Spin\(^c\) structures and homotopy equivalences

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Abstract

We show that a homotopy equivalence between manifolds induces a correspondence between their spin\(^c\) structures, even in the presence of 2-torsion. This is proved by generalizing spin\(^c\) structures to Poincare complexes. A procedure is given for explicitly computing the correspondence under reasonable hypotheses.

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1 Introduction

The theory of spin\(\text{c}\)\{structures has attained new importance through its recent application to the topology of smooth 4\{manifolds. Among smooth, closed, oriented 4\{manifolds (with \(b_1 + b_\nu\) odd) a typical homeomorphism type contains many di\{eomorphism types. The only invariants known to distinguish such di\{eomorphism types are those arising from gauge theory, as pioneered by Donaldson (eg [1]). The most ef\{cient approach currently known is to assign a Seiberg\{Witten invariant (eg [6]) to any such 4\{manifold \(X\) with a \{xed spin\(\text{c}\)\{structure. To extract the most information from these invariants, one must understand how spin\(\text{c}\)\{structures transform under homeomorphisms. This is straightforward if \(H^2(X; \mathbb{Z})\) has no 2\{torsion (for example, if \(X\) is simply connected), for then the Chern class will distinguish any two spin\(\text{c}\)\{structures on \(X\). The general case is less obvious, however. In high dimensions, a homeomorphism between smooth manifolds need not be covered by an isomorphism of their tangent bundles. While such isomorphisms always exist in dimension 4, they are not canonical, and automorphisms of the tangent bundle covering \(\text{id}_X\) may permute the spin\(\text{c}\)\{structures on \(X\). (For example, such an automorphism over \(\mathbb{R}P^3\) or \(\mathbb{R}P^3 \times S^1\) can be constructed from the di\{eomorphism \(\mathbb{R}P^3 \to SO(3)\).) In this note, we show how to canonically assign to any orientation\{preserving proper homotopy equivalence \(X_1 \to X_2\) between manifolds a correspondence between spin\(\text{c}\)\{structures on \(X_1\) and those on \(X_2\).

Our approach is to generalize the theory of spin and spin\(\text{c}\)\{structures from \(SO(n)\) to more general structure groups \(H\). Most of the homotopy of \(SO(n)\) does not enter into the theory. In fact, it suffices for \(H\) to be path connected with a nontrivial double cover so that we can generalize the definition \(\text{spin}^c(n) = (\text{spin}(n) \times \text{spin}(2)) = \mathbb{Z}_2\). The resulting theory generalizes the classical theory in the obvious way, for example, with spin\(\text{c}\)\{structures on a bundle over \(X\) classified by \(H^2(X; \mathbb{Z})\) whenever \(W_3(\cdot) = 0\) (Proposition 1). Ultimately, the map \(BSO \to BSG\) of classifying spaces allows us to generalize spin\(\text{c}\)\{structures from smooth manifolds to Poincare complexes, and the latter theory has the required functoriality with respect to homotopy equivalences by naturality of the Spivak normal \{bration (Theorem 5). Under reasonable hypotheses, one can explicitly compute the correspondence of spin\(\text{c}\)\{structures induced by a homotopy equivalence; a procedure is given following Theorem 5. The concluding remarks include other characterizations of classical spin\(\text{c}\)\{structures.

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2 Generalized spin\(^c\)\{structures

