots , N\} \) is defined to be \( \rho^{*i}(i) \). If we sum this quantity over all \( \tilde{A} \) and all \( \rho \) that induce the same permutation \( \tilde{\rho} \) of \( \{1, \ldots , N\} \), we find that (17) becomes simply \( \langle h^{*}(c_i), c_{\tilde{\rho}(i)} \rangle \). Therefore the quantity in brackets is a sum of terms like
\[
(-1)^{\text{sgn}(\rho) + N} \langle h^{*}c_{s_1}, c_{\tilde{\rho}(1)} \rangle \cdots \langle h^{*}c_{s_N}, c_{\tilde{\rho}(N)} \rangle,
\]
where we have fixed \( s_1, \ldots , s_N \) and \( \tilde{\rho} \) and carried out the sum over all \( \rho \) such that
\begin{enumerate}
\item \( \min\{ m > 0 | \rho^m(i) \in \{1, \ldots , N\} \} = s_i \), and
\item The permutation \( i \mapsto \rho^{s_i}(i) \) of \( \{1, \ldots , N\} \) is \( \tilde{\rho} \).
\end{enumerate}

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(As we will see, \( \text{sgn}(\rho) \) depends only on \( \tilde{\rho} \) and \( |\tilde{A}| \).) It remains to sum over partitions \( s_1 + \cdots + s_N \) of \( s = |\tilde{A}| = r + N \) and over permutations \( \tilde{\rho} \). But from Corollary 6.4 and Lemma 6.5, the result of those two summations is precisely \( [\tau(M^*)]_r \), if we can see just that \( \text{sgn}(\tilde{\rho}) = \text{sgn}(\rho) + |\tilde{A}| \mod 2 \). That is the content of the next lemma.

**Lemma 7.1** Let \( A = \{1, \ldots, N\} \) and \( \tilde{A} = \{1, \ldots, s\} \) for some \( s \geq N \). Let \( \rho \in \mathfrak{S}_{s;N} \) and define
\[
\tilde{\rho}(i) \in \mathfrak{S}_N, \quad \tilde{\rho}(i) = \rho^{s_i}(i)
\]
where \( s_i \) is defined as above. Then \( \text{sgn}(\rho) = \text{sgn}(\tilde{\rho}) + m \mod 2 \).

**Proof** Suppose \( \rho = \rho_1 \cdots \rho_p \) is an expression of \( \rho \) as a product of disjoint cycles; we may assume that the initial elements \( a_1, \ldots, a_p \) of \( \rho_1, \ldots, \rho_p \) are elements of \( A \) since \( \rho \in \mathfrak{S}_{m;N} \). For convenience we include any 1–cycles among the \( \rho_i \), noting that the only elements of \( \tilde{A} \) that may be fixed under \( \tilde{\rho} \), so the expression of \( \tilde{\rho} \) as a product of disjoint cycles is \( \tilde{\rho} = \tilde{\rho}_1 \cdots \tilde{\rho}_p \) where each \( \tilde{\rho}_i \) has \( a_i \) as its initial element. For \( a \in A \), define
\[
n(a) = \min\{m > 0|\rho^m(a) \in A\}
\]
\[
\tilde{n}(a) = \min\{m > 0|\rho^{m}(a) = a\}.
\]
Note that \( n(a_i) = s_i \) for \( i = 1, \ldots, N \), \( \sum s_i = s \), and \( \tilde{n}(a_i) \) is the length of the cycle \( \tilde{\rho}_i \). The cycles \( \rho_i \) are of the form
\[
\rho_i = (a_i \cdots \tilde{\rho}(a_i) \cdots \tilde{\rho}^2(a_i) \cdots \tilde{\rho}^{s_i(a_i)}(a_i) \cdots)
\]
where “\( \cdots \)” stands for some number of elements of \( \tilde{A} \). Hence the cycles \( \rho_i \) have length
\[
\tilde{\ell}(\rho_i) = \sum_{m=0}^{\tilde{n}(a_i)-1} (n(\tilde{\rho}^m(a_i)) + 1) = \tilde{n}(a_i) + \sum_{m=0}^{\tilde{n}(a_i)-1} n(\tilde{\rho}^m(a_i)).
\]
Modulo 2, then, we have
\[
\text{sgn}(\rho) = \sum_{i=1}^{p} (\tilde{\ell}(\rho_i) - 1)
\]
\[
= \sum_{i=1}^{p} \left[ \left( \tilde{n}(a_i) + \sum_{m=0}^{\tilde{n}(a_i)-1} n(\tilde{\rho}^m(a_i)) \right) - 1 \right]
\]
\[
= \sum_{i=1}^{p} (\tilde{n}(a_i) - 1) + \sum_{i=1}^{p} \sum_{m=0}^{\tilde{n}(a_i)-1} n(\tilde{\rho}^m(a_i))
\]
\[
= \text{sgn}(\tilde{\rho}) + s,
\]
since because \( \rho \in \mathfrak{S}_{s;N} \) we have \( \sum_{i=1}^{p} \sum_{m=0}^{n(a_i)-1} n(\rho^m(a_i)) = \sum_{i=1}^{N} s_i = s \). 

