We consider properties of the total absolute geodesic curvature functional on circle immersions into a Riemann surface. In particular, we study its behavior under regular homotopies, its infima in regular homotopy classes, and the homotopy types of spaces of its local minima.

53C42; 53A04, 57R42

1 Introduction

An immersion of manifolds is a map with everywhere injective differential. Two immersions are regularly homotopic if there exists a continuous 1–parameter family of immersions connecting one to the other. The Smale–Hirsch $h$–principle [8; 4] says that the space of immersions $M \to N$, $\dim(M) < \dim(N)$ is homotopy equivalent to the space of injective bundle maps $TM \to TN$. In contrast to differential topological properties, differential geometric properties of immersions do not in general satisfy $h$–principles, see [3, (A) on page 62]. In this paper and the sequel [2], we study some aspects of the differential geometry of immersions and regular homotopies in the most basic cases of codimension one immersions. We investigate whether or not it is possible to perform topological constructions while keeping control of certain geometric quantities.

Let $\Sigma$ be a Riemann surface, ie, an orientable 2–manifold with a Riemannian metric, and let $c: S^1 \to \Sigma$ be an immersion of the circle parameterized by arc length. If $\bar{c}: S^1 \to U \Sigma$, where $U \Sigma$ is the unit tangent bundle of $\Sigma$, denotes the natural lift of $c$, then the $h$–principle mentioned above implies that the map $c \mapsto \bar{c}$ induces a weak homotopy equivalence between the space of circle immersions into $\Sigma$ and the space of continuous circle maps into $U \Sigma$. In particular, regular homotopy classes of circle immersions into $\Sigma$ are in one to one correspondence with the homotopy classes of (free) loops in $U \Sigma$.

Published: 23 March 2006

DOI: 10.2140/agt.2006.6.459
The total absolute geodesic curvature $\kappa$ of a circle immersion $c$ into a Riemann surface is given by the integral
\[ \kappa(c) = \int_c |k_g| \, ds, \]
where $k_g$ is the geodesic curvature of $c$, and where $ds$ denotes the arc length element along $c$. We study properties of the functional $\kappa$ on the space of circle immersions, starting with the following question. If $c_0$ and $c_1$ are regularly homotopic circle immersions into a Riemann surface, what is the infimum, over all regular homotopies $c_t$, $0 \leq t \leq 1$, connecting $c_0$ to $c_1$, of $\max_{0 \leq t \leq 1} \kappa(c_t)$?

Theorem 1.1 answers this question for the simplest Riemann surfaces of constant curvature. We use the following notational conventions: All Riemann surfaces are assumed to be complete unless otherwise explicitly stated. If $\Sigma$ is a Riemann surface then $K: \Sigma \to \mathbb{R}$ denotes its Gaussian curvature function. For topological spaces $X$ and $Y$, we write $X \approx Y$ to indicate that $X$ is homeomorphic to $Y$.

**Theorem 1.1** Let $\Sigma$ be a Riemann surface and let $c_0, c_1: S^1 \to \Sigma$ be regularly homotopic.

(a) If $\Sigma$ has constant curvature $K = 0$ (the completeness assumption then implies $\Sigma \approx \mathbb{R}^2$, $\Sigma \approx S^1 \times \mathbb{R}$, or $\Sigma \approx T^2$), or if $\Sigma \approx \mathbb{R}^2$ and has constant curvature $K < 0$, then there exists a regular homotopy $c_t$, $0 \leq t \leq 1$, connecting $c_0$ to $c_1$ with
\[ \kappa(c_t) \leq \max\{\kappa(c_0), \kappa(c_1)\}, \quad 0 \leq t \leq 1. \]

(b) If $\Sigma$ is the 2–sphere with a constant curvature metric (with $K > 0$) then, for any $\epsilon > 0$, there exists a regular homotopy $c_t$, $0 \leq t \leq 1$, connecting $c_0$ to $c_1$ such that
\[ \kappa(c_t) \leq \max\{\kappa(c_0), \kappa(c_1), 2\pi + \epsilon\}, \quad 0 \leq t \leq 1. \]

Moreover, if $c_0: S^1 \to \Sigma$ runs $m$ times around a geodesic and $c_1: S^1 \to \Sigma$ runs $m + 2$ times around a geodesic then any regular homotopy $c_t$, $0 \leq t \leq 1$, connecting $c_0$ to $c_1$ has an instant $c_\tau$, $0 < \tau < 1$, with
\[ \kappa(c_\tau) > 2\pi. \]

Theorem 1.1 is proved in Section 5.3. The proof of (b) uses Arnol’d’s $J^–$–invariant for immersed curves on the sphere, see Arnol’d [1], Inshakov [5] and Tchernov [9]. In Remark 5.1 we present a metric on $\mathbb{R}^2$ with $K \leq 0$ for which the conclusion in (a) does not hold.

The proof of Theorem 1.1 also gives information about infima of $\kappa$. To state these results we first introduce some notation. If $\Sigma$ is a flat Riemann surface then parallel
translation gives a trivialization of $U^*$ and the free homotopy classes of curves in $U^*$ are in natural one to one correspondence with $\pi_1(\Sigma) \times \mathbb{Z}$, where $\pi_1(\Sigma)$ encodes the homotopy class of a circle immersion and $\mathbb{Z}$ its tangential degree. Thus, if $\Sigma \approx \mathbb{R}^2$, then we denote a regular homotopy class of circle immersions by the integer $m$ which equals the tangential degree of any of its representatives, and, similarly, if $\Sigma \approx S^1 \times \mathbb{R}$ or $\Sigma \approx T^2$, then we denote a regular homotopy class by $(\xi, m) \in \pi_1(\Sigma) \times \mathbb{Z}$, where $\xi$ and $m$ is the homotopy class in $\Sigma$ and the tangential degree, respectively, of any of its representatives. If $\Sigma$ is the 2–sphere then there are exactly two regular homotopy classes: one represented by a simple closed curve, the other by such a curve traversed twice. Finally, if $\alpha$ is a regular homotopy class of curves in a Riemann surface then let $\hat{k}(\alpha) = \inf_{c \in \alpha} \kappa(c)$.

**Theorem 1.2**

(a) Let $\Sigma$ be a Riemann surface with $K(p) < 0$ for all $p \in \Sigma$ and assume that either $\Sigma$ is closed or $\Sigma \approx \mathbb{R}^2$. Then the infimum $\hat{k}(\alpha)$ is attained at some curve in the regular homotopy class $\alpha$ if and only if $\alpha$ is representable by (a multiple of) a closed geodesic. Moreover, if $K(p) = K < 0$ is constant and $\Sigma \approx \mathbb{R}^2$ then $\hat{k}(m), m \in \mathbb{Z}$, satisfies

$$\hat{k}(m) = \begin{cases} 2\pi & \text{for } m = 0, \\ \pi(|m| + 1) & \text{for } m \neq 0. \end{cases}$$

(b) Let $\Sigma$ be a Riemann surface of constant curvature $K = 0$ (the completeness assumption then implies $\Sigma \approx \mathbb{R}^2$, $\Sigma \approx S^1 \times \mathbb{R}$, or $\Sigma \approx T^2$). Then the infimum $\hat{k}((\xi, m))$ is attained at some curve in the regular homotopy class $(\xi, m) \in \pi_1(\Sigma) \times \mathbb{Z}$ if and only if $\xi \neq *$ or $m \neq 0$, where $*$ denotes the homotopy class of the constant loop. Moreover, $\hat{k}((\xi, m)), (\xi, m) \in \pi_1(\Sigma) \times \mathbb{Z}$, satisfies

$$\hat{k}((\xi, m)) = \begin{cases} 2\pi & \text{for } (\xi, m) = (*, 0), \\ 2\pi(|m|) & \text{otherwise}. \end{cases}$$

(c) Let $\Sigma$ be the 2–sphere with any metric and let $\alpha$ be a regular homotopy class of circle immersions into $\Sigma$. Then the infimum $\hat{k}(\alpha)$ equals 0 and is attained at some curve in $\alpha$.

**Theorem 1.2** is proved in Section 5.1.

A curve in a Riemann surface $\Sigma$ with $K(p) \neq 0$ for all $p \in \Sigma$, which is a local minimum of $\kappa$ is in fact a closed geodesic, see Proposition 2.2 (a). For flat Riemann surfaces this is not the case. Here any local minimum of $\kappa$ is a locally convex curve, see Proposition 2.2 (b). We say that a curve $c$ is locally convex if $k_g \geq 0$ everywhere.
for some orientation of \( c \). If \( k_g > 0 \) everywhere, we say that \( c \) is \textit{strictly locally convex}. In the terminology of Gromov \cite[page 8]{gromov}, strictly locally convex curves are called free curves.

The following result describes the homotopy types of the spaces of local minima of \( \kappa \) for a flat Riemann surface. (Here we think of circle immersions as oriented unit speed curves parameterized by arc length.)

\textbf{Theorem 1.3} \textit{On a flat Riemann surface} \( \Sigma \) \textit{(the completeness assumption implies} \( \Sigma \cong \mathbb{R}^2 \), \( \Sigma \cong S^1 \times \mathbb{R} \), or \( \Sigma \cong T^2 \)), \textit{the space} \( \Omega_{(\xi,m)} \) \textit{of (strictly) locally convex curves of regular homotopy class} \( (\xi,m) \in \pi_1(\Sigma) \times \mathbb{Z} \) \textit{satisfies}

\[
\Omega_{(\xi,m)} \simeq \begin{cases} 
\emptyset & \text{if } (\xi, m) = (\ast, 0), \\
\Sigma & \text{if } \xi \neq \ast \text{ and } m = 0, \\
U \Sigma & \text{if } m \neq 0,
\end{cases}
\]

where \( \simeq \) \textit{denotes weak homotopy equivalence}.

\textbf{Theorem 1.3} is proved in \textbf{Section 5.2}.

\textbf{Acknowledgements}

During the preparation of this paper, the author was a research fellow of the Royal Swedish Academy of Sciences sponsored by the Knut and Alice Wallenberg foundation. He also acknowledges support from the Alfred P Sloan Foundation and from NSF–grant DMS-0505076. The author thanks B Pausader for reading and commenting on earlier versions of the paper.

\section{First variation and local minima}

In this section we compute the first variation of total absolute geodesic curvature. We use the result to classify local minima. Since the absolute value function is not differentiable at zero, the first variation is expressed as a statement about differences rather than as a statement about derivatives.