A naive approach to generalizing the theory of spin and spin\(^c\)\{structures would be to define spin(H) to be a preassigned double cover of a path connected topological group H, and let spin\(^c\)(H) denote the group spin(H) \(\bowtie\) SO(2) diagonally double covering \(H \bowtie\) SO(2). One could then generalize the theory in the obvious way, using principal spin(H) and spin\(^c\)(H)\{bundles, the natural epimorphisms from spin\(^c\)(H) to H and SO(2), and the involution of spin\(^c\)(H) induced by conjugation on SO(2) = U(1). However, to avoid the difficulties of adapting principal bundle theory to spherical fibrations, we translate the argument into the language of classifying spaces, replacing epimorphisms of groups with kernel \(\mathbb{Z}_2\) or SO(2) by fibrations of the corresponding classifying spaces with fiber \(B\mathbb{Z}_2 = K(\mathbb{Z}_2; 1) = \mathbb{RP}^1\) or \(B\text{SO}(2) = K(\mathbb{Z}; 2) = \mathbb{CP}^1\), respectively. We remove the groups from the theory while keeping the suggestive notation, obtaining a theory of spin and spin\(^c\)\{structures on bundles or fibrations classified by a universal bundle (fibration) \(H \rightarrow BH\), where \(BH\) is homotopy equivalent to a simply connected CW\{complex, and a nonzero class \(w 2 H^2(BH; \mathbb{Z}_2)\) is specified (corresponding to a choice of double cover of \(H\)). We can recover the classical theory by setting \(BH = B\text{SO}(n) (n \geq 2)\), with w the unique nonzero class \(w 2 H^2(B\text{SO}(n); \mathbb{Z}_2) = \mathbb{Z}_2\).

Recall [8] that any map \(f: X \rightarrow Y\) can be transformed into a fibration by replacing \(X\) by the space \(P\) of paths from \(X\) to \(Y\) in the mapping cylinder of \(f\). The initial point fibration \(p_0: P \rightarrow X\) has contractible fiber, and the endpoint fibration \(p_1: P \rightarrow Y\) is homotopic to \(f \circ p_0\). The fiber \(F\) of \(p_1\) is homotopy equivalent to a CW\{complex if \(X\) and \(Y\) are [4], and \(p_0|F\) is a fibration with fiber the loop space \(\Omega Y\).

Now let \((BH; w)\) be as above. Then we define epimorphisms \(H^2(BH; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2\) and hence \(\sim\) \(w, \sim (BH) \rightarrow \mathbb{Z}_2\). We apply the previous paragraph to the map \(BH \rightarrow K(\mathbb{Z}_2; 2)\) induced by \(\sim w\), and let \(B\text{spin}(H; w)\) denote the fiber \(F\). The fiber \(B\text{spin}(H; w)\) \(\rightarrow BH\) induces isomorphisms \(i_{BH}(B\text{spin}(H; w))\) with kernel \(\mathbb{Z}_2\) for \(i = 2\) and \(i(BH)\) otherwise, and its fiber is \(K(\mathbb{Z}_2; 1) = \mathbb{RP}^1\). Now we define \(B\text{spin}(H; w)\) to be \(B\text{spin}(H; w + w)\), where \(BH = BH \bowtie\text{SO}(2)\). We immediately obtain fibrations \(p_H\) and \(p_{H(2)}\) of \(B\text{spin}(H; w)\) over \(BH\) and \(B\text{SO}(2)\), whose fibers are \(B\text{spin}(\text{SO}(2); w) = K(\mathbb{Z}; 2)\) and \(B\text{spin}(H; w), respectively, and each fibration restricted to the opposite fiber is the map arising from the definition of \(B\text{spin}(H)\). (Compare with the projections of \(B\text{spin}(H; w)\) to \(H\) and \(\text{SO}(2)\) on the level of groups.) By obstruction theory, complex conjugation on the second factor \(B\text{SO}(2) = \mathbb{CP}^1\) of \(BH\) lifts uniquely from \(BH\) to a map on \(B\text{spin}(H; w)\) whose square is fiber homotopic to the identity, and the map is homotopic to conjugation on each \(\mathbb{CP}^1\) fiber of \(p_H\).

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To define spin\(^c\) structures over \(H\), recall that an \(H\) bundle (or fibration) \(\rightarrow X\) over a CW complex is classified by a bundle map

\[
\begin{array}{c}
\rightarrow
\\
\uparrow
\\
\rightarrow
\\
\uparrow
\\
X \quad BH
\end{array}
\]

For two choices of classifying map \(\sim\), there is a canonical homotopy (up to homotopy rel 0,1) between the corresponding maps \(f\), characterized by lifting to a homotopy of the maps \(f\) through bundle maps. This allows us to define spin\(^c\) structures in a manner independent of the choice of \(\sim\).

