Dieomorphisms, symplectic forms and Kodaira fibrations

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Abstract

As was recently pointed out by McMullen and Taubes [7], there are 4-manifolds for which the dihomorphism group does not act transitively on the deformation classes of orientation-compatible symplectic structures. This note points out some other 4-manifolds with this property which arise as the orientation-reversed versions of certain complex surfaces constructed by Kodaira [3]. While this construction is arguably simpler than that of McMullen and Taubes, its simplicity comes at a price: the examples exhibited herein all have large fundamental groups.

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Let $M$ be a smooth, compact oriented 4-manifold. If $M$ admits an orientation-compatible symplectic form, meaning a closed 2-form $\omega$ such that $\omega^2$ is an orientation-compatible volume form, one might well ask whether the space of such forms is connected. In fact, it is not difficult to construct examples where the answer is negative. A more subtle question, however, is whether the group of orientation-preserving diffeomorphisms $\text{Diff}(M)$ acts transitively on the set of connected components of the orientation-compatible symplectic structures of $M$. As was recently pointed out by McMullen and Taubes [7], there are 4-manifolds $M$ for which this subtler question also has a negative answer. The purpose of the present note is to point out that many examples of this interesting phenomenon arise from certain complex surfaces with Kodaira fibrations.

A Kodaira fibration is by definition a holomorphic submersion $f: M \to B$ from a compact complex surface to a compact complex curve, with base $B$ and fiber $F_z = f^{-1}(z)$ both of genus 2. (In $C^1$ terms, $f$ is thus a locally trivial fiber bundle, but nearby fibers of $f$ may well be non-isomorphic as complex curves.) One says that $M$ is a Kodaira-bered surface if it admits such a fibration $f$. Now any Kodaira-bered surface $M$ is algebraic, since $K_M \otimes f^*K_B$ is obviously positive for sufficiently large $\epsilon$. On the other hand, recall that a holomorphic map from a curve of lower genus to a curve of higher genus must be constant. If $f: M \to B$ is a Kodaira fibration, it follows that $M$ cannot contain any rational or elliptic curves, since composing $f$ with the inclusion would result in a constant map, and the curve would therefore be contained in a fiber of $f$; contradiction. The Kodaira-Enriques classification [2] therefore tells us that $M$ is a minimal surface of general type. In particular, the only non-trivial Seiberg-Witten invariants of the underlying oriented 4-manifold $M$ are [8] those associated with the canonical and anti-canonical classes of $M$. Any orientation-preserving self-diffeomorphism of $M$ must therefore preserve $f^*c_1(M)$.

We have just seen that $M$ is of Kähler type, so let $\omega$ denote some Kähler form on $M$, and observe that $\omega$ then of course a symplectic form compatible with the usual 'complex' orientation of $M$. Let $\psi$ be any area form on $B$, compatible with its complex orientation, and, for sufficiently small $\epsilon > 0$, consider the closed 2-form

$$\omega' = \omega - \epsilon \psi.$$

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\[\]Indeed, by Poincaré duality, a continuous map $h: X \to Y$ of non-zero degree between compact oriented manifolds of the same dimension must induce inclusions $h_*: H^j(Y;\mathbb{R}) \to H^j(X;\mathbb{R})$ for all $j$. Such a map $h$ therefore cannot exist whenever $b_j(X) < b_j(Y)$ for some $j$. 

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Then
\[
\langle \langle f ', i \rangle \rangle \cap = \langle 2(f ') \rangle \cap + \langle i \rangle \cap = \langle \langle \langle \langle \langle \langle h f ', i \rangle \rangle \rangle \rangle \rangle \rangle.
\]
where the inner product is taken with respect to the Kähler metric corresponding to \( \langle \rangle \). Now \( \langle h f ', i \rangle \) is a positive function, and, because \( M \) is compact, therefore has a positive minimum. Thus, for a sufficiently small \( \langle \rangle > 0, \langle \rangle \cap \langle \rangle \) is a volume form compatible with the non-standard orientation of \( M \); or, in other words, \( \langle \rangle \) is a symplectic form for the reverse-oriented 4-manifold \( M \). For related constructions of symplectic structures on fiber-bundles, cf [6].

If follows that \( M \) carries a unique deformation class of almost-complex structures compatible with \( \langle \rangle \). One such almost-complex structure can be constructed by considering the (non-holomorphic) orthogonal decomposition
\[
TM = \ker(f) f (TB)
\]
induced by the given Kähler metric, and then reversing the sign of the complex structure on the 'horizontal' bundle \( f (TB) \). The first Chern class of the resulting almost-complex structure is thus given by
\[
C_1(M; \langle \rangle) = C_1(M) - 4(1 - g)F;
\]
where \( g \) is the genus of \( B \), and where \( F \) now denotes the Poincare dual of a fiber of \( f \). For further discussion, cf [4, 5, 9].

Of course, the product \( B \times F \) of two complex curves of genus 2 is certainly Kodaira fibered, but such a product also admits orientation-reversing diffeomorphisms, and so, in particular, has signature \( \langle \rangle = 0 \). However, as was first observed by Kodaira [3], one can construct examples with \( \langle \rangle > 0 \) by taking branched covers of products; cf [1, 2].