\subsection{First variation of total absolute geodesic curvature}

Let \( c: [0, L] \to \Sigma \) be an immersion into a Riemann surface, parameterized by arc length. Let \( e_1 \) be the unit tangent vector field of \( c \) and let \( e_2 \) be a unit vector field along \( c \) everywhere orthogonal to \( e_1 \). We consider variations \( \omega: [0, L] \times (-\delta, \delta) \to \Sigma \).
of \( c \) with the following three properties: \( \omega(s, 0) = c(s) \), \( \partial_\epsilon \omega(s, 0) = \alpha(s)e_2(s) \), where \( \partial_\epsilon \) denotes differentiation with respect to the second variable, for some function \( \alpha: [0, L] \to \mathbb{R} \), and the curves \( \omega_\epsilon(s) = \omega(s, \epsilon) \) (\( \epsilon \) fixed) are immersions for \( \epsilon \in (-\delta, \delta) \).

We also introduce the sign function \( \sigma: \mathbb{R} \to \mathbb{R} \) as follows

\[
\sigma(x) = \begin{cases} 
1 & \text{for } x > 0, \\
0 & \text{for } x = 0, \\
-1 & \text{for } x < 0.
\end{cases}
\]

**Lemma 2.1**  For \( \epsilon \in (-\delta, \delta) \),

\[
\kappa(\omega_\epsilon) - \kappa(c) = \epsilon \left( \int_{\{k_g \neq 0\}} \sigma(k_g) (\tilde{\alpha} + \alpha K) \, ds \right) \\
+ |\epsilon| \left( \int_{\{k_g = 0\}} |\tilde{\alpha} + \alpha K| \, ds \right) + \mathcal{O}(\epsilon^2),
\]

where \( \tilde{\alpha} = \frac{d^2 \alpha}{ds^2} \), and where \( \mathcal{O}(\epsilon^2) \) denotes a function such that \( \epsilon^a \mathcal{O}(\epsilon^2) \to 0 \) as \( \epsilon \to 0 \) for all \( a > -2 \).

**Proof**  To simplify notation, let \( \frac{\partial \omega}{\partial s} = \dot{\omega} \) and \( \frac{\partial \omega}{\partial \epsilon} = \omega' \). Let \( \nabla \) denote the Levi–Civita connection and let \( \nabla_s = \nabla_\dot{\omega} \) and \( \nabla_\epsilon = \nabla_{\omega'} \). If \( d\tau \) denotes the arc length element of the curve \( \omega_\epsilon \) and \( k_g(s, \epsilon) \) denotes the geodesic curvature of \( \omega_\epsilon \) at \( s \) then

\[
|k_g| \, d\tau = |k_g(s, \epsilon)| \, |\dot{\omega}| \, ds = \frac{\langle \nabla_s \dot{\omega}, t \dot{\omega} \rangle}{|\dot{\omega}|^2} \, ds,
\]

where \( t \) denotes rotation by \( \frac{\pi}{2} \). Assuming \( k_g(s, 0) \neq 0 \) and remembering that \( c \) is parameterized by arc length, we compute

\[
\partial_\epsilon (|k_g(s, 0)| |\dot{\omega}|) = \left( \partial_\epsilon |\dot{\omega}|^{-2} \right) |k_g| + \sigma(k_g) \left( \partial_\epsilon \left( \nabla_s \dot{\omega}, t \dot{\omega} \right) \right) \\
= -2 \langle \nabla_\epsilon \dot{\omega}, \dot{\omega} \rangle |k_g| + \sigma(k_g) \left( \langle \nabla_\epsilon \nabla_s \dot{\omega}, t \dot{\omega} \rangle + \langle \nabla_\epsilon \dot{\omega}, t \nabla_s \dot{\omega} \rangle \right) \\
= -2 \langle \nabla_\epsilon \omega', \dot{\omega} \rangle |k_g| \\
+ \sigma(k_g) \left( \langle \nabla_s \nabla_s \omega', t \dot{\omega} \rangle + \langle R(\omega', \dot{\omega}) \dot{\omega}, t \dot{\omega} \rangle + \langle \nabla_\epsilon \dot{\omega}, t \nabla_s \omega' \rangle \right),
\]
where $R$ is the curvature tensor. Noting that $\omega' = \alpha e_2$, $\nabla_s e_1 = k_g e_2$, and $\nabla_s e_2 = -k_g e_1$, we conclude

$$\partial_\epsilon |k_g(s, 0)| = 2\alpha k_g |k_g| + \sigma(k_g)(\bar{\alpha} - \alpha k_g^2 + K \alpha - \alpha k_g^2)$$

\hfill (2-1)

$$= \sigma(k_g)(\bar{\alpha} + K \alpha).$$

A similar calculation at $s$ where $k_g(s, 0) = 0$ gives

$$\partial_\epsilon (k_g(s, 0)|\omega|) = \bar{\alpha} + K \alpha.$$ 

Hence for such $s$,

$$|k_g(s, \epsilon)||\omega| = |\epsilon||\bar{\alpha} + K \alpha| + O(\epsilon^2).$$ 

The result follows by integration of (2-1) and (2-2).

\hfill $\square$

### 2.2 Local minima of $\kappa$

**Proposition 2.2** Let $\Sigma$ be a Riemann surface.

(a) If $K(p) \neq 0$ for all $p \in \Sigma$, then an immersion $c: S^1 \to \Sigma$ is a local minimum of $\kappa$ if and only if it is a geodesic.

(b) If $\Sigma$ is flat then an immersion $c: S^1 \to \Sigma$ is a local minimum of $\kappa$ if and only if it is a locally convex curve.

In particular, in both cases (a) and (b), any local minimum of $\kappa$ is a global minimum in its regular homotopy class.

**Proof** Consider case (a). A curve $c: S^1 \to \Sigma$ is a geodesic if and only if $k_g(s) = 0$ for all $s \in S^1$ and geodesics are global minima of $\kappa$. Let $c: S^1 \to \Sigma$ be a local minimum of $\kappa$. Note that $U = \{s \in S^1: k_g(s) \neq 0\}$ is open. Assume $U$ is nonempty. Then there exists a nonempty open subinterval $J \subset U$. For any variation $\omega(s, \epsilon)$ of $c$ with $\partial_\epsilon \omega = \alpha e_2$ where $\alpha: S^1 \to \mathbb{R}$ is supported in $J$ we have

$$\int_J \sigma(k_g)\bar{\alpha} \; ds = 0.$$ 

Thus, since $c$ is a local minimum, we conclude from Lemma 2.1 that

$$\int_J \sigma(k_g)\alpha K \; ds = 0.$$ 

This contradicts $K(p) \neq 0$ for all $p \in \Sigma$. It follows that $U$ is empty and thus $c$ is a geodesic.

*Algebraic & Geometric Topology, Volume 6 (2006)*
Consider case (b). Let $c: S^1 \to \Sigma$ be a local minimum. We show that $k_g$ cannot change sign along $c$. Assume it does, then there exist two disjoint open subintervals $J_+$ and $J_-$ of $S^1$ such that $k_g > 0$ on $J_+$ and $k_g < 0$ on $J_-$. Let $A$ be a subinterval of $S^1$ containing both $J_+$ and $J_-$. Let $\alpha: A \to \mathbb{R}$ be a function such that $\bar{\alpha}$ is supported in small subintervals of $J_+ \cup J_-$ and such that $\bar{\alpha} = r \neq 0$, where $r$ is a non-zero constant, between $J_+$ and $J_-$. For a variation $\omega$ of $c$ with $\partial_\epsilon \omega = \alpha e_2$, Lemma 2.1 implies that

$$\kappa(\omega_\epsilon) - \kappa(c) = \pm 2r \epsilon + O(\epsilon^2).$$

This contradicts $c$ being a local minimum. Consequently, $k_g$ does not change sign along $c$ and $c$ is locally convex.

It remains to show that $c$ is a global minimum. Fix a unit speed parametrization of $c$ so that $k_g(s) \geq 0$ for all $s \in S^1$. As in Section 1, we construct an orthonormal trivialization of $T\Sigma$ by parallel translation with respect to the flat metric. This identifies the unit tangent bundle $U\Sigma$ of $\Sigma$ with $\Sigma \times S^1$ and the regular homotopy class of $c$ is determined by its homotopy class in $\Sigma$ and the degree of $\pi_2 \circ \bar{c}: S^1 \to S^1$, where $\pi_2: \Sigma \times S^1 \to S^1$. (Recall that $\bar{c}: S^1 \to U\Sigma$ denotes the natural lift of the unit speed curve $c$.) Moreover, $\kappa(c)$ is simply the length of the curve $\pi_2 \circ \bar{c}$. Now, $k_g(s) \geq 0$ for all $s \in S^1$ implies that the length of $\pi_2 \circ \bar{c}$ equals $2\pi$ times the degree of $\pi_2 \circ \bar{c}$ and it follows that local minima are global minima also in this case. □

**Remark 2.3** Proposition 2.2 does not hold for arbitrary Riemann surfaces. Consider for example the boundary of a convex body in $\mathbb{R}^3$ which agrees with the standard 2–sphere except that it has a flat region near the north pole. Any locally convex curve in this flat region is a local minimum of $\kappa$ but it is certainly not a global minimum in its regular homotopy class.

## 3 Curvature concentrations and approximations

In this section we define piecewise geodesic curves with curvature concentrations and show that circle immersions can be approximated by such curves without increasing the total absolute geodesic curvature.

### 3.1 Piecewise geodesic curves with curvature concentrations

Let $\Sigma$ be a Riemann surface. A *piecewise geodesic curve* in $\Sigma$ is a continuous curve $c: S^1 \to \Sigma$ which is a finite union of geodesic segments. More formally such a curve $c$ can be described as follows. Consider a finite collection of geodesics $c_j: [0, 1] \to \Sigma$, $j = 1, \ldots, m$, with $c_j(1) = c_{j+1}(0)$ for each $j$ (here $c_{m+1} = c_1$). Let $I_j$ be $[0, 1]$...
thought of as the domain of $c_j$ and let $I_{m+1} = I_1$. Then the space obtained by identifying $1 \in I_j$ with $0 \in I_{j+1}$ is a circle $S^1$ which can be considered as the domain of a continuous map $c: S^1 \to \Sigma$ such that if $p \in S^1$ is the image of $p' \in I_j$ under the quotient projection then $c(p) = c_j(p')$. We say that the points $p \in S^1$ with two preimages under the quotient projection are the vertices of the piecewise geodesic curve $c$. We will often deal with images of vertices and we call also these image points vertices of $c$, when no confusion can arise.