**Definition** A spin structure on an \(H\) bundle (or fibration) \(\rightarrow X\) (relative to \(w\)) is a function assigning to each classifying bundle map \(\sim\) a homotopy class of lifts \(\sim f\) of \(f\), such that for two choices of \(\sim\) the canonical homotopy between the maps \(f\) lifts to a homotopy of the corresponding maps \(\sim f\). A spin\(^c\) structure is defined similarly with spin replaced by spin\(^c\).

We denote the sets of spin and spin\(^c\) structures on an \(H\) bundle by \(S(\ ;w)\) and \(S^c(\ ;w)\), respectively. Note that in either case, any lift of a single \(f\) with a specified \(\sim\) uniquely determines such a structure, but changing \(\sim\) with \(f\) fixed may result in an automorphism of \(S(\ ;w)\) or \(S^c(\ ;w)\).

To define characteristic classes, let \(Y \subseteq X\) be a possibly empty subcomplex, and let \(\sim\) be a trivialization of \(jY\). Then we can assume that the classifying map \(f: X \rightarrow BH\) of \(\sim\) is constant on \(Y\), and that \(f\sim jY\) determines the restriction \(f\sim jY: jY 
\rightarrow H\). Set \(w_2(\ ; ) = f (w) 2 H^2(X;Y;\mathbb{Z}_2)\) and \(W_3(\ ; ) = w_2(\ ; ) 2 H^3(X;Y;\mathbb{Z})\), where \(\tau\) is the Bockstein homomorphism. Any spin\(^c\) structure \(s\) on \(X\) determines a homotopy class of lifts \(\sim f\) of \(\sim\), and we define a trivialization \(\sim\) of \(sjY\) over \(Y\) to be a choice of \(\sim f\) (within the given homotopy class) that is constant on \(Y\), up to homotopies through such maps. (Equivalently, \(\sim\) is a spin\(^c\) structure on \(X\) that pulls back to \(s\) on \(X\).) We denote Chern classes by setting \(c_1(s;\tau) = f_\tau p_{SO(2)}(c) 2 H^2(X;Y;\mathbb{Z})\), where \(c 2 H^2(BSO(2);\mathbb{Z}) = \mathbb{Z}\) is the generator \(c_1(SO(2))\). If \(Y\) is empty, we use the notation \(w_2(\ )\), \(W_3(\ )\), \(c_1(s)\).

**Proposition 1** The set \(S(\ ;w)\) of spin structures on an \(H\) bundle (or fibration) \(\rightarrow X\) is nonempty if and only if \(w_2(\ ) = 0\). If so, then \(H^1(X;\mathbb{Z}_2)\) acts freely and transitively on \(S(\ ;w)\). The set \(S^c(\ ;w)\) is nonempty if and only if \(W_3(\ ) = 0\), and if so, then \(H^2(X;\mathbb{Z})\) acts freely and transitively on it. For
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Thus, choosing a base point in S( ; ) or S^c( ; ) (if nonempty) identifies it with H^2(X; Z_2) or H^2(X; Z).

Proof The first two sentences are immediate from obstruction theory, since the fiber of BSpin(H; w)! BH is K(Z_2; 1). In fact, w_2( ; ) is the obstruction to lifting f to a map f: X! BSpin(H; w) with fY constant, as can be seen by rst considering the case where Y contains the 1-skeleton of X. Similarly, H^2(X; Z) acts as required on S^c( ; w) (when nonempty) via difference classes, since the fiber of p_1 is K(Z; 2). Now recall that BSpin^c(H; w) = BSpin(H; w + w ) with BH = BSO(2). Thus, a lift of f to f: X! BSpin^c(H; w) with fY constant is the same as a choice of complex line bundle L! X with a trivialization 1 over Y, together with a spin structure on the bundle L! X (classified by BH BSO(2)) whose defining lift f is constant on Y. The resulting spin^c-structure s with trivialization 1 over satisfies c_1(s; ) = c_1(L; 1), since p_{SO(2)} f is the classifying map of L. Such a structure exists if and only if f = w_2(L; 1) = w_2( ; ) + w_2(L; 1), or equivalently w_2( ; ) = w_2(L; 1) = c_1(L; 1);j_2. Thus, S^c( ; w) is nonempty if and only if w_2( ; ) has a lift to H^2(X; Z), i.e. W_2( ) = 0, and any c_1(s; ) reduces mod 2 to w_2( ; ). Given s; s^0 2 S^c( ; w), the difference class d(s; s^0) takes coefficients in 2(K(Z; 2)), where K(Z; 2) is the fiber of p_1. Since (p_{SO(2)}): 2(K(Z; 2)) ! 2(BO(2)) is multiplication by 2, we have 2d(s; s^0) = c_1(s^0) − c_1(s). Equivalently, c_1(s + a) = c_1(s) + 2a for a = d(s; s^0). The assertion about conjugation is clear from the way it lifts to BSpin^c(H; w).