8 Proof of Theorem 2.5

The theorem of Hutchings and Lee quoted at the beginning of this work can be seen as (or more precisely, the logarithmic derivative of formula (1) can be seen as) a kind of Lefschetz fixed-point theorem for partially-defined maps, specifically the return map \( F \), in which the torsion \( \tau(M) \) appears as a correction term (see [6]). Now, the Lefschetz number of a homeomorphism \( h \) of a closed compact manifold \( M \) is just the intersection number of the graph of \( h \) with the diagonal in \( M \times M \); such consideration motivates the proof of Theorem 2.3 in [6]. With the results of Section 5, we can give another construction.

Given \( \phi : X = M(g, N, h) \to S^1 \) our circle-valued Morse function, cut along \( \phi^{-1}(-1) \) to obtain a cobordism \( W_\phi \) between two copies of \( \Sigma_{g+N} \). Write \( \gamma_i, i = 1, \ldots, N \) for the intersection of the ascending manifolds of the index-1 critical points with \( \partial_+ W \) and \( \delta_i \) for the intersection of the descending manifolds of the index-2 critical points with \( \partial_- W \). Since the homology classes \([\gamma_i]\) and \([\delta_i]\) are the same (identifying \( \partial_+ W = \partial_- W = \Sigma_{g+N} \)), we may perturb the curves \( \gamma_i \) and \( \delta_i \) to be parallel, ie, so that they do not intersect one another (or any other \( \gamma_j, \delta_j \) for \( j \neq i \) either). Choose a complex structure on \( \Sigma_{g+N} \) and use it to get a complex structure on the symmetric powers \( \Sigma_{g+N}^{(k)} \) for each \( k \). Write \( T_\gamma \) for the \( N \)-torus \( \gamma_1 \times \cdots \times \gamma_N \) and let \( T_\delta = \delta_N \times \cdots \times \delta_1 \). Define a function

\[
\psi : T_\gamma \times \Sigma_{g+N}^{(n)} \times T_\delta \to \Sigma_{g+N}^{(n+N)} \times \Sigma_{g+N}^{(n+N)}
\]

by mapping the point \((q_1, \ldots, q_N; \sum p_i, q'_1, \ldots, q'_1)\) to \((\sum p_i + \sum q'_i, \sum p_i + \sum q'_i)\).

The perhaps unusual-seeming orders on the \( \delta_i \) and in the domain of \( \psi \) are chosen to obtain the correct sign in the sequel.

**Proposition 8.1** \( \psi \) is a smooth embedding, and \( D = \text{Im} \psi \) is a totally real submanifold of \( \Sigma_{g+N}^{(n+N)} \times \Sigma_{g+N}^{(n+N)} \).

The submanifold \( D \) plays the role of the diagonal in the Lefschetz theorem.