Let $U \Sigma$ denote the unit tangent bundle of $\Sigma$. Note that at each vertex $c(p)$ of a piecewise geodesic curve $c$ as above, there is an incoming unit tangent vector $\dot{c}(1)/|\dot{c}(1)| \in U_{c(p)} \Sigma$ and an outgoing unit tangent vector $\dot{c}_{j+1}(0)/|\dot{c}_{j+1}(0)| \in U_{c(p)} \Sigma$. A piecewise geodesic curve with curvature concentrations is a piecewise geodesic curve $c$ together with a vertex curve $\gamma: [0, 1] \to U_{c(p)} \Sigma$ for each vertex $p$ which connects the incoming–to the outgoing unit tangent of $c$ at $p$ and which satisfies the following condition: $\gamma$ is a continuous piecewise geodesic curve (with finitely many geodesic arcs) in the fiber circle $U_{c(p)} \Sigma$ equipped with the metric induced by the Riemannian metric on $\Sigma$. We use the abbreviations PGC–curve to denote piecewise geodesic curves with curvature concentrations, and we often write $(c_1, \gamma_1, \ldots, c_m, \gamma_m)$ for a PGC–curve with geodesic segments $c_j$ and vertex curves $\gamma_j$. We say that the length $l(\gamma_j)$ of the vertex curve $\gamma_j$ is the curvature concentration of the PGC–curve $c$ at the vertex $c(p) = c_j(1) = c_{j+1}(0)$, and that the piecewise geodesic curve with geodesic arcs $c_1, \ldots, c_m$ is the underlying curve of $c$.

We note that any PGC–curve $c = (c_1, \gamma_1, \ldots, c_m, \gamma_m)$ in $\Sigma$ has a natural continuous lift $\tilde{c}: S^1 \to U \Sigma$ which consists of the usual lifts $\tilde{c}_j$ of $c_j$, $j = 1, \ldots, m$, connected by the curves $\gamma_j$, $j = 1, \ldots, m$, in the fibers of $U \Sigma \to \Sigma$ over vertices of $c$. The lift $\tilde{c}$ of $c$ is thus a piecewise smooth curve. In particular, its derivative is smooth except for finitely many jump discontinuities where the curve has left and right derivatives. If $c$ is a PGC–curve we consider $\tilde{c}: S^1 \to U \Sigma$ as a parameterized curve with its natural arc length parametrization scaled by a suitable factor so that its domain becomes the unit circle.

**Definition 3.1** The total absolute geodesic curvature of a PGC–curve

$$c = (c_1, \gamma_1, \ldots, c_m, \gamma_m)$$

is

$$\kappa(c) = \sum_{j=1}^{m} l(\gamma_j),$$

where $l(\gamma_j)$ is the length of the vertex curve $\gamma_j$. 

*Algebraic & Geometric Topology, Volume 6 (2006)*
Let $\text{Pgc}(S^1, \Sigma)$ denote the set of all PGC–curves in $\Sigma$. We define the distance between two elements $b$ and $c$ in $\text{Pgc}(S^1, \Sigma)$ to be the $C^0$–distance (with respect to the metric on $U \Sigma$ induced from the metric on $\Sigma$) between their lifts $\tilde{b}$ and $\tilde{c}$ endowed with parameterizations proportional to arc length, as discussed above. A PGC–homotopy is a continuous 1–parameter family of PGC–curves or equivalently a continuous map from the interval to $\text{Pgc}(S^1, \Sigma)$.

### 3.2 Approximation

If $\Sigma$ is a Riemann surface then let $\text{Imm}(S^1, \Sigma)$ denote the space of circle immersions into $\Sigma$ with the $C^2$–topology. Let $c: S^1 \to \Sigma$ be a circle immersion, let $\pi = (p_0, \ldots, p_m)$ be a partition of $S^1$ and let $|\pi| = \max_j d(p_j, p_{j+1})$ ($d$ is the distance function on $S^1$ and we use the convention $p_{m+1} = p_0$). If $|\pi|$ is sufficiently small then we associate a PGC–curve $c^\pi$ to $c$, as follows: $c^\pi$ is the PGC–curve with underlying piecewise geodesic curve consisting of the shortest geodesic segments between $c(p_j)$ and $c(p_{j+1})$, and with vertex curves $\gamma_j$ which are the shortest arcs in $U_{c(p_j)} \Sigma$ connecting the incoming– to the outgoing unit tangent of the underlying piecewise geodesic curve. We note that $\max_j l(\gamma_j) \to 0$ and $|\kappa(c) - \kappa(c^\pi)| \to 0$ as $|\pi| \to 0$.

If $f: \Lambda \to \text{Imm}(S^1, \Sigma)$ is a continuous family of circle immersions parameterized by a compact space $\Lambda$ and if $\epsilon > 0$ is arbitrary, then there exists $\delta > 0$ such that for all partitions $\pi$ with $|\pi| < \delta$, $f^\pi: \Lambda \to \text{Pgc}(S^1, \Sigma)$, defined by $f^\pi(\lambda) = (f(\lambda))^\pi$, is a continuous family of PGC–curves, and there exists an $\epsilon$–small homotopy connecting the family of continuous curves $\tilde{f}: \Lambda \to U \Sigma$ to the family of continuous curves $\tilde{f^\pi}: \Lambda \to U \Sigma$.

#### Lemma 3.2

Let $c_0: S^1 \to \Sigma$ be a circle immersion into a Riemann surface with $K < 0$, $K > 0$, or $K = 0$ everywhere. Then there exists a regular homotopy $c_t$, $0 \leq t \leq 1$, of $c_0$ such that

\[(3-1) \quad \kappa(c_t) \leq \kappa(c_0), \quad 0 < t \leq 1, \]

and such that the PGC–curve $c_t^\pi$ (defined using any sufficiently fine partition $\pi$), satisfies

\[(3-2) \quad \kappa(c_t^\pi) \leq \kappa(c_0). \]

Moreover, if $c$ is not a local minimum of $\kappa$ then the non-strict inequalities in (3–1) and (3–2) can be replaced by strict inequalities.
Assume $K \neq 0$ everywhere. Then Proposition 2.2 implies that $c_0$ is a local minimum if and only if $c_0$ is a geodesic. Hence, if $c_0$ is not a geodesic then there exists a $\kappa$–decreasing regular homotopy connecting $c_0$ to some curve $c_1$. For sufficiently fine partition $\pi$, the curve $c_\pi^\pi$ then satisfies (3–2).

Assume $K = 0$. If $c$ is not locally convex then the above argument can be repeated. Recall from Section 1 that parallel translation in the flat metric gives $U\Sigma = \Sigma \times S^1$ and let $\pi_2: U\Sigma \rightarrow S^1$ denote the projection. If $c$ is locally convex it is elementary to see that for sufficiently fine partitions $\pi$, $\pi_2 \circ c_\pi$ is monotone and thus $\kappa(c_\pi) = \kappa(c)$. 

**Remark 3.3** Lemma 3.2 does not hold for general metrics. Consider for example $\mathbb{R}^2$ with coordinates $(x, y)$ and a metric given by

$$ds^2 = \exp\left(2\left(\sqrt{x^2 + y^2} - 1\right)\right) (dx^2 + dy^2)$$

in a small neighborhood of $c = \{x^2 + y^2 = 1\}$. Then $K(x, y) < 0$ for $x^2 + y^2 > 1$ and $K(x, y) > 0$ for $x^2 + y^2 < 1$ and the geodesic curvature of $c$ is identically equal to 1. Let $b$ be any curve which is a $C^1$–small perturbation of $c$. Assume that $b$ meets $c$ transversely in $2m$ points. These intersection points subdivide $b$ and $c$ into unions of arcs $c = \gamma_\text{out} \cup \gamma_\text{in}$ and $b = \beta_\text{out} \cup \beta_\text{in}$, where $\beta_\text{out} \subset \{x^2 + y^2 \geq 1\}$ and $\beta_\text{in} \subset \{x^2 + y^2 \leq 1\}$ and where the endpoints of an arc in $\gamma_\text{in}$ ($\gamma_\text{out}$) agree with the endpoints of some arc in $\beta_\text{out}$ ($\beta_\text{in}$). Let $\Omega_j, j = 1, \ldots, m$, be the $m$ regions bounded by an arc in $\beta_\text{out}$ and an arc in $\gamma_\text{in}$ and let $\Omega_j, j = 1, \ldots, m$ be the $m$ region bounded by an arc in $\gamma_\text{out}$ and an arc in $\beta_\text{in}$. Then the Gauss–Bonnet theorem implies that

$$\int_{\beta_\text{out}} k_g \, ds - \int_{\gamma_\text{in}} k_g \, ds + \sum_{j=1}^{m} \int_{\Omega_j} K \, dA + \sum_{j=1}^{2m} \alpha_j = 2\pi m,$$

$$\int_{\gamma_\text{out}} k_g \, ds - \int_{\beta_\text{in}} k_g \, ds + \sum_{j=1}^{m} \int_{\Omega_j} K \, dA + \sum_{j=1}^{2m} \alpha_j = 2\pi m,$$

where $\alpha_j$ is the exterior angle at the $j$th intersection point between $b$ and $c$. Thus

$$\kappa(b) \geq \int_b k_g \, ds > \int_c k_g \, ds = \kappa(c).$$

By approximation we conclude that for any PGC–curve $e$ which lies in a sufficiently small tubular neighborhood of $c$, $\kappa(e) > \kappa(c)$. 

_Tobias Ekholm_
4 Curvature non-increasing homotopies, smoothing, and locally convex curves

In this section we construct special PGC–homotopies of PGC–curves on flat Riemann surfaces and on the hyperbolic plane which decrease the total absolute geodesic curvature of a given initial curve and which ends at a curve of certain standard shape. We construct similar special PGC–homotopies of curves on the 2–sphere with a constant curvature metric. We show that these special PGC–homotopies can be smoothed to regular homotopies, increasing the total curvature arbitrarily little. We also study the space of locally convex curves on flat surfaces.

4.1 Flat surfaces

Let \( \Sigma \) be a Riemann surface. Let \( \Pi: \hat{\Sigma} \to \Sigma \) be a smooth covering map and endow \( \hat{\Sigma} \) with the pull-back metric. Define a lift of a PGC–curve \( c \) in \( \Sigma \) to be a PGC–curve \( b \) in \( \hat{\Sigma} \) such that \( \hat{b} \) is a lift of \( \hat{c} \) with respect to the induced covering \( \Pi_U: U \hat{\Sigma} \to U \Sigma \).

Let \( c \) be a PGC–curve in \( \Sigma \) with lift \( b \) in \( \hat{\Sigma} \) and let \( b_t, 0 \leq t \leq 1 \), be a PGC–homotopy of \( b \) with the following properties: the start-point \( b_t(\alpha) \) and endpoint \( b_t(\omega) \) of \( b_t \) satisfy \( \Pi(b_t(\alpha)) = \Pi(b_t(\omega)) \) and \( \Pi_U(b_t(\alpha)) = \Pi_U(b_t(\omega)) \), for all \( t \), where \( \hat{b}_t(\alpha) \) and \( \hat{b}_t(\omega) \) are the outgoing– and incoming tangent vectors of \( b_t \) at \( b_t(\alpha) \) and \( b_t(\omega) \), respectively. Then \( b_t \) induces a PGC–homotopy \( c_t \) of \( c \) by transporting geodesic segments of \( b_t \) in \( \hat{\Sigma} \) to geodesic segments of \( c_t \) in \( \Sigma \) with the projection \( \Pi: \hat{\Sigma} \to \Sigma \), and by transporting vertex curves of \( b_t \) to vertex curves of \( c_t \) with the induced projection \( \Pi_U: U \hat{\Sigma} \to U \Sigma \). (Note that the conditions on the PGC–homotopy \( b_t \) holds if \( \hat{b}_t \) is a closed curve for each \( t \).)