Now suppose we are given pairs (BH; w) and (BH^0; w^0) as before, and a map h: BH! BH^0 covered by a bundle map h: BH! H^0, with h w^0 = w. Then any H{bundle ! X determines an H^0{bundle ! X with the same w_2 and W_3, and h determines maps BSpin(H; w)! BSpin(H^0; w^0) and BSpin^c(H; w)! BSpin^c(H^0; w^0). We obtain canonical equivariant identifications S( ; w) = S^c( ; w^0) and S^c( ; w) = S^c( ; w^0), and the latter preserves Chern classes and conjugation. On the other hand, given an H{bundle map g: 1! 2 covering g: X_1! X_2, we have induced maps g: S( ; w) ! S( ; w) and g: S^c( ; w) ! S^c( ; w) that are equivariantly equivalent to g: H^1(X_2; Z_2)! H^1(X_1; Z_2) and g: H^2(X_2; Z)! H^2(X_1; Z) when the domains are nonempty, and characteristic classes and conjugation are preserved in the obvious way. If g is a homotopy equivalence, then the maps g are isomorphisms.

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Examples 2  (a) If $h: BSO(m) \to BSO(n)$, $2 \leq m < n$, is induced by the usual inclusion of groups, we recover the stabilization-invariance of classical spin and spin$^c$ structures. We are free to pass to the limiting group $SO$, eliminating the dependence on $n$.

(b) An oriented topological $n$-manifold $X$ has the homotopy type of a CW complex, and it has a tangent bundle classified by a map into the universal bundle over $BSTOP(n)$ (e.g. [3]). There is a canonical map $h: BSO(n) \to BSTOP(n)$ that corresponds to interpreting $SO(n)$ as a topological bundle and is a $\mathbb{Z}_2$-isomorphism of simply connected spaces. We immediately obtain a theory of spin and spin$^c$ structures on oriented topological manifolds by using their tangent bundles (stabilized if $n < 2$). As before, the theory is stabilization-invariant, and we can pass to the limiting case of $BSTOP$. On smooth manifolds, the new theory canonically reduces via $h$ to the classical theory. However, any orientation-preserving homeomorphism $g: X_1 \to X_2$ induces an isomorphism of topological tangent bundles, hence, isomorphisms $g: S(X_2) = S(X_1)$ and $g: S^c(X_2) = S^c(X_1)$ as above.

To generalize to homotopy equivalences, we need one further construction. Suppose we are given a bundle map

$$
\begin{array}{ccc}
H & \xrightarrow{k} & H^0 \\
\gamma & \mapsto & \gamma \\
B H & \xrightarrow{h} & B H^0
\end{array}
$$

with $k(w^0) = w + w^0$. Then a pair of bundles $\mathcal{E}$ classified by $B H; B H^0$ determine an $H^0$ bundle $\mathcal{E}$ on $X$, and $w_2$ and $w_3$ add.

**Proposition 3** A trivialization of $\mathcal{E}$ induces equivariant isomorphisms $k: S(\mathcal{E}; w) \to S(\mathcal{E}; w^0)$ and $k: S^c(\mathcal{E}; w) \to S^c(\mathcal{E}; w^0)$, and the latter preserves conjugation and Chern classes.

