Diffeomorphisms, symplectic forms and Kodaira fibrations

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Abstract

As was recently pointed out by McMullen and Taubes [7], there are 4–manifolds for which the diffeomorphism group does not act transitively on the deformation classes of orientation-compatible symplectic structures. This note points out some other 4–manifolds with this property which arise as the orientation-reversed versions of certain complex surfaces constructed by Kodaira [3]. While this construction is arguably simpler than that of McMullen and Taubes, its simplicity comes at a price: the examples exhibited herein all have large fundamental groups.

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Let $M$ be a smooth, compact oriented 4–manifold. If $M$ admits an orientation-compatible symplectic form, meaning a closed 2–form $\omega$ such that $\omega \wedge \omega$ is an orientation-compatible volume form, one might well ask whether the space of such forms is connected. In fact, it is not difficult to construct examples where the answer is negative. A more subtle question, however, is whether the group of orientation-preserving diffeomorphisms $M \to M$ acts transitively on the set of connected components of the orientation-compatible symplectic structures of $M$. As was recently pointed out by McMullen and Taubes [7], there are 4–manifolds $M$ for which this subtler question also has a negative answer. The purpose of the present note is to point out that many examples of this interesting phenomenon arise from certain complex surfaces with Kodaira fibrations.

A Kodaira fibration is by definition a holomorphic submersion $f: M \to B$ from a compact complex surface to a compact complex curve, with base $B$ and fiber $F_z = f^{-1}(z)$ both of genus $\geq 2$. (In $C^\infty$ terms, $f$ is thus a locally trivial fiber bundle, but nearby fibers of $f$ may well be non-isomorphic as complex curves.) One says that $M$ is a Kodaira-fibered surface if it admits such a fibration $f$. Now any Kodaira-fibered surface $M$ is algebraic, since $K_M \otimes f^*K_B^\ell$ is obviously positive for sufficiently large $\ell$. On the other hand, recall that a holomorphic map from a curve of lower genus to a curve of higher genus must be constant.\footnote{Indeed, by Poincaré duality, a continuous map $h: X \to Y$ of non-zero degree between compact oriented manifolds of the same dimension must induce inclusions $h^*: H^j(Y,\mathbb{R}) \hookrightarrow H^j(X,\mathbb{R})$ for all $j$. Such a map $h$ therefore cannot exist whenever $b_j(X) < b_j(Y)$ for some $j$.} If $f: M \to B$ is a Kodaira fibration, it follows that $M$ cannot contain any rational or elliptic curves, since composing $f$ with the inclusion would result in a constant map, and the curve would therefore be contained in a fiber of $f$; contradiction. The Kodaira–Enriques classification [2] therefore tells us that $M$ is a minimal surface of general type. In particular, the only non-trivial Seiberg–Witten invariants of the underlying oriented 4–manifold $M$ are [8] those associated with the canonical and anti-canonical classes of $M$. Any orientation-preserving self-diffeomorphism of $M$ must therefore preserve $\{\pm c_1(M)\}$.

We have just seen that $M$ is of Kähler type, so let $\psi$ denote some Kähler form on $M$, and observe that $\psi$ is then of course a symplectic form compatible with the usual ‘complex’ orientation of $M$. Let $\varphi$ be any area form on $B$, compatible with its complex orientation, and, for sufficiently small $\varepsilon > 0$, consider the closed 2–form

$$\omega = \varepsilon \psi - f^* \varphi.$$
Then
\[ \omega \land \omega = -2(f^* \varphi) \land \psi + \varepsilon \psi \land \psi = (\varepsilon - \langle f^* \varphi, \psi \rangle) \psi \land \psi, \]
where the inner product is taken with respect to the Kähler metric corresponding to \( \psi \). Now \( \langle f^* \varphi, \psi \rangle \) is a positive function, and, because \( M \) is compact, therefore has a positive minimum. Thus, for a sufficiently small \( \varepsilon > 0 \), \( \omega \land \omega \) is a volume form compatible with the non-standard orientation of \( M \); or, in other words, \( \omega \) is a symplectic form for the reverse-oriented 4-manifold \( \overline{M} \).