Lemma 4.1 Let \( \Sigma \) be a Riemann surface with constant curvature \( K = 0 \) and let \( c \) be a PGC–curve in \( \Sigma \). Then there exists a PGC–homotopy \( c_t, 0 \leq t \leq 1 \), with \( c_0 = c \), with \( \kappa(c_t) \leq \kappa(c_0) \), for all \( t \), and with the following property: the underlying curve of \( c_1 \) is a geodesic and any curvature concentration of \( c_1 \) equals \( \pi \).

Proof Consider the case \( \Sigma = \mathbb{R}^2 \). We claim that any PGC–curve which does not have underlying curve a line segment admits a PGC–homotopy which does not increase \( \kappa \) and which decreases the number of vertices with curvature concentration not a multiple of \( \pi \). Together with an obvious inductive argument this shows that any PGC–curve is PGC–homotopic through an homotopy with properties as above to a curve with all curvature concentrations integral multiples of \( \pi \). Noting that any curvature concentration at a vertex of magnitude \( m\pi \), \( m \) a positive integer, can be split up into...
Consider the claim. If the underlying curve of a PGC–curve is not a line segment then it has three consecutive line segments connected by two vertices with curvature concentrations which are not integral multiples of \( \pi \). At such a vertex the curvature concentration is either the sum of the exterior angle of the underlying curve and a multiple of \( 2\pi \) or the sum of the interior angle of the curve and an odd multiple of \( \pi \). We call the former type of vertices exterior and the latter interior. We also need to distinguish two types of configurations of the three consecutive segments. We say that the configuration is convex if all three segments are contained in the closure of one of the half planes determined by the line containing the middle segment, otherwise we say it is non-convex. To establish the claim, consider three consecutive segments \( e_1, e_2, e_3 \) as above connected at vertices \( v_1 \) and \( v_2 \). We separate the cases:

**Case 1** If both \( v_1 \) and \( v_2 \) are exterior, then move \( v_2 \) along \( e_3 \) until it reaches the next vertex following it. It is straightforward to check that this PGC–homotopy preserves \( \kappa \) in the convex case and decreases \( \kappa \) in the non-convex case.

**Case 2** If \( v_1 \) is exterior and \( v_2 \) interior, then move \( v_1 \) backwards along \( e_1 \) until it reaches the vertex preceding it. This PGC–homotopy preserves \( \kappa \) in the non-convex case and decreases it in the convex case.

**Case 3** Assume that both \( v_1 \) and \( v_2 \) are interior. If the configuration is convex, then move \( v_2 \) along \( e_3 \) until it reaches the vertex following it. This PGC–homotopy preserves \( \kappa \) in the non-convex case and decreases \( \kappa \) in the convex case.

Assume first that \( l_1 \) and \( l_3 \) are not parallel. Note that the segments \( e_1 \) and \( e_3 \) lie in different components of \( \mathbb{R}^2 \setminus l_2 \). If the point \( l_1 \cap l_3 \) lies in the component of \( \mathbb{R}^2 \setminus l_2 \) which contains \( e_3 \) then move \( v_1 \) along \( l_1 \) to \( l_1 \cap l_3 \). Otherwise, move \( v_2 \) along \( l_3 \) to the intersection point. This PGC–homotopy strictly decreases \( \kappa \). In the case that \( l_1 \) and \( l_3 \) are parallel, start by moving \( v_1 \) as described above. Note that this decreases \( \kappa \). Thus we may change \( e_3 \) slightly without increasing \( \kappa \) past \( \kappa(c_0) \) so that \( l_1 \) and \( l_3 \) intersect. We then apply the above.

Note that in either case, the PGC–homotopy described reduces the number of vertices with curvature concentration not an integral multiple of \( \pi \). The claim follows.

Consider the non-simply connected case. Since \( \Sigma \) is a flat Riemann surface, there is a covering map \( \Pi: \mathbb{R}^2 \to \Sigma \) with deck transformations which are translations of \( \mathbb{R}^2 \). Let \( b \) be a lift of \( c \). If \( b \) is closed, we apply the above result to construct a PGC–homotopy \( b_t, 0 \leq t \leq 1 \), with properties as in the formulation of the lemma. The induced PGC–homotopy \( c_t, 0 \leq t \leq 1 \), of \( c \) then satisfies the lemma. If \( b \) is
We consider separate cases. Consider first the case when both \( e \) vertices. Rotate the line containing \( A \) b \( \text{Let} \) two line segments and two vertices. \( \text{Thus, to complete the proof it remains only to study curves with two line segments and two vertices.} \)

Let \( b \) be the lift of a curve \( c \) with two vertices. Let \( v_0 \) and \( v_1 \) be the endpoints of the lift \( b \) and let \( l \) be the straight line segment in \( \mathbb{R}^2 \) connecting \( v_0 \) to \( v_1 \). Let \( e_0 \) and \( e_1 \) be the straight line segments of \( b \) with one endpoint at \( v_0 \) and at \( v_1 \), respectively. Let \( e \) be the segment of \( b \) connecting the endpoint of \( e_0 \) which is not equal to \( v_0 \) to the endpoint of \( e_1 \) not equal to \( v_1 \). Since we lift \( c \) at a midpoint of a geodesic segment it follows that \( e \) passes the midpoint \( p \) of \( l \).

Let \( w_0 \) and \( w_1 \) be the vertices where \( e_0 \) and \( e \) meet and where \( e_1 \) and \( e \) meet, respectively. Choose an orienting basis \( (\hat{e}, \hat{f}) \) of the plane where \( \hat{e} \) is a unit vector in direction of \( e \) oriented from \( v_0 \) to \( v_1 \) and where \( \hat{f} \) is a vector perpendicular to \( \hat{e} \) oriented so that the direction vector of \( e_0 \) has positive \( \hat{f} \) component. If \( \Delta(r, s, t) \) is a triangle with corners at \( r, s, t \in \mathbb{R}^2 \) we write \( \alpha(r; r, s, t) \) and \( \beta(r; r, s, t) \) for the exterior– respectively interior angle of \( \Delta(r, s, t) \) at the corner \( r \).

We consider separate cases. Consider first the case when both \( w_0 \) and \( w_1 \) are exterior vertices. Rotate the line containing \( e \) in the positive directions around \( p \), and rotate the lines containing \( e_0 \) and \( e_1 \) in the negative direction around \( v_0 \) and \( v_1 \), respectively, so that angles change at linear speed. More precisely if \( w_0' \) and \( w_1' \) denotes the intersection points between the rotated line containing \( e \) and the rotated line containing \( e_0 \) and \( e_1 \), respectively. Then \( \beta(p; p, w_j', j) = (1-t)\beta(p; p, w_j, j) \), \( j = 0, 1 \). Note that the segments in the triangles \( \Delta(p, w_j', j) \), \( j = 0, 1 \), which are not parallel to \( l \) give a PGC–homotopy \( b_t \) of \( b \) which via the projection \( \Pi \) induces a PGC–homotopy \( c_t \) of \( c \). Moreover,

\[
\frac{d}{dt}(\kappa(c) - \kappa(c_t)) = 2(\beta(p; p, v_0, w_0) + \beta(v_0; p, v_0, w_0)) > 0,
\]

and the curvature concentrations of \( c_1 \) are integral multiples of \( 2\pi \).

Consider second the case when both \( w_0 \) and \( w_1 \) are interior angles. In this case we repeat the construction above with the important difference that we rotate also the lines containing \( e_0 \) and \( e_1 \) in the positive direction. Again the projection gives a PGC–homotopy \( c_t \) of \( c \) and we have

\[
\frac{d}{dt}(\kappa(c_t) - \kappa(c)) = 2(-\alpha(v_0; p, v_0, w_0) + \beta(p; p, v_0, w_0)).
\]
but $\alpha(v_0; p, v_0, w_0) = \beta(p; p, v_0, w_0) + \beta(w_0; p, v_0, w_0)$ and hence $\kappa(c_t) < \kappa(c)$. Again $c_1$ has the desired form.

Finally, if one of $w_0$ and $w_1$ is an exterior vertex and the other one is an interior vertex, then any one of the above procedures may be used. The result is a PGC–homotopy which does not change $\kappa$: the sum of the interior angle and the exterior angle at $w_0$ and $w_1$ is constantly equal to $\pi$. This finishes the proof in the non-simply connected case.

\begin{lemma}
If $c_0$ and $c_1$ are two PGC–homotopic PGC–curves on a Riemann surface as in Lemma 4.1 then there exists a PGC–homotopy $c_t$, $0 \leq t \leq 1$, connecting them such that
\[ \kappa(c_t) \leq \max\{\kappa(c_0), \kappa(c_1)\}, \]
for all $t$.
\end{lemma}

\begin{proof}
After Lemma 4.1 it is sufficient to consider the case when $c_0$ and $c_1$ both have underlying curves geodesics and all curvature concentrations equal to $\pi$. Any vertex curve is thus either a positive or a negative $\pi$–rotation. It is easy to see that neighboring vertex curves of different orientations cancel. The lemma follows.
\end{proof}

\section{The hyperbolic plane}

The counterpart of Lemma 4.1 for curves in the hyperbolic plane differs from the flat case in an essential way: the limit curve which arises as the end of a $\kappa$–decreasing PGC–homotopy, is not a PGC–curve in the hyperbolic plane, in fact it often has infinite length. To deal with this phenomenon we define a generalized PGC–curve in the hyperbolic plane as a PGC–curve which is allowed to have vertices at infinity. More concretely, consider the disk model of the hyperbolic plane,

\begin{equation}
D = \{x = (x_1, x_2) \in \mathbb{R}^2: |x| < 1\}, \quad ds^2 = \frac{4(dx_1^2 + dx_2^2)}{(1-|x|^2)^2},
\end{equation}

and add to it the circle at infinity $\partial D$. Define a \textit{generalized PGC–curve} as a piecewise smooth curve in $\bar{D} = D \cup \partial D$ which consists of geodesic segments (ie, arcs of circles meeting $\partial D$ at right angles), which is allowed to have vertices on $\partial D$, and which have vertex curves at all vertices connecting the incoming– to the outgoing unit tangent. Note that the length of any vertex curve at a vertex on $\partial D$ is an odd multiple of $\pi$. We extend $\kappa$ to generalized PGC–curves by defining it as the sum of the lengths of all vertex curves (the sum of all curvature concentrations).