**Proof** By obstruction theory, the map $k$ uniquely determines a map $\hat{k}$ making the diagram

$$
\begin{array}{ccc}
B \text{spin}(H; w) & \xrightarrow{k} & B \text{spin}(H^0; w^0) \\
\gamma_{p_1} & \mapsto & \gamma_{p_2} \\
B H & \xrightarrow{h} & B H^0
\end{array}
$$

commute, and a similar diagram is induced for $\text{spin}^c$ via the map $k \circ k_0$, where $k_0: BSO(2) \to BSO(2)$ induces addition on $\mathbb{Z}_2$. The diagrams
determine a map \( k_\#: S( ; w) \rightarrow S( 0, w^0) \) and similarly for \( S^c \). In the latter case, \( k_\# \) commutes with conjugation and adds Chern classes. In either case, \( k_\# \) restricts to addition on the homotopy groups of the fibers of \( p_1 \) and \( p_2 \), so difference classes add under \( k_\# \), and for suitably chosen base points \( k_\# \) is given by addition on \( H^1(X; \mathbb{Z}_2) \) or \( H^2(X; \mathbb{Z}) \) whenever its domain is nonempty. Now a trivialization of \( c_0 \) determines a trivial spin\(^c\) structure \( s^0 \) on \( X \). Since \( W_3( ) + W_3( 0) = W_3( 0) = 0 \), it follows that \( S^c( ; w) \) is nonempty if and only if \( S^c( 0, w^0) \) is. For each \( s \ 2 S^c( ; w) \) there is a unique \( \text{inverse}^\prime \) \( s^0 \ 2 S^c( 0, w^0) \) with \( k_\#(s; s^0) = s^0 \). Let \( k(s) = \text{conjugate of } s^0 \). Then \( k : S^c( ; w) \rightarrow S^c( 0, w^0) \) is an equivariant isomorphism, and it preserves conjugation and Chern classes since \( s^0 \) is conjugation-invariant with \( c_1(s^0) = 0 \). A similar procedure (with \( k(s) = s^0 \) ) works for spin structures. \( \square \)

**Example 4** Any oriented, smooth \( n \) -manifold \( X \) admits a unique isotopy class of proper embeddings in \( \mathbb{R}^N \) for \( N \) sufficiently large. This determines a normal bundle \( X \) that is unique up to stabilization. Since the tangent bundle \( X \) satisfies \( X \rightarrow X = \mathbb{R}^N \times X \) and the latter bundle is canonically trivial, the obvious map \( \text{BSO}(n) \rightarrow \text{BSO}(N - n) \) ! \( \text{BSO}(N) \) determines canonical equivariant identifications \( S( X; w) = S( X; w) \) and \( S^c( X; w) = S^c( X; w) \), the latter preserving Chern classes and conjugation.

**Theorem 5** Let \((X; @X)\) be an oriented, possibly noncompact Poincare pair. There is a canonical procedure for defining sets \( S(X) \) and \( S^c(X) \) of spin and spin\(^c\) structures on \( X \) having the structure described in Proposition 1 (with respect to the usual classes \( w_2(X) \) and \( W_3(X) \)). For \((X; @X)\) a smooth manifold, the theory is canonically equivariantly equivalent to the standard one (preserving Chern classes and conjugation). For pairs \((X_1; @X_1)\) as above, any orientation-preserving, pairwise, proper homotopy equivalence \( g : (X_1; @X_1) \rightarrow (X_2; @X_2) \) induces equivariant isomorphisms \( g : S(X_2) = S(X_1) \) and \( g : S^c(X_2) = S^c(X_1) \), the latter preserving Chern classes and conjugation, and the construction is functorial for such maps \( g \).

**Proof** The pair \((X; @X)\) has a canonical Spivak normal fibration \([7]\) defined by embedding \((X; @X)\) pairwise and properly in half-space \( \mathbb{R}^N \) \((0; 1); 0) \) (uniquely for \( N \) sufficiently large), and making a fibration out of the collapsing map of the boundary of a regular neighborhood. The resulting oriented spherical fibration over \( X \) is classified by a fiber-preserving map into the universal spherical fibration, whose base space stabilizes to \( \text{BSG} \). As in Example 2(b), there is a canonical map \( h : \text{BSO} ! \text{BSG} \) induced by the spherical fibrations \( \text{SO}(n) - (0\{\text{section}) \), and \( h \) is a \{isomorphism of simply connected spaces. We

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immediately obtain $S(X)$, $S^c(X)$ and characteristic classes satisfying Proposition 1, using the Spivak fibration and BSG. (The resulting classes $w_2(X)$ and $W_2(X)$ are well known.) For $(X; gX)$ a smooth manifold, the theory is canonically equivalent (via $h$) to that of the stable normal bundle, which is the usual theory over the tangent bundle by Example 4. A homotopy equivalence $g$ as above induces a fiber-preserving map of the corresponding Spivak fibrations, and hence, the required maps $g$.