**Proof** That \( \psi \) is one-to-one is clear since the \( \gamma_i \) and \( \delta_j \) are all disjoint. For smoothness, we work locally. Recall that the symmetric power \( \Sigma_{g}^{(k)} \) is locally
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isomorphic to \( \mathbb{C}^{(k)} \), and a global chart on the latter is obtained by mapping a point \( \sum w_i \) to the coefficients of the monic polynomial of degree \( k \) having zeros at each \( w_i \). Given a point \( (\sum p_i + \sum q_j, \sum p_i + \sum q_j') \) of \( \text{Im}(\psi) \) we can choose a coordinate chart on \( \Sigma_{g+N} \) containing all the points \( p_i, q_j, q_j' \) so that the \( \gamma_i \) and \( \delta_j \) are described by disjoint curves in \( \mathbb{C} \). Thinking of \( q_j \in \gamma_j \subset \mathbb{C} \cong \mathbb{C}^{(1)} \) and similarly for \( q_j' \), we have that locally \( \psi \) is just the multiplication map:

\[
\left( (z - q_1), \ldots, (z - q_N), \prod_{i=1}^{n} (z - p_i), (z - q_1'), \ldots, (z - q_N') \right) \mapsto \left( \prod_{i=1}^{n} (z - p_i) \prod_{j=1}^{N} (z - q_j), \prod_{i=1}^{n} (z - p_i) \prod_{j=1}^{N} (z - q_j') \right)
\]

It is clear that the coefficients of the polynomials on the right hand side depend smoothly on the coefficients of the one on the left and on the \( q_j, q_j' \).

On the other hand, if \((f(z), g(z))\) are the polynomials whose coefficients give the local coordinates for a point in \( \text{Im}(\psi) \), we know that \( f(z) \) and \( g(z) \) share exactly \( n \) roots since the \( \gamma_i \) and \( \delta_j \) are disjoint. If \( p_1 \) is one such shared root then we can write \( f(z) = (z - p_1) \bar{f}(z) \) and similarly for \( g(z) \), where \( \bar{f}(z) \) is a monic polynomial of degree \( n + N - 1 \) whose coefficients depend smoothly (by polynomial long division!) on \( p_1 \) and the coefficients of \( f \). Continue factoring in this way until \( f(z) = f_0(z) \prod_{i=1}^{n} (z - p_i) \), using the fact that \( f \) and \( g \) share \( n \) roots to find the \( p_i \). Then \( f_0 \) is a degree \( N \) polynomial having one root on each \( \gamma_i \), hence having all distinct roots. Those roots (the \( q_j \)) therefore depend smoothly on the coefficients of \( f_0 \), which in turn depend smoothly on the coefficients of \( f \). Hence \( D \) is smoothly embedded.

That \( D \) is totally real is also a local calculation, and is a fairly straightforward exercise from the definition.

We are now ready to prove the “algebraic” portion of Theorem 2.5.

**Theorem 8.2** Let \( \Gamma \) denote the graph of the map \( h^{(n+N)} \) induced by the gluing map \( h \) on the symmetric product \( \Sigma_{g+N} \). Then

\[
D \cdot \Gamma = \text{Tr} \kappa_n.
\]

**Proof** Using the notation from the previous section, we have that in cohomol-
ogy the duals of $D$ and $\Gamma$ are

\begin{align*}
D^* &= \sum_{\beta \in B_{g+1}^{(n)}} (-1)^{\epsilon_1(\beta)} (c_1 \wedge \cdots \wedge c_N \wedge \beta^0) \times (c_1 \wedge \cdots \wedge c_N \wedge \beta) \\
\Gamma^* &= \sum_{\alpha \in B_{g+1}^{(n+N)}} (-1)^{\deg(\alpha)} \alpha^0 \times h^{*-1}(\alpha).
\end{align*}

Here $\epsilon_1(\beta) = \deg(\beta)(N + 1) + \frac{1}{2} N(N - 1)$. Indeed, since the diagonal is the pushforward of the graph by $1 \times h^{-1}$, we get that the dual of the graph is the pullback of the diagonal by $1 \times h^{-1}$. We will find it convenient to write

\begin{equation}
D^* = \sum_{\beta} (-1)^{\epsilon_1(\beta) + \epsilon_2(\beta)} (c_1 \wedge \cdots \wedge c_N \wedge \beta) \times (c_1 \wedge \cdots \wedge c_N \wedge \beta^0),
\end{equation}

by making the substitution $\beta \mapsto \beta^0$ in the previous expression. Since $\beta^{oo} = \pm \beta$, the result is still a sum over the monomial basis with an additional sign denoted by $\epsilon_2$ in the above but which we will not specify.