To connect generalized PGC–curves to PGC–curves we measure the distance between generalized PGC–curves as the $C^0$–distance between their lifts in $U \bar{D}$ with respect to

\begin{equation}
\end{equation}
the metric on $U \overline{D}$ induced by the Euclidean metric on the plane. Using this metric we define a generalized PGC–homotopy as a continuous $1$–parameter family of generalized PGC–curves. Moreover, it is clear that the following approximation result holds: if $b_\lambda, \lambda \in \Lambda$ is any continuous family of generalized PGC–curves parameterized by a compact space $\Lambda$ and if $\epsilon > 0$ is arbitrary then there exists a family of PGC–curves $c_\lambda, \lambda \in \Lambda$, such that the distance between $b_\lambda$ and $c_\lambda$ is less than $\epsilon$ and such that $|\kappa(b_\lambda) - \kappa(c_\lambda)| < \epsilon$ for every $\lambda \in \Lambda$.

Lemma 4.3 Let $\Sigma \approx \mathbb{R}^2$ be a Riemann surface of constant curvature $K < 0$ and let $c$ be a PGC–curve in $\Sigma$. Then there exists a generalized PGC–homotopy $c_t$, $0 \leq t \leq 1$, with $c_0 = c$, with $\kappa(c_t)$ a non-increasing function of $t$, and with the following property: $c_1$ is a generalized PGC–curve with all its vertices on the circle at infinity, and with all curvature concentrations equal to $\pi$.

Proof Note that $\kappa$ is invariant under scaling. Therefore we may assume that $K = -1$. Then $\Sigma = D$, where $D$ is as in (4–1). Let $c$ be a PGC–curve in $\Sigma$. Consider a vertex curve $\gamma$ of $c$ at the vertex $p \in D$. Let $e_1$ be the incoming geodesic arc of $c$ at $p$. We consider two cases separately.

Case 1 Assume that $l(\gamma) = a + n2\pi$ where $0 < a < \pi$ and $n \geq 0$. In this case the vertex curve is, modulo $2\pi$–rotations, the exterior angle of the curve. We push the endpoint of $e_1$ backwards along $e_1$ until we reach its start-point. At this moment we have reduced the number of non-infinite vertices of $c$ with curvature concentrations not a multiple of $\pi$ by one. To see that this generalized PGC–homotopy does not increase $\kappa$ we calculate with notation as in Figure 1,

$$\alpha' + \beta' = \alpha + \int_{\Omega} K \, dA \leq \alpha,$$

and hence $\kappa$ does not increase.

![Figure 1: Removing an exterior angle](image)
**Case 2** Assume that $l(y) = a + n2\pi$ where $\pi \leq a < 2\pi$ and $n \geq 0$. In this case the vertex curve is, modulo $2\pi$–rotations, the complementary angle to the exterior angle. We push the vertex in the positive direction along $e_1$ until it hits $\partial D$, thereby reducing the number of non-infinite vertices by one. To see that this PGC–homotopy does not increase $\kappa$ we calculate with notation as in Figure 2,

$$\alpha' + \beta' = \alpha + \int_{\Omega} K \, dA \leq \alpha,$$

and hence $\kappa$ does not increase.

![Figure 2: Removing an interior angle](image)

Repeating this argument a finite number of times we remove all non-infinite vertices of $c$. To finish the proof we note that any vertex curve at infinity has length $\pi + 2m\pi$, for some integer $m \geq 0$, and that the $2\pi m$–concentration can be pushed along one of the geodesics which ends at the infinite vertex so that it lies in $D$ (not in $\partial D$). Finally, such a curvature concentration of magnitude $2\pi m$ in the finite part of the disk can be split up and pushed to $2m$ curvature concentrations at infinity, each of length $\pi$.

In Figure 3 this is illustrated for $m = 1$. (If the multiplicity of the geodesic on the left hand picture in Figure 3 is $k$, then the multiplicity between the two curvature concentrations on the right hand side is $k + 2$.)

Our next goal is to deform any generalized PGC–curve with its vertices at infinity to a standard form. To this end, we distinguish two different vertices at infinity: let $x$ be a vertex on $\partial D$ of a generalized PGC–curve with incoming geodesic segment along the geodesic $c_i$ and outgoing geodesic segment along the geodesic $c_o$. Note that $c_i$ subdivides $D$ into two components. Let $D_+$ be the component with inward normal $v$ along $c_i$ such that if $n(x)$ is the outward normal of $\partial D$ at $x$ then $n(x), v(x)$ is a positively oriented basis of $\mathbb{R}^2$. Let $D_-$ be the other component. If $c_i = c_o$ then we say $x$ is a **degenerate vertex**. Assume that $x$ is a non-degenerate vertex and that the vertex...
curve at $x$ is a positive (negative) $\pi$–rotation. Then we say that $x$ is an over-rotated vertex if $c_0$ lies in $D_-$ ($D_+$), otherwise we say it is an under-rotated vertex.

We say that a generalized PGC–curve of tangential degree (Whitney index) $m$ in the hyperbolic plane is in standard position if it has the form of the curve in Figure 4, where if, $|m| \neq 0$, all vertex curves have length $\pi$ and have the same orientation. Clearly any two PGC–homotopic generalized PGC–curves in standard position are PGC–homotopic through such curves.

**Lemma 4.4** If $c_0$ and $c_1$ are two PGC–homotopic PGC–curves in the hyperbolic plane each with at least one curvature concentration not an integral multiple of $\pi$, then there exists a PGC–homotopy $c_t$, $0 \leq t \leq 1$, from $c_0$ to $c_1$ such that

$$\kappa(c_t) \leq \max\{\kappa(c_0), \kappa(c_1)\},$$

**Figure 3:** Splitting a curvature concentration

**Figure 4:** A curve in standard position
for all $t$.

**Proof** Using Lemma 4.3, we deform any given PGC–curve to a generalized PGC–curve with all its vertices at infinity and all curvature concentrations equal to $\pi$ without increasing $\kappa$. In fact, the condition that some curvature concentration is not an integral multiple of $\pi$ implies that $\kappa$ is decreased by this deformation. It is thus sufficient to show that any generalized PGC–curve with all its vertices at infinity and all curvature concentrations equal to $\pi$ may be deformed to a curve in standard position by a generalized PGC–homotopy which increases $\kappa$ arbitrarily little.

Note that any generalized PGC–curve with all its vertices at infinity which has only two infinite vertices is automatically in standard position. Assume inductively that any such curve with $< m$ vertices can be brought to standard position by a PGC–homotopy which increases $\kappa$ arbitrarily little and consider a curve with $m$ vertices.

If the curve has an under-rotated vertex then this vertex can be removed using the method of Case 1 in the proof of Lemma 4.3. The inductive assumption finishes the proof in that case. We thus assume that the curve does not have any under-rotated vertices.

Consider a curve which satisfies this assumption and which has two curvature concentrations of opposite signs. Such a curve must have two vertices $a$ and $b$ with curvature concentrations of opposite signs which are connected by a geodesic arc $C$. Let $A$ and $B$ be the other geodesics with endpoints at $a$ and $b$, respectively. Since every vertex is over-rotated, it follows that $C$ separates $A$ from $B$. Push the second endpoint of $B$ which is not equal to $b$ until it is very close to $a$ and the endpoint of $A$ which is not equal to $a$ until it is very close to $b$. With this done, push the curvature concentrations inwards along $C$ and cancel them increasing $\kappa$ arbitrarily little. The initial phase of the latter homotopy is shown in Figure 5.

![Figure 5: Canceling opposite concentrations](image)

Finally, assume that there are no under-rotated vertices and that all of the vertices have curvature concentrations of the same sign. In this case we pick a first vertex and move
the second toward it, eventually creating a small geodesic. We then move the third vertex toward the second and so on. We claim that either this process creates a standard curve or it creates an under-rotated vertex. To see this, let \( p_j \) be the first vertex which has not been moved. There are two cases, either \( p_j \) can be moved without passing \( p_j \). In this case the construction continues. Or, \( p_j \) must pass \( p_j \). We then move the third vertex toward the second and so on. We claim that either this process creates a standard curve or it creates an under-rotated vertex. To see this, let \( p_j \) be the first vertex which has not been moved. There are two cases, either \( p_j \) can be moved without passing \( p_j \). In this case the construction continues. Or, \( p_j \) must pass \( p_j \). The claim and the lemma follow.

4.3 Constant curvature spheres

**Lemma 4.5** Let \( \Sigma \) be the 2–sphere with a constant curvature metric and let \( c \) be a PGC–curve in \( \Sigma \). Then there exists a PGC–homotopy \( c_t \), \( 0 \leq t \leq 1 \), with \( c_0 = c \), with \( \kappa(c_t) \) a non-increasing function of \( t \), and with the following property: the underlying curve of \( c_1 \) is a geodesic and any curvature concentration of \( c_1 \) equals \( \pi \).

**Proof** As in Lemma 4.3, scaling invariance of \( \kappa \) implies we may assume \( K = 1 \). We claim that any PGC–curve \( c \) which does not have underlying curve a geodesic admits a PGC–homotopy which does not increase \( \kappa \) and which decreases the number of vertices with curvature concentration not a multiple of \( \pi \). As in the proof of Lemma 4.1 this finishes the proof.

Let \( p_0 \) denote the start and \( p_1 \) denote the endpoint of a geodesic \( e_1 \) of \( c \). Let \( e_0 \) and \( e_2 \) be the other geodesic segments connecting to \( p_0 \) and \( p_1 \), respectively, and let \( p_2 \) denote the other vertex of \( e_2 \). Let \( G_j \) be the great circle in which \( e_j \) has its image. If \( g \) is a geodesic on \( \Sigma \) we let \( l(g) \) denote its length. We must consider two separate cases. In the first case \( l(e_1) \neq m\pi \) for all integers \( m > 0 \). In this case we deform the curve by moving \( p_1 \) along \( G_2 \) in such a way that the length \( l(\gamma_0) \) of the vertex curve \( \gamma_0 \) at \( p_0 \) decreases. We stop this deformation the first time \( p_1 \) hits \( p_2 \) or one of the points in \( G_0 \cap G_2 \), or, when \( l(\gamma_0) = n\pi \), for some integer \( n \). At this instant we obtain a curve with the number of vertices with curvature concentration not an integral multiple of \( \pi \) one smaller than the corresponding number for \( c \).