The map $g : S^c(X_2) = S^c(X_1)$ induced by a homotopy equivalence can frequently be computed explicitly. We consider the case where $X_2$ contains a 1-dimensional subcomplex with a regular neighborhood $N_2$ that is a manifold, such that $H^2(X_2; N_2; \mathbb{Z})$ has no 2-torsion. We also assume that $g : X_1 \to X_2$ restricts to a homeomorphism from $N_1 = g^{-1}(N_2)$ to $N_2$. These conditions are always satisfied if $g$ is a homeomorphism between smooth manifolds, for example by taking $N_2$ to be a neighborhood of the 1-skeleton of $X_2$. Now the map $g : H(X_2; N_2) = H(X_1; N_1)$ is an isomorphism. A (stable) trivialization of the tangent bundle of $N_2$ (or equivalently, of the stable normal bundle) pulls back via $g|_{N_1}$ to a trivialization over $N_1$, and $g w_2(X_2; s) = w_2(X_1; g|_{N_1}(s))$. Given spin structures $s_1, 2 S^c(X_1)$, pick any trivializations $\gamma$ of $s_1$ on $N_1$. Then by Proposition 1, $g c_i(s_2; \gamma) - c_i(s_1; \gamma)$ reduces to zero mod 2. Since $H^2(X_1; N_1; \mathbb{Z})$ has no 2-torsion, there is a unique class $(s_1; s_2) \in H^2(X_1; N_1; \mathbb{Z})$ such that $d(s_1; s_2) = g c_i(s_2; \gamma) - c_i(s_1; \gamma)$. If we change $\gamma$ with fixed, then $(s_1; s_2)$ changes by the coboundary of a cochain in $N_1$, so it represents a class $d(s_1; s_2) \in H^2(X_1; \mathbb{Z})$ that depends only on $s_1$ and $s_2$ (fixed). But $(s_1; s_2)$ vanishes for $s_1 = g s_2$ and $\gamma$ given by pulling back $\gamma$, and a change of $s_1$ changes 2 $(s_1; s_2)$ by twice the corresponding relative difference class (by the addition formula of Proposition 1 applied to $X_1 = N_1$). Thus, $d(s_1; s_2)$ is precisely the difference class $d(s_1; g s_2)$, in a form accessible to computation.

Remarks (a) Spin structures have several other convenient characterizations. As we observed in proving Proposition 1, a spin structure on $X$ is the same as a line bundle $L$ and spin structure on $X$. For a different approach, recall that Milnor [5] observed that a spin structure on an oriented vector bundle over a CW complex is equivalent (after stabilizing if necessary) to a trivialization over the 1-skeleton that can be extended over the 2-skeleton, just as an orientation is a trivialization over the 0-skeleton that extends over the 1-skeleton. Similarly, a spin structure over an oriented vector bundle is equivalent (after stabilizing if the fiber dimension is odd or 2) to a complex structure over the 2-skeleton that can be extended over the 3-skeleton. To see this, observe that the map of classifying spaces induced by inclusion $i : U(n) \to SO(2n)$

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lifts canonically to a map \( j : BU(n) \to BU(2n) \) by rst lifting the map \( \text{id} \to \text{det} : BU(n) \to BU(2n) \to BSO(2) \to BSO^2(\mathbb{H}) \). (In fact, the corresponding diagram exists on the group level.) Thus, any complex structure determines a spin\(^c\)-structure (and the correspondence preserves \( c_1 \) and conjugation). For \( n = 2 \), this correspondence is bijective for 2{complexes and surjective for 3complexes, since the map \( j \) has a 2{connected \( \pi_0 \). The observation now follows from the fact that restriction induces a bijection from spin\(^c\)-structures to those over the 2{skeleton extending over the 3{skeleton. The same remark applies to bundles classi ed by \( H\) or \( \text{BSG} \) if we de ne a complex structure to be a lift of the classifying map to \( BU \).