Therefore the intersection number is

\begin{equation*}
D^* \cup \Gamma^* = \sum_{\alpha, \beta} (-1)^{\epsilon_1 + \epsilon_2 + \epsilon_3(\alpha, \beta)}
\end{equation*}

(18)

\begin{align*}
\left( \alpha^0 \cup (c_1 \wedge \cdots \wedge c_N \wedge \beta) \right) \times \left( h^{*-1} \alpha \cup (c_1 \wedge \cdots \wedge c_N \wedge \beta^0) \right)
\end{align*}

where $\epsilon_3(\alpha, \beta) = \deg(\alpha)(1 + \deg(\beta) + N)$. Since this is a sum over a monomial basis $\alpha$, the first factor in the cross product above vanishes unless $\alpha = c_1 \wedge \cdots \wedge c_N \wedge \beta$, and in that case is 1. Therefore $\deg(\alpha) = \deg(\beta) + N$, which gives $\epsilon_3(\alpha, \beta) \equiv 0 \mod 2$, and (18) becomes

\begin{align*}
D^* \cup \Gamma^* &= \sum_{\beta} (-1)^{\epsilon_1 + \epsilon_2} h^{*-1} (c_1 \wedge \cdots \wedge c_N \wedge \beta) \cup (c_1 \wedge \cdots \wedge c_N \wedge \beta^0) \\
&= \sum_{\beta} (-1)^{\epsilon_1 + \epsilon_2} (c_1 \wedge \cdots \wedge c_N \wedge \beta) \cup h^*(c_1 \wedge \cdots \wedge c_N \wedge \beta^0) \\
&= \sum_{\beta} (-1)^{\epsilon_1} (c_1 \wedge \cdots \wedge c_N \wedge \beta^0) \cup h^*(c_1 \wedge \cdots \wedge c_N \wedge \beta)
\end{align*}

(19)

where we have again used the substitution $\beta \mapsto \beta^0$ and therefore cancelled the sign $\epsilon_2$. Now, some calculation using the cup product structure of $H^*(\Sigma_{g+1}^{(n+N)})$ derived in [9] shows that

\begin{equation*}
c_1 \wedge \cdots \wedge c_N \wedge \beta^0 = (-1)^{\epsilon_4(\beta)} (d_1 \wedge \cdots \wedge d_N \wedge \beta)^0.
\end{equation*}

where $\epsilon_4(\beta) = N \deg(\beta) + \frac{1}{2} N(N + 1) \equiv \epsilon_1(\beta) + \deg(\beta) + N \mod 2$. Note that $(\cdot)^0$ refers to duality in $H^*(\Sigma_{g+1}^{(n)})$ on the left hand side and in $H^*(\Sigma_{g+1}^{(n+N)})$ on the right hand side.
the right. Returning with this to (19) gives
\[ D^* \cup \Gamma^* = \sum_{\beta} (-1)^{\deg(\beta)+N} (d_1 \wedge \cdots \wedge d_N \wedge \beta)^\circ \cup h^*(c_1 \wedge \cdots \wedge c_N \wedge \beta), \]
which is Tr $\kappa_n$ by (14). Theorem 8.2 follows.

To complete the proof of Theorem 2.5, we recall that we have already shown that $D$ is a totally real submanifold of $\Sigma_{g+1} \times \Sigma_{g+1}$. The graph of $h^{(n+N)}$, however, is not even smooth unless $h$ is an automorphism of the chosen complex structure of $\Sigma_{g+1}$: in general the set-theoretic map induced on a symmetric power by a diffeomorphism of a surface is only Lipschitz continuous. Salamon [16] has shown that if we choose a path of complex structures on $\Sigma$ between the given one $J$ and $h^*(J)$, we can construct a symplectomorphism of the moduli space $M(\Sigma, J) \cong \Sigma^{(n+N)}_{g+1}$ that is homotopic to the induced map $h^{(n+N)}$. Hence $\Gamma$ is homotopic to a Lagrangian submanifold of $\Sigma^{(n+N)}_{g+1} \times -\Sigma^{(n+N)}_{g+1}$. Since Lagrangians are in particular totally real, and since intersection numbers do not change under homotopy, Theorem 2.5 is proved.