A straightforward case by case check shows that this deformation does not increase \( \kappa \). More precisely, there are 16 subcases to check. They arise as follows. First \( l(e_1) = a + 2\pi m \), \( 0 < a < \pi \) where \( m \geq 0 \) is even or odd, second the tangent vectors of \( e_0 \) and \( e_2 \) at \( p_0 \) and \( p_1 \), respectively, points into different components of \( \Sigma - G_1 \) or into the same component, third and fourth the lengths of the vertex curve \( \gamma_j \) at \( p_j \) satisfies \( l(\gamma_j) = a + 2\pi m \), \( 0 < a < \pi \), where \( m \geq 0 \) is even or odd, \( j = 0, 1 \). However, the fact that \( \kappa \) does not increase follows in all of these subcases from one of the following two computations.
First, with notation as in Figure 6, we calculate

\[ \beta' + (\pi - (\alpha - \alpha')) + \pi - \beta + \int_{\Omega} K dA = 2\pi. \]

Thus,

\[ \Delta \kappa = (\alpha' + \beta') - (\alpha + \beta) = -\int_{\Omega} K dA < 0. \]

Second, with notation as in Figure 7, we calculate

\[ \beta + (\pi - \beta') + (\pi - (\alpha - \alpha')) + \int_{\Omega} K dA = 2\pi. \]

Thus,

\[ \Delta \kappa = (\alpha' + \beta') - (\alpha + \beta) = \int_{\Omega} K dA - 2(\alpha - \alpha') < 0, \]

where the last inequality follows since \( 2(\alpha - \alpha') = \int_{\Gamma} K dA \) where \( \Gamma \) is the angular region between the geodesics connecting antipodal points and intersecting at an angle \( \alpha - \alpha' \), and since \( \Omega \subset \Gamma \).

In the second case \( l(e_1) = n\pi \) for some integer \( n > 0 \). In this case we rotate \( e_1 \) in such a way that the length of at least one of the vertex curves at the endpoints of \( e_1 \) decreases. The process stops when one of the vertex curves becomes an integral multiple of \( \pi \). This finishes the proof. \( \Box \)
Lemma 4.6 Let $c_0$ and $c_1$ be any two PGC–homotopic curves on a Riemann surface $\Sigma$ as in Lemma 4.5. Then there exists a PGC–homotopy $c_t$, $0 \leq t \leq 1$, connecting $c_0$ to $c_1$ with
\[
\kappa(c_t) \leq \min\{\kappa(c_0), \kappa(c_1), 2\pi\},
\]
for all $t$.

Proof Lemma 4.5 shows that it is enough to consider two curves with the properties of $c_1$ there. We first show how to deform such a curve to a multiple of a closed geodesic. Fix a first vertex $p$ and the orientation of the great circle of its incoming geodesic. Move the second vertex $q$ to the antipodal point of the first and rotate the arc which connects $p$ and $q$ an angle $\pi$. Note that after the rotation, the orientation of the arc agrees with the fixed one. If the orientation of the vertex curves at $p$ and $q$ are the same then this rotation removes two vertices and decreases $\kappa$. If the two vertex curves have opposite orientations this rotation does not change $\kappa$ and one vertex with vertex curve of length $2\pi$ is created. Splitting this new born vertex into two and repeating the above argument removes them. In this way, we eventually remove all vertices.

To finish the proof we need only show how to increase the number of times a curve encircles a geodesic by 2 not increasing $\kappa$ by more than $2\pi$. We use the following procedure. Create two curvature concentrations of length $\pi$ and of opposite orientations keeping $\kappa$ not larger than $2\pi$. Applying the procedure described above, we arrive at a curve which goes two more times around the geodesic. $\Box$
4.4 Curves with self-tangencies on the sphere

A generic circle immersion into a surface has only transverse double points. In generic regular homotopies there appear isolated instances of triple points and self-tangencies. The self-tangencies are of two kinds \textit{direct} when the tangent vectors at the tangency point agree and \textit{opposite} when they do not. Let \( \Sigma \) denote the \( 2 \)-sphere with a constant curvature metric throughout this subsection.

\textbf{Lemma 4.7} Any circle immersion \( c : S^1 \to \Sigma \) with an opposite self-tangency satisfies \( \kappa(c) > 2\pi \).

\textbf{Proof} Since \( c \) is not a geodesic we may decrease \( \kappa(c) \) by a small deformation, keeping the self-tangency, see the proof of Proposition 2.2. Let \( b \) be the curve resulting from such a deformation. Fix a partition \( \pi \) of the circle such that the PGC–approximation \( b^\pi \) of \( b \) satisfies \( \kappa(b^\pi) < \kappa(c) \), see Lemma 3.2. Let \( \tilde{b} \) be a PGC–curve close to \( b^\pi \) which contains two segments of the same geodesic close to the self-tangency point of \( c \) and such that \( \kappa(\tilde{b}) < \kappa(c) \).

Apply the deformation in the proof of Lemma 4.5 to \( \tilde{b} \) with the endpoint of one of the self-tangency segments as \( p_1 \). Note that as this process reaches the endpoint of the other self-tangency segment, the PGC–curve constructed must contain a geodesic segment with at least one vertex curve of length \( \pi \). Since the process does not increase \( \kappa \) and since the lift of a PGC–curve is closed, it follows that \( \kappa(\tilde{b}) \geq 2\pi \) and therefore \( \kappa(c) > 2\pi \). \( \square \)

In order to prove Theorem 1.1 (b) we are going to apply Arnol’d’s \( J^- \)-invariant of immersed curves on \( S^2 \). Arnol’d introduced his invariant for circle immersions in the plane, see [1]. Its existence for curves on more general surfaces was established by Inshakov [5] and Tchernov [9]. The existence of the \( J^- \)-invariant stems from the following fact: if \( c_0 \) and \( c_1 \) are two self-transverse regularly homotopic curves on \( S^2 \) then the algebraic numbers of opposite self-tangencies in any two generic regular homotopies \( c_t \), \( 0 \leq t \leq 1 \), connecting them are equal. This number is called the relative \( J^- \)-invariant of \( c_0 \) and \( c_1 \), we will denote it \( \Delta J^-(c_0, c_1) \). To compute the algebraic number of self-tangencies each self-tangency moment is equipped with a sign as follows: it is a positive moment if it increases the number of double points of the curve, otherwise it is negative.

\textbf{Lemma 4.8} Let \( k > 0 \) be an integer and consider the circle immersions \( S^1 \to \Sigma \) which go \( k \) and \( k + 2 \) times respectively around a geodesic (a great circle). Let \( c_0 \) and \( c_1 \) be any small perturbations of these curves. Then \( |\Delta J^-(c_0, c_1)| = 2 \).
Proof. Any sufficiently small perturbation of a curve going around a geodesic can be thought of as a multi-graph over the 0–section in a tubular neighborhood of a great circle. This implies that there exist deformations without opposite self-tangencies connecting any two such perturbations. To compute $\Delta J^-(c_0, c_1)$ it is thus sufficient to pick two perturbations and count the algebraic number of opposite self-tangencies in one regular homotopy connecting them. In Figure 8, the first picture illustrates a curve in a neighborhood of the north pole which is obtained by shrinking a perturbed multiple of the equator and in the first deformation one kink is pulled over the south pole. The lemma follows from Figure 8.

\[ \begin{array}{c}
\text{\includegraphics[width=0.7\textwidth]{figure8.png}} \\
\text{Figure 8: Computation of $\Delta J^-$}
\end{array} \]

4.5 Smoothing PGC–homotopies

The PGC–homotopies in Lemma 4.1, Lemma 4.3 and Lemma 4.5 have very special forms. We show that such PGC–homotopies can be made smooth in a standard way, increasing the total curvature arbitrarily little.

Consider first two unit vectors $v_{in} \in \mathbb{R}^2$ and $v_{out} \in \mathbb{R}^2$ and let $\gamma$ be a geodesic in $S^1$ connecting $v_{in}$ to $v_{out}$ and let $\delta > 0$ be given. Fix a family of reference curves $b(v_{in}, v_{out}, \gamma)$ inside the unit disk such that $b$ agrees with the straight line in direction $v_{in}$ ($v_{out}$) near the endpoints. The tangent map of $b$ is homotopic to $\gamma$ with endpoints
fixed and the total curvature of $b$ exceeds the length of $\gamma$ by at most $\delta$. Clearly there exists such families which depend continuously on the data for all $\delta > 0$.

We first discuss smoothing of a fixed PGC–curve. Let $c = (c_1, \gamma_1, \ldots, c_m, \gamma_m)$ be a PGC–curve such that all vertex curves are immersions. For sufficiently small $\epsilon > 0$, we define an $\epsilon$–smoothing of $c$ as follows. Fix a disk of radius $\epsilon > 0$ around each vertex of $c$ . Under the inverse of the exponential map at a vertex the curve $c$ looks like the model discussed above. More precisely, in the tangent space of the surface $\Sigma$ at a vertex $p$ we have an incoming unit tangent vector $v_{in}$ and an outgoing one $v_{out}$, and we glue in the curve $b(v_{in}, v_{out}, \gamma)$ in this tangent space. We then scale the glued in curve by $\epsilon$ and map it back into $\Sigma$ with the exponential map at $p$. Applying this procedure at each vertex we get a smoothing $\tilde{c}$ of $c$. Note that $\kappa(\tilde{c})$ can be made arbitrarily close to $\kappa(c)$ by choosing $\delta$ and $\epsilon$ sufficiently small. (The deviation from the flat case is measured by a curvature integral over a region with area going to 0 with $\epsilon$.) We illustrate this smoothing process in Figures 9 and 10.

![Figure 9: Smoothing of exterior angle](image1)

![Figure 10: Smoothing of interior angle](image2)

We next note that all PGC–homotopies in the proofs of Lemma 4.1, Lemma 4.3 and Lemma 4.5 have one of the following forms

- One geodesic segment of a PGC–curve moves along a consecutive segment.
- Two curvature concentrations of magnitude $\pi$ and with opposite orientations are created or annihilated somewhere along a PGC–curve.
- A homotopy of the form presented in the last part of Lemma 4.1.
We call such PGC–homotopies simple.

**Lemma 4.9** Let $c_t$, $0 \leq t \leq 1$, be a simple PGC–homotopy between two PGC–curves $c_0$ and $c_1$ with immersed vertex curves. Then, for any $\epsilon > 0$ there exists a regular homotopy $\tilde{c}_t$ connecting the $\epsilon$–smoothings $\tilde{c}_0$ and $\tilde{c}_1$ with

$$\max_{0 \leq t \leq 1} \kappa(\tilde{c}_t) \leq \max_{0 \leq t \leq 1} \kappa(c_t) + 10\epsilon.$$  

**Proof** The proof is straightforward. Consider a PGC–homotopy of the first type. We define $\tilde{c}_t$ as the $\epsilon$–smoothing of $c_t$ for $t$ outside a neighborhood of 1. Inside a neighborhood of 1 we may again use local flat models and define $\tilde{c}_t$ by composing with the exponential map. A picture of such a local model is shown in *Figure 11*. For the local model of a simple homotopy of the second type, see *Figure 12*.