For related constructions of symplectic structures on fiber-bundles, cf [6].

If follows that \( \overline{M} \) carries a unique deformation class of almost-complex structures compatible with \( \omega \). One such almost-complex structure can be constructed by considering the (non-holomorphic) orthogonal decomposition
\[ TM = \ker(f_*) \oplus f^*(TB) \]
induced by the given Kähler metric, and then reversing the sign of the complex structure on the ‘horizontal’ bundle \( f^*(TB) \). The first Chern class of the resulting almost-complex structure is thus given by
\[ c_1(\overline{M}, \omega) = c_1(M) - 4(1 - g)F, \]
where \( g \) is the genus of \( B \), and where \( F \) now denotes the Poincaré dual of a fiber of \( f \). For further discussion, cf [4, 5, 9].

Of course, the product \( B \times F \) of two complex curves of genus \( g \geq 2 \) is certainly Kodaira fibered, but such a product also admits orientation-reversing diffeomorphisms, and so, in particular, has signature \( \tau = 0 \). However, as was first observed by Kodaira [3], one can construct examples with \( \tau > 0 \) by taking branched covers of products; cf [1, 2].

**Example** Let \( C \) be a compact complex curve of genus \( k \geq 2 \), and let \( B_1 \) be a curve of genus \( g_1 = 2k - 1 \), obtained as an unbranched double cover of \( C \). Let \( \iota: B_1 \rightarrow B_1 \) be the associated non-trivial deck transformation, which is a free holomorphic involution of \( B_1 \). Let \( p: B_2 \rightarrow B_1 \) be the unique unbranched cover of order \( 2^{4k-2} \) with \( p_*[\pi_1(B_2)] = \ker[\pi_1(B_1) \rightarrow H_1(B_1, \mathbb{Z})] \); thus \( B_2 \) is a complex curve of genus \( g_2 = 2^{4k-1}(k-1) + 1 \). Let \( \Sigma \subset B_2 \times B_1 \) be the union of the graphs of \( p \) and \( \iota \circ p \). Then the homology class of \( \Sigma \) is divisible by \( 2 \). We may therefore construct a ramified double cover \( M \rightarrow B_2 \times B_1 \) branched over \( \Sigma \). The projection \( f_1: M \rightarrow B_1 \) is then a Kodaira fibration, with fiber \( F_1 \) of genus \( 2^{4k-2}(4k-3)+1 \). The projection \( f_2: M \rightarrow B_2 \) is also a Kodaira fibration, with fiber \( F_2 \) of genus \( 4k - 2 \). The signature of this doubly Kodaira-fibered complex surface is \( \tau(M) = 2^{4k}(k-1) \).

We now axiomatize those properties of these examples which we will need.

**Definition** Let $M$ be a complex surface equipped with two Kodaira fibrations $f_j: M \to B_j$, $j = 1, 2$. Let $g_j$ denote the genus of $B_j$, and suppose that the induced map

$$f_1 \times f_2: M \to B_1 \times B_2$$

has degree $r > 0$. We will then say that $(f_1, f_2)$ is a **Kodaira double-fibration** of $M$ if $\tau(M) \neq 0$ and

$$(g_2 - 1) \not| r(g_1 - 1).$$

In this case, $(M, f_1, f_2)$ will be called a **Kodaira doubly-fibered** surface.

Of course, the last hypothesis depends on the ordering of $(f_1, f_2)$, and is automatically satisfied, for fixed $r$, if $g_2 \gg g_1$. The latter may always be arranged by simply replacing $M$ and $B_2$ with suitable covering spaces.

Note that $r = 2$ in the explicit examples given above.