**Example** Let \( C \) be a compact complex curve of genus 2, and let \( B_1 \) be a curve of genus \( g_2 = 2k - 1 \), obtained as an unbranched double cover of \( C \). Let \( B_1 \). \( B_2 \) be the associated non-trivial deck transformation, which is a free holomorphic involution of \( B_2 \). Let \( p : B_2 \). \( B_1 \) be the unique unbranched cover of order \( 2^{4k-2} \) with \( p_1(B_2) = \ker(1(B_1) ! H_1(B_1; \mathbb{Z}_2)) \); thus \( B_2 \) is a complex curve of genus \( g_2 = 2^{4k-2}(k-1) + 1 \). Let \( B_2 \). \( B_1 \) be the union of the graphs of \( p \) and \( p \). Then the homology class of \( \langle \rangle \) is divisible by 2. We may therefore construct a ramified double cover \( M \) of \( B_2 \). \( B_1 \) branched over \( \langle \rangle \). The projection \( f_1 : M \). \( B_1 \) is then a Kodaira fibration, with fiber \( F_1 \) of genus \( 2^{4k-2}(4k-3) + 1 \). The projection \( f_2 : M \). \( B_2 \) is also a Kodaira fibration, with fiber \( F_2 \) of genus \( 4k - 2 \). The signature of this doubly Kodaira-fibered complex surface is \( \langle (M) \rangle = 2^{4k}(k-1) \).
We now axiomatize those properties of these examples which we will need.

**Definition** Let $M$ be a complex surface equipped with two Kodaira fibrations $f_j: M \to B_j$, $j = 1, 2$. Let $g_j$ denote the genus of $B_j$, and suppose that the induced map

$$f_1, f_2: M \to B_1 \cong B_2$$

has degree $r > 0$. We will then say that $(f_1, f_2)$ is a Kodaira double-fibration of $M$ if $(M) \not\equiv 0$ and

$$(g_2 - 1) \not\equiv r(g_1 - 1):$$

In this case, $(M; f_1, f_2)$ will be called a Kodaira doubly-bered surface.

Of course, the last hypothesis depends on the ordering of $(f_1; f_2)$, and is automatically satisfied, for fixed $r$, if $g_2 \geq g_1$. The latter may always be arranged by simply replacing $M$ and $B_2$ with suitable covering spaces.

Note that $r = 2$ in the explicit examples given above.

Given a Kodaira doubly-bered surface $(M; f_1, f_2)$, let $\overline{M}$ denote $M$ equipped with the non-standard orientation, and observe that we now have two different symplectic structures on $\overline{M}$ given by

$$!_1 = -f_1', \quad !_2 = -f_2'$$

for any given area forms $'j$ on $B_j$ and any sufficiently small $" > 0$.

**Theorem 1** Let $(M; f_1, f_2)$ be any Kodaira doubly-bered complex surface. Then for any self-diffeomorphism $g: M \to M$, the symplectic structures $!_1$ and $!_2$ are deformation inequivalent.

That is, $!_1$, $-!_1$, $!_2$, and $-!_2$ are always in different path components of the closed, non-degenerate 2-forms on $\overline{M}$. (The fact that $!_1$ and $-!_1$ are deformation inequivalent is due to a general result of Taubes [10], and holds for any symplectic 4-manifold with $b^+ > 1$ and $c_1 \not\equiv 0$.)

Theorem 1 is actually a corollary of the following result:

**Theorem 2** Let $(M; f_1, f_2)$ be any Kodaira doubly-bered complex surface. Then for any self-diffeomorphism $g: M \to M$,

$$[c_1(\overline{M}; !_2)] \not\equiv c_1(\overline{M}; !_1):$$

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Proof Because \((M) \& 0\), any self-diffeomorphism of \(M\) preserves orientation. Now \(M\) is a minimal complex surface of general type, and hence, for the standard 'complex' orientation of \(M\), the only Seiberg-Witten basic classes \([8]\) are \(c_1(M)\). Thus any self-diffeomorphism of \(M\) satisfies

\[[c_1(M)] = c_1(M)\]

Letting \(F_j\) be the Poincare dual of the fiber of \(f_j\), and letting \(g_j\) denote the genus of \(B_j\), we have

\[c_1(M; !_j) = c_1(M) + 4(g_j - 1)F_j\]

for \(j = 1, 2\). The adjunction formula therefore tells us that

\[[c_1(M; !_j)] [c_1(M)] = (2 + 3)(M) - 2(M) = 3(M) \& 0;\]

where the intersection form is computed with respect to the 'complex' orientation of \(M\).

If we had a diffeomorphism \(M \to M\) with \([c_1(M; !_2)] = c_1(M; !_1)\), this computation would tell us that that

\[[c_1(M)] = c_1(M) = [c_1(M; !_2)] = c_1(M; !_1)\]

and that

\[[c_1(M)] = -c_1(M) = [c_1(M; !_2)] = -c_1(M; !_1);\]

In either case, we would then have

\[4(g_1 - 1)F_1 = c_1(M; !_1) - c_1(M) = [c_1(M; !_2) - c_1(M)] = 4(g_2 - 1) (F_2);\]

On the other hand, \(F_1 \cdot F_2 = r\), so intersecting the previous formula with \(F_2\) yields

\[4(g_1 - 1)r = 4(g_1 - 1)F_1 \cdot F_2 = 4(g_2 - 1)[(F_2) \cdot F_2];\]

and hence

\[(g_2 - 1) \cdot r(g_1 - 1);\]

in contradiction to our hypotheses. The assumption that \([c_1(M; !_1)] = c_1(M; !_2)\) is therefore false, and the claim follows. \(\Box\)

Theorem 1 is now an immediate consequence, since the first Chern class of a symplectic structure is deformation-invariant.

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References