![Figure 11: A regular homotopy near two meeting vertices](image)

![Figure 12: Creation of curvature concentrations](image)

For simple homotopies of the third kind one may simply take $\tilde{c}_t$ to equal the $\epsilon$–smoothing of $c_t$ for all $t$. $\square$
4.6 Locally convex curves on flat surfaces

Let $\Sigma$ be a flat Riemann surface ($\Sigma \approx T^2$, $\Sigma \approx S^1 \times \mathbb{R}$, or $\Sigma \approx \mathbb{R}^2$). For $p \in \Sigma$, $v \in U_p \Sigma$, and $(\xi, m) \in \pi_1(\Sigma) \times \mathbb{Z}$ let $\Omega_{(\xi,m)}(p,v)$ (or $\hat{\Omega}_{(\xi,m)}(p,v)$) denote the space of all (strictly) locally convex circle immersions $c: S^1 \to \Sigma$ of regular homotopy class $(\xi, m)$, see Section 1, such that $c(1) = p$ and $\dot{c}(1) = v$. (As usual we use the $C^2$–topology on the space of circle immersions.)

Lemma 4.10 The spaces $\Omega_{(\xi,m)}(p,v)$ and $\hat{\Omega}_{(\xi,m)}(p,v)$ are weakly contractible.

Proof We start in the simply connected case, $\Sigma = \mathbb{R}^2$. Let $F$ denote $\Omega_m(p,v)$ or $\hat{\Omega}_m(p,v)$. Let $\Gamma: S^n \to F$ be a continuous map from the $n$–sphere, $\Gamma(x) = c_x: S^1 \to \mathbb{R}^2$. We think of $S^1$ as an interval $[0, L]$ with endpoints identified. Thus $c_x(0) = p$ and $\dot{c}_x(0) = v$, for some fixed point $p$, some unit vector $v$, and all $x \in S^n$. To prove the lemma we must extend $\Gamma$ continuously to the $(n + 1)$–ball $B^{n+1}$.

Fix a small $\epsilon > 0$ and a unit vector $w$ such that $\langle v, w \rangle = 0$ and such that in the orientation of the plane induced by the basis $w, v$ the tangential degree of the curves $c_x$ are positive. We claim that there exist continuous maps $t_j: S^n \to \mathbb{R}$, $j = 1, 2$, with $0 < t_1(x) < t_2(x) < L$, and with the following properties: $\dot{c}_x(t_1(x))$ lies in the short sub-arc $A_{\epsilon}$ of $S^1$ between $(\cos \epsilon)v + (\sin \epsilon)w$ and $(\cos 2\epsilon)v + (\sin 2\epsilon)w$, $\dot{c}_x(t_2(x))$ lies in the short sub-arc $B_{\epsilon}$ between $(\cos \epsilon)v - (\sin \epsilon)w$ and $(\cos 2\epsilon)v - (\sin 2\epsilon)w$, $t_1(x)$ lies in the component of $\dot{c}_x^{-1}(A_{\epsilon})$ closest to 0, and $t_2(x)$ lies in the component of $\dot{c}_x^{-1}(B_{\epsilon})$ closest $L$.

In the strictly locally convex case this claim is obviously true: consider preimages under $\dot{c}_x$ of fixed points in $A_{\epsilon}$ and $B_{\epsilon}$ to define $t_1(x)$ and $t_2(x)$, respectively. To see that it holds also in the non-strictly locally convex case we argue as follows. By continuity, the subset $\Lambda \subset S^n \times S^1$,

$$\Lambda = \{(x, t): \dot{c}_x(t) \in \text{int}(A_{\epsilon})\},$$

where $\text{int}(X)$ denotes the interior of $X$, is open. In particular, for each $x \in S^n$ there exists $r_x > 0$ such that

$$\bigcap_{y \in B(x, r_x)} \dot{c}_y^{-1}(\text{int}(A_{\epsilon}))$$

contains an interval $I_x$. Cover $S^n$ by balls $B(x, r_x)$ with this property. This cover has a Lebesgue number $\delta > 0$. Triangulate $S^n$ by simplices which are so small that for every vertex in the triangulation, the union of all simplices in which this vertex lies is a subset of diameter less than $\delta$. It is then straightforward to construct $t_1: S^n \to \mathbb{R}$: if $v$ is a vertex take $t_1(v)$ as any point in $I_x$ where $x$ is some point such that the
union of all simplices containing \( v \) lies in \( B(x, r_x) \). By contractibility of the interval we can now inductively extend this function over higher dimensional skeleta of the triangulation. In the final step we get the desired function on \( S^n \).

Let \( q_x \) be the intersection point of the tangent lines to \( c_x \) at \( c_x(t_1(x)) \) and \( c_x(t_2(x)) \). Define an initial deformation of the curves \( c_x \) in the family which pushes \( c_x \) toward the piecewise linear curve obtained by replacing \( c_x([0, t_1(x)]) \cup c_x([t_2(x), L]) \) with the curve \( c_x(t_1(x))q_x \cup c_x(t_2(x))q_x \), where \( \overline{ab} \) denotes the line segment between \( a \) and \( b \). To stay in the space of pointed curves we also translate and slightly rotate the curves to ensure that they pass through \( p \) with the right tangent, see Figure 13. Clearly, this deformation can be made continuous in \( x \) and chosen in such a way that the resulting curves, still denoted \( c_x \), are strictly locally convex at \( c_x(0) = p \) for each \( x \in S^n \).

The next step is to deform the curves so that the marked point is a global minimum of the height function in direction \( w \). To this end, let \( T_j : S^n \to \mathbb{R}, j = 1, 2, 0 < T_1(x) < T_2(x) < L \) be continuous functions on \( S^n \) such that \( \partial_x(T_1(x)) (\partial_x(T_2(x))) \) lies in an \( \eta \)-arc \( C_\eta \) ending at \( w \) (beginning at \( -w \)) in the orientation of \( S^1 \) determined by \( \partial_x \) and such that \( T_1(x) (T_2(x)) \) lies in the component of \( \partial_x^{-1}(C_\eta) \) closest to 0 (to \( L \)). In the strictly locally convex case such functions are easily constructed using suitable points in the preimage \( \partial_x^{-1} (\pm w) \). In the non-strictly locally convex case, such functions can be constructed using the same arguments that were used in the construction of the functions \( t_j, j = 1, 2 \), just given. Let \( w_1(x) = \partial_x(T_1(x)) \) and let \( w_2(x) = -\partial_x(T_2(x)) \).

Pick \( M > 0 \) such that the minimum point \( c_x(q) \) in the \( w \)-direction of any curve \( c_x, x \in S^n \), satisfies \( \langle w, c_x(q) - p \rangle > -\frac{1}{2} M \). Pick \( \eta > 0 \) sufficiently small so that the intersection point \( r \) of the lines in direction \( w_1(x) \) and \( w_2(x) \) passing through \( c_x(T_1(x)) \) and \( c_x(T_2(x)) \) respectively satisfies \( \langle w, r - c_x(q) \rangle < -20 M \). Consider the subdivision of the source circle of \( c_x \) into two arcs \( D_1(x) \) and \( D_2(x) \) as follows: the endpoints of the arcs are \( T_1(x) \) and \( T_2(x) \), \( D_1(x) \) contains the marked point, and \( D_2(x) \) does not contain it.
For $0 \leq s \leq \frac{1}{2}$, define $c_{sx}$ as the curve which consists of the following four pieces: the curve $c_x(D_2(x))$, the line segment $l_1^s$ in direction $w_1(x)$ starting at $c_x(T_1(x))$ and such that the projection of this segment to a line in the $w$–direction has length $10sM$, a suitably scaled version of the translate of the curve $c_x(D_1(x))$ along $l_1^s$, which is tangent to the line through $c_x(T_2(x))$ in direction $w_2(x)$, and finally the line segment $l_2^s$ in direction $-w_2(x)$ connecting the scaled and translated $c_x(D_1(x))$ to $c_x(D_2(x))$ at $c_x(T_2(x))$, see Figure 14. In the case when $F$ is a space of strictly convex curves

![Figure 14](image)

Figure 14: Making the marked a global extremum

we replace the straight line segments in $c_{sx}(t)$ above with slightly curved circular arcs. (In fact, since $S^n$ is compact we can take these arcs to have curvature smaller than $\min_{x \in S^n} \min_{t \in [0,L]} (|k_2(t)|)$.) Finally, to have the curves mapping the marked point to $p$ we also compose with a suitable translation.

The second step in the deformation takes all the curves in the family to curves with image in a large circle. Let $C$ be a circle through $p$ with tangent $v$ at $p$. Let $D$ be the bounded component of the plane with boundary $C$. If the radius of $C$ is sufficiently big then we may choose $C$ so that all curves $c_{\frac{1}{2}x}$ has image in $D$ and $C \cap c_{\frac{1}{2}x}([0,L]) = c_{\frac{1}{2}x}(0)$. For $0 \leq s \leq 1$ let $c_{(-\frac{s}{2} + \frac{1}{2})x}$ be the continuous family of convex curves which is the union of a curve in $C$ starting at $p$ and ending at the intersection point of the negative tangent half-line of $c_{\frac{1}{2}x}$ at $c_{\frac{1}{2}x}(sL)$, follows this
half-line, and then goes along $c_{\frac{1}{2}x}$. For $s = 1$, we get a curve wrapping around $C$, see Figure 15. The construction is continuous in $x$ and therefore gives a continuous extension of the map $\Gamma: S^n \to F$ over $B^{n+1}$, as desired. (Again, in the case of strictly locally convex curves we replace the tangent half-line with a very slightly curved circular arc.)

We next consider the non-simply connected case. In this case we have a covering $\Pi: \mathbb{R}^2 \to \Sigma$ where the deck transformations are translations. We lift the curves $c_x$ at the marked point. In the case when $F$ is the space of non strictly locally convex curves a similar local deformation as the initial deformation above makes all curves strictly locally convex at the marked point. Let $b_x$ denote the lifted curves. We have $b_x(0) = 0$, $b_x(L) = q$, $\dot{b}_x(0) = v = \hat{b}_x(L)$.

Let $l_0$ and $l_q$ be the straight lines through $0$ and $q$ respectively with tangent vector $v$. By strict local convexity, for one of the unit vectors $w$ orthogonal to $v$, we have $\langle b_x(t), w \rangle > 0$ for $t$ in a (punctured) neighborhood of $0$ and for $t$ in a (punctured) neighborhood of $L$. For convenience, assume that $\langle q, w \rangle > 0$ (otherwise change coordinates in $\mathbb{R}^2$ so that $q = 0$). We construct $T_j: S^n \to \mathbb{R}$, $j = 1, 2$, analogous to the functions with the same names above, in a similar way as above so that $\dot{b}_x(T_1(x)) \approx w$ and so that $\dot{b}_x(T_2(x)) \approx -w$. As above, we add straight line segments to $b_x$ so that $b_x(0)$ (or $b_x(q)$) is the global minimum of the height function in direction $w$ and so that $b_x$ intersects the region between the two lines $l_0$ and $l_q$ in an arc.