Given a Kodaira doubly-fibered surface $(M, f_1, f_2)$, let $\overline{M}$ denote $M$ equipped with the non-standard orientation, and observe that we now have two different symplectic structures on $\overline{M}$ given by

$$\omega_1 = \varepsilon \psi - f_1^* \varphi_1$$

$$\omega_2 = \varepsilon \psi - f_2^* \varphi_2$$

for any given area forms $\varphi_j$ on $B_j$ and any sufficiently small $\varepsilon > 0$.

**Theorem 1** Let $(M, f_1, f_2)$ be any Kodaira doubly-fibered complex surface. Then for any self-diffeomorphism $\Phi: M \to M$, the symplectic structures $\omega_1$ and $\pm \Phi^* \omega_2$ are deformation inequivalent.

That is, $\omega_1, -\omega_1, \Phi^* \omega_2$, and $-\Phi^* \omega_2$ are always in different path components of the closed, non-degenerate 2-forms on $\overline{M}$. (The fact that $\omega_1$ and $-\omega_1$ are deformation inequivalent is due to a general result of Taubes [10], and holds for any symplectic 4-manifold with $b^+ > 1$ and $c_1 \neq 0$.)

Theorem 1 is actually a corollary of the following result:

**Theorem 2** Let $(M, f_1, f_2)$ be any Kodaira doubly-fibered complex surface. Then for any self-diffeomorphism $\Phi: M \to M$,

$$\Phi^*[c_1(\overline{M}, \omega_2)] \neq \pm c_1(\overline{M}, \omega_1).$$
Proof Because $\tau(M) \neq 0$, any self-diffeomorphism of $M$ preserves orientation. Now $M$ is a minimal complex surface of general type, and hence, for the standard ‘complex’ orientation of $M$, the only Seiberg–Witten basic classes [8] are $\pm c_1(M)$. Thus any self-diffeomorphism $\Phi$ of $M$ satisfies

$$\Phi^*[c_1(M)] = \pm c_1(M).$$

Letting $F_j$ be the Poincaré dual of the fiber of $f_j$, and letting $g_j$ denote the genus of $B_j$, we have

$$c_1(M, \omega_j) = c_1(M) + 4(g_j - 1)F_j$$

for $j = 1, 2$. The adjunction formula therefore tells us that

$$[c_1(M, \omega_j)] \cdot [c_1(M)] = (2\chi + 3\tau)(M) - 2\chi(M) = 3\tau(M) \neq 0,$$

where the intersection form is computed with respect to the ‘complex’ orientation of $M$.

If we had a diffeomorphism $\Phi: M \to M$ with $\Phi^*[c_1(M, \omega_2)] = \pm c_1(M, \omega_1)$, this computation would tell us that

$$\Phi^*[c_1(M)] = c_1(M) \implies \Phi^*[c_1(M, \omega_2)] = c_1(M, \omega_1)$$

and that

$$\Phi^*[c_1(M)] = -c_1(M) \implies \Phi^*[c_1(M, \omega_2)] = -c_1(M, \omega_1).$$

In either case, we would then have

$$4(g_1 - 1)F_1 = c_1(M, \omega_1) - c_1(M) = \pm \Phi^*[c_1(M, \omega_2) - c_1(M)] = \pm 4(g_2 - 1)\Phi^*(F_2).$$

On the other hand, $F_1 \cdot F_2 = r$, so intersecting the previous formula with $F_2$ yields

$$4(g_1 - 1)r = 4(g_1 - 1)F_1 \cdot F_2 = 4(g_2 - 1)[\pm \Phi^*(F_2) \cdot F_2],$$

and hence

$$(g_2 - 1)| r(g_1 - 1),$$

in contradiction to our hypotheses. The assumption that $\Phi^*[c_1(M, \omega_1)] = \pm c_1(M, \omega_2)$ is therefore false, and the claim follows.

Theorem 1 is now an immediate consequence, since the first Chern class of a symplectic structure is deformation-invariant.

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References