9 Proof of Lemma 6.2

We restate the lemma:

Assume that there are no “short” gradient flow lines between critical points, that is, every flow line between critical points intersects $\Sigma_g$ at least once. Given a symmetric pair $(g_0, \phi)$ on $M(g, N, h)$ and suitable genericity hypotheses on $h$, there exists a $C^0$–small perturbation of $g_0$ to a metric $\tilde{g}$ such that for given $n \geq 0$

(1) The gradient flow of $\phi$ with respect to $\tilde{g}$ is Morse–Smale; in particular the hypotheses of Theorem 2.3 are satisfied.

(2) The quantity $[\zeta(F)\tau(M^*)]_m$, $m \leq n$ does not change under this perturbation.

Proof Alter $g_0$ in a small neighborhood of $\Sigma_g \subset M(g, N, h)$ as follows, working in a half-collar neighborhood of $\Sigma_g$ diffeomorphic to $\Sigma_g \times (-\epsilon, 0]$ using the flow of $\nabla_{g_0} \phi$ to obtain the product structure on this neighborhood.

Let $p_1, \ldots, p_{2N}$ denote the points in which the ascending manifolds (under gradient flow of $f$ with respect to the symmetric metric $g_0$) of the index-2
critical points intersect $\Sigma_g$ in $W_\phi$. Since $g_0$ is symmetric, these points are
the same as the points $q_1, \ldots, q_{2N}$ in which the descending manifolds of the
index-1 critical points intersect $\Sigma_g$. Let $\mathcal{O}$ denote the union of all closed orbits
of $\nabla \phi$ (with respect to $g_0$) of degree no more than $n$, and all gradient flow
lines connecting index-1 to index-2 critical points. We may assume that this is
a finite set. Choose small disjoint coordinate disks $U_i$ around each $p_i$ such that
$U_i \cap (\mathcal{O} \cap \Sigma_g) = \emptyset$.

In $U_i \times (-\epsilon, 0]$, we may suppose the Morse function $f$ is given by projection
onto the second factor, $(u, t) \mapsto t$, and the metric is a product $g_0 = g_{\Sigma_g} \oplus (1)$. Let
$X_i$ be a nonzero constant vector field in the coordinate patch $U_i$ and $\mu$
a cutoff function that is equal to 1 near $p_i$ and zero off a small neighborhood
of $p_i$ whose closure is in $U_i$. Let $\nu(t)$ be a bump function that equals 1 near $t = \epsilon/2$ and vanishes near the ends of the interval $(-\epsilon, 0]$. Define the vector
field $v$ in the set $U_i \times (-\epsilon, 0]$ by $v(u, t) = \nabla_{g_0} \phi + \nu(t)\mu(u)X(u)$. Now define
the metric $g_{X_i}$ in $U_i \times (-\epsilon, 0]$ by declaring that $g_{X_i}$ agrees with $g_0$ on tangents
to slices $U_i \times \{t\}$, but that $v$ is orthogonal to the slices. Thus, with respect to
$g_{X_i}$, the gradient $\nabla \phi$ is given by a multiple of $v(u, t)$ rather than $\partial / \partial t$.

It is easy to see that replacing $g_0$ by $g_{X_i}$ in $U_i \times (-\epsilon, 0]$ for each $i = 1, \ldots, 2N$
produces a metric $g_X$ for which upward gradient flow of $\phi$ on $W_\phi$ does not
connect index-2 critical points to index-1 critical points with "short" gradient
flow lines. Elimination of gradient flows of $\phi$ from index-2 to index-1 points that
intersect $\Sigma_{g+\phi}$ is easily arranged by small perturbation of $h$, as are transverse
intersection of ascending and descending manifolds and nondegeneracy of fixed
points of $h$ and its iterates. Hence the new metric $g_X$ satisfies condition (1) of
the Lemma.

For condition (2), we must verify that we have neither created nor destroyed
either closed orbits of $\nabla \phi$ or flows from index-1 critical points to index-2 critical
points. The fact that no such flow lines have been destroyed is assured by our
choice of neighborhoods $U_i$. We now show that we can choose the vector fields
$X_i$ such that no fixed points of $F^k$ are created, for $1 \leq k \leq n$.