(b) The Wu relations are known to hold for Poincare complexes. In particular, for a compact, oriented 4{dimensional Poincare complex \( X \) (without boundary) we have \( w_2(X) = 0 \). The usual argument [2] then shows that \( W_3(X) = 0 \), so all such complexes admit spin\(^c\)-structures.

(c) As in the classical case, we have a canonical map \( i : B\text{spin}(\mathbb{H}) \to B\text{spin}(\mathbb{H};w) \) via the classifying map \( i : \mathbb{RP}^1 \to \mathbb{C P}^1 \), respectively, both of which represent the unique nontrivial homotopy class of maps \( [\mathbb{RP}^1 : \mathbb{C P}^1] \). For any \( \mathbb{C P}^1 \)-classifying map \( f : X \to \mathbb{C P}^1 \), spin structures \( s_1; s_2 \) determine lifts \( f_1; f_2: X \to B\text{spin}(\mathbb{H};w) \). We can assume that these agree over the 0{skeleton and that \( p \) \( f_1 \) and \( p \) \( f_2 \) agree over the 1{skeleton, giving us obstruction cochains \( d(s_1; s_2) \to C^1(X; \mathbb{Z}) \) and \( d(s_1; s_2) \to C^2(X; \mathbb{Z}) \). Now \( d(s_1; s_2) \) evaluates on a 2{cell \( c \) and the element of \( 2(\mathbb{CP}^1) = \mathbb{Z} \) given by \( p \) \( f_2(c) - p \) \( f_1(c) \). Since the boundary operator \( 2(\mathbb{CP}^1) \to 1(\mathbb{RP}^1) \) of \( p \) is multiplication by 2, the same coefficient is obtained as \( \frac{1}{2}d(s_1; s_2) \). Thus, we obtain the required equivalence \( d(s_1; s_2) = d(s_1; s_2) \). To compute \( \text{Im} \), rst note that any \( s \) 2 \( \text{Im} \) is conjugation-invariant (since \( i \) is) with \( c_1 = 0 \). If \( S(\mathbb{C P}^1) \) is nonempty, \( x \in \text{Im} \) and \( s^0 \) be any spin\(^c\)-structure that either is conjugation-invariant or satis es \( c_1(s^0) = 0 \). By Proposition 1, \( 2d(s^0) = 0 \), so \( d(s^0) \to 2\text{Im} \) is empty, no spin\(^c\)-structure has \( c_1 = 0 \) or is conjugation-invariant. The rst assertion is obvious since \( c_1(j_2) = w_2 \neq 0 \). For the remaining assertion, choose \( s \) 2 \( S^c \to \mathbb{C P}^1 \) with conjugate \( s \). Since \( 1(\mathbb{CP}^1; \mathbb{RP}^1) = 0 \), we
can assume that the lift $f: X \to B\text{spin}(H;w)$ determined by $s$ maps the $1$-skeleton $X_1$ into $i(B\text{spin}(H;w))$, which is fixed by conjugation. Thus, $f$ and its conjugate determine a difference cochain $d(s; s) \in C^2(X; \mathbb{Z})$. Since $\pi_2(\mathbb{C}P^1; \mathbb{R}P^1)$ is multiplication by 2 on $\mathbb{Z}$, we can change $d(s; s)$ by any coboundary by changing $f|X_1$: $X_1 \to i(B\text{spin}(H;w))$. Thus, if $s = s$ we can assume that $d(s; s) = 0$, so over each 2-cell, $f$ is conjugation-invariant up to homotopy rel @. But conjugation fixes only 0 in $\pi_2(\mathbb{C}P^1; \mathbb{R}P^1)$, so $f$ can then be homotoped into $i(B\text{spin}(H;w))$, i.e. $s \in \text{Im}$.

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