Let $F_1 : \Sigma_g \to \Sigma_{g+\phi} = \partial_+ W_\phi$ be the map induced by gradient flow with
respect to $g_0$, defined away from the $q_j$, and let $F_2 : \Sigma_{g+N} = \partial_- W_\phi \to \Sigma_g$
be the similar map from the bottom of the cobordism, defined away from the
cubes. Then the flow map $F$, with respect to $g_0$, is given by the composition
$F = F_2 \circ h \circ F_1$ where this is defined. The return map with respect to the $g_X$
gradient, which we will write $\tilde{F}$, is given by $F$ away from the $U_i$ and by $F + cX$
in the coordinates on $U_i$ where $c$ is a nonnegative function on $U_i$ depending on
$\mu$ and $\nu$, vanishing near $\partial U_i$. 

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Consider the graph $\Gamma_{F_k} \subset \Sigma_g \times \Sigma_g$. Since $F^k$ is not defined on all of $\Sigma_g$, the graph is not closed, nor is its closure a cycle since $F^k$ in general has no continuous extension to all of $\Sigma$. Indeed, the boundary of $\Gamma_{F_k}$ is given by a union of products of “descending slices” (i.e., the intersection of a descending manifold of a critical point with $\Sigma_g$) with ascending slices. Restrict attention to the neighborhood $U$ of $p$, where for convenience $p$ denotes any of the $p_1, \ldots, p_{2N}$ above. We have chosen $U$ so that there are no fixed points of $F^k$ in this neighborhood, i.e., the graph and the diagonal are disjoint over $U$. If there is an open set around $\Gamma_{F_k} \cap (U \times U)$ that misses the diagonal $\Delta \subset U \times U$, then any sufficiently small choice of $X$ will keep $\Gamma_{F_k}$ away from $\Delta$ and therefore produce no new closed orbits of the gradient flow. However, it may be that $\partial \Gamma_{F_k}$ has points on $\Delta$. Indeed, if $c \subset \partial_+ W_0 = \Sigma_{g+N}$ is the ascending slice of the critical point corresponding to $p = q$, suppose $h^k(c) \cap c \neq \emptyset$. Then it is not hard to see that $(p,p) \in \partial \Gamma_{F_k}$, and this situation cannot be eliminated by genericity assumptions on $h$. Essentially, $p$ is both an ascending slice and a descending slice, so $\partial \Gamma_{F_k}$ can contain both $\{p\} \times \text{(asc.slice)}$ and $\text{(desc.slice)} \times \{p\}$, and ascending and descending slices can have $p$ as a boundary point.

Our perturbation of $F$ using $X$ amounts, over $U$, to a “vertical” isotopy of $\Gamma_{F_k} \subset U \times U$. The question of whether there is an $X$ that produces no new fixed points is that of whether there is a vertical direction to move $\Gamma_{F_k}$ that results in the “boundary-fixed” points like $(p,p)$ described above remaining outside of $\text{int}(\Gamma_{F_k})$. The existence of such a direction is equivalent to the jump-discontinuity of $F^k$ at $p$. This argument is easy to make formal in the case $k = 1$, and for $k > 1$ the ideas are the same, with some additional bookkeeping.

We leave the general argument to the reader.

Turn now to the question of whether any new flow lines between critical points are created. Let $D = (h \circ F_1)^{-1}(\bigcup c_i)$ denote the first time that the descending manifolds of the critical points intersect $\Sigma_g$, and let $A = F_2 \circ h(\bigcup c_i)$ be the similar ascending slices. Then except for short flows, the flow lines between critical points are in 1–1 correspondence with intersections of $D$ and $F^k(A)$, for various $k \geq 0$. We must show that our perturbations do not introduce new intersections between these sets. It is obvious from our constructions that only $F^k(A)$ is affected by the perturbation, since only $F_2$ is modified.

Since there are no short flows by assumption, there are no intersections of $h^{-1}(c_j)$ with $c_i$ for any $i$ and $j$. This means that $D$ consists of a collection of embedded circles in $\Sigma_g$, where in general it may have included arcs connecting various $q_i$. Hence, we can choose our neighborhoods $U_i$ small enough that $U_i \cap D = \emptyset$ for all $i$, and therefore the perturbed ascending slices $F^k(A)$ stay away from $D$. Hence no new flows between critical points are created.
This concludes the proof of Lemma 6.2.

References


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