Pick two circles $C_0$ and $C_q$ through $0$ and $q$, respectively, both with tangents $v$ at these points. Let $D_0$ and $D_q$ be the bounded components of the plane which are
bounded by $C_0$ and $C_q$, respectively. If the circles have sufficiently large radii then $b_X([0, L]) \subset D_0$ for each $x$ and $b_X$ intersects $D_0 - D_q$ in an arc. Let $C$ be the boundary of the convex hull of $C_0$ and $C_q$. As above, we deform $b_x$ using its negative tangent half-line. For $0 \leq s \leq \tau$ where $\tau$ is the smallest number such that the negative tangent half-line at $b_X(\tau)$ intersects $C$ in a point in $C_q$. We let $b_{sx}$ be a part of $C$ followed by the negative tangent half-line, in turn followed by the rest of $b_x$. For $s \geq \tau$ we let $b_{sx}$ be the curve with initial part as above followed by a curve in $C_q$, in turn followed by the tangent half-line, and finally the rest of $b_x$. At $s = L$ we find that every curve in the family is a curve which is a segment in $C$ followed by a curve with image in $C_q$, see Figure 16. In the case of strictly locally convex curves it is easy to modify the deformation described so that it keeps all curves strictly locally convex. This finishes the proof.

**Lemma 4.11** The inclusion $\hat{\Omega}_{(\xi, m)} \subset \Omega_{(\xi, m)}$ induces surjections

$$\pi_r(\hat{\Omega}_{(\xi, m)}) \to \pi_r(\Omega_{(\xi, m)}).$$

on homotopy groups, for all $r$.

**Proof** The proof of Lemma 4.10 homotopes an arbitrary family of (based) locally convex curves to a family of strictly locally convex curves. Hence we need only consider what happens to the base point. Let $\Gamma: S^n \to \Omega_{(\xi, m)}$ be a family of curves. First use the initial deformation of the proof above to make all curves strictly locally convex in a neighborhood of $1 \in S^1$. With this accomplished we lift all curves in the family to $\mathbb{R}^2$, by lifting at $c(1)$. (If $n \geq 2$ then we can lift the whole family in this way however when $n = 1$ the start- and endpoints of our lifts may differ by a translation.) Now apply the procedure of the proof of Lemma 4.10. Note that (in the case $n = 1$) the procedure behaves well with respect to translations. Hence, any $\Gamma: S^n \to \Omega_{(\xi, m)}$ can be homotoped to a map $\Gamma': S^n \to \hat{\Omega}_{(\xi, m)}$. This finishes the proof. \qed
5 Proofs

In this section we prove the theorems stated in Section 1.

5.1 Infima

Proof of Theorem 1.2 Consider first case (c). A well-known theorem of Lyusternik and Fet [7] says that the 2–sphere with any metric (actually any closed Riemannian manifold) has a non-constant simple closed geodesic, see also Jost [6, Section 5.5]. This geodesic traversed once and twice respectively gives representatives with $\kappa = 0$ for both regular homotopy classes on the 2–sphere.

In case (b) it is easy to construct a locally convex curve in any regular homotopy class $(\xi, m)$ with $\xi \neq *$ or $m \neq 0$. Note that the class $(*, 0)$ can neither be represented by a closed geodesic nor by a locally convex curve. Hence $\kappa (c)$ can be decreased for each $c$ of regular homotopy class $(*, 0)$. Moreover, the curvature decreasing procedure in the proof of Lemma 4.1 gives in this case a $PGC$–curve with underlying curve a geodesic segment. Since the lift of that $PGC$–curve is closed it must have at least two curvature concentrations of magnitude $\pi$. Hence $\hat{\kappa} (\ast, 0) \geq 2\pi$. Approximating a segment traversed twice with two vertex curves which are rotations $\pi$ and $-\pi$ we find that $\hat{\kappa} (\ast, 0) = 2\pi$.

Finally, in case (a), it follows as above that the infimum is not attained in classes not representable by geodesics. To find the infima in case $\Sigma \approx \mathbb{R}^2$ we apply Lemma 4.3 to conclude that it is enough to find the minimal $\kappa$ of a generalized $PGC$–curve in standard form in a given regular homotopy class. It is straightforward to check that a curve in standard form representing the regular homotopy class $m$ has two vertex curves of length $\pi$ if $|m| = 0$ and $|m| + 1$ vertex curves of length $\pi$ otherwise.

5.2 Locally convex curves

Proof of Theorem 1.3 Let $v$ be a (covariantly) constant unit vector field on $\Sigma$. Let $\hat{\Omega}_{(\xi, m)} (v)$ and $\hat{\Omega}_{(\xi, m)} (v)$ denote the space of strictly locally convex–, respectively, locally convex curves $c$ with $\dot{c}(1) = v$. Consider the evaluation maps $e (c) = c(1)$,

$$e : \hat{\Omega}_{(\xi, m)} (v) \to \Sigma \quad \text{and} \quad e : \hat{\Omega}_{(\xi, m)} (v) \to \Sigma.$$ 

These maps are clearly Serre fibrations: using translations we can lift any map from an $n$–disk into $\Sigma$. The fibers of these fibrations are $\hat{\Omega}_{(\xi, m)} (p, v)$ and $\Omega_{(\xi, m)} (p, v)$, respectively, which are both weakly contractible by Lemma 4.10. Hence

$$\pi_r (\hat{\Omega}_{(\xi, m)} (v)) \approx \pi_r (\Sigma) \approx \pi_r (\Omega_{(\xi, m)} (v))$$

Algebraic & Geometric Topology, Volume 6 (2006)
with isomorphisms induced by evaluation.
Assume \( m \neq 0 \) and consider the fibration
\[
e'(\hat{\Omega}(\xi,m)) \to S^1,
\]
\( e'(c) = \hat{c}(0) \). This is a Serre fibration since \( \hat{c}: S^1 \to S^1 \) is a covering map (here it is essential that the curves are strictly locally convex). Since the fiber of \( e' \) is \( \hat{\Omega}(\xi,m)(v) \) we find that
\[
\pi_r(\hat{\Omega}(\xi,m)) = \pi_r(U \Sigma),
\]
with isomorphism induced by the evaluation map. Since \( \pi_r(U \Sigma) = 0 \) if \( r > 1 \) we conclude from Lemma 4.11 that \( \pi_k(\hat{\Omega}(\xi,m)) = \pi_k(\hat{\Omega}(\xi,m)) \) for all \( k \).

We finally consider \( m = 0 \). In this case the space under consideration is the space of closed geodesics in a fixed homotopy class. If \( \Sigma \approx S^1 \times \mathbb{R} \) any element in such a space is uniquely determined by its intersection with \( \mathbb{R} \times \{1\} \) and \( \pi_{S^1}(c(1)) \), where \( \pi_{S^1}: S^1 \times \mathbb{R} \to S^1 \) is the natural projection. Thus \( \hat{\Omega}(\xi,0) \simeq \Sigma \). Similar arguments give the same result when \( \Sigma \approx T^2 \).

### 5.3 Regular homotopies

**Proof of Theorem 1.1** Consider first case (a). If \( \Sigma \) is flat and both \( c_0 \) and \( c_1 \) are closed geodesics or locally convex curves then the theorem follows from Theorem 1.3.

In all other cases we may first decrease the total curvature a little by Proposition 2.2 and then approximate by a PGC–curve as in Lemma 3.2. In the flat case the theorem then follows from Lemma 4.1, which allows us to deform both curves to a standard form without increasing \( \kappa \), Lemma 4.2 which allows us to connect these curves, and Lemma 4.9 which allows us to smooth the entire homotopy keeping control of \( \kappa \). In the case of negative curvature the theorem follows in a similar way from Lemmas 4.3, 4.4 and 4.9.

In case (b) we argue in the same way to prove the first statement using Lemmas 3.2, 4.5, 4.6 and 4.9. The second statement follows from Lemma 4.8 which shows that any regular homotopy \( c_t \), \( 0 \leq t \leq 1 \), connecting the two multiple geodesics of different multiplicities must have an instant \( c_T \) which is a curve with an opposite self-tangency and Lemma 4.7 which shows that \( \kappa(c_T) > 2\pi \).

We end the paper by demonstrating that Theorem 1.1 (a) does not hold for Riemann surfaces with metrics of non-constant curvature with \( K \leq 0 \).

**Remark 5.1** Consider the upper half-plane with coordinates \((x, y), \ y > 0\) and metric
\[
ds^2 = e^{2f(y)}(dx^2 + dy^2),
\]
where \( f: (0, \infty) \to \mathbb{R} \) is a (weakly) convex function such that its derivative \( f' \) satisfies \( f'(y) = 0 \) for \( y \in [1, 2] \cup [3, 4] \), and such that its second derivative \( f'' \) satisfies \( f''(y) > 0 \) for \( y \in (0, 1) \cup (4, \infty) \) and such that \( f(y) = -\log(y) \) for \( y \) in some neighborhood of 0 and of \( \infty \).

Let \( A_0 = \mathbb{R} \times [1, 2] \), \( A_1 = \mathbb{R} \times [3, 4] \), and \( B = \mathbb{R} \times (0, \infty) - (A_0 \cup A_1) \). Then \( K(p) = 0 \) for \( p \in A_0 \cup A_1 \) and \( K(p) < 0 \) for \( p \in B \). Let \( c_0 \) and \( c_1 \) be convex curves in \( A_0 \) and \( A_1 \), respectively. Then \( c_0 \) and \( c_1 \) are regularly homotopic. We claim that for every regular homotopy \( c_t \), \( 0 \leq t \leq 1 \) connecting \( c_0 \) to \( c_1 \), there exists an instant \( c_t \) with \( \kappa(c_t) > 2\pi \). To see this note that as long as the curve \( c_t \) stays in \( A_0 \) it must remain convex, otherwise \( \kappa > 2\pi \). In particular, the curve must remain embedded. Let \( t_0 \) be the last moment when the curve lies completely inside the closure of \( A_0 \). Since embeddedness is an open condition we see that \( c_t \) is embedded for all \( \tau > t_0 \) sufficiently close to \( t_0 \). Note that \( c_{t_0} \cap \partial A_0 \neq \emptyset \). Pick some line \( l \) parallel to \( \partial A_0 \), and close to a point in \( c_{t_0} \cap \partial A_0 \) and such that \( c_{t_0} \) intersects it transversely in two points. (The existence of such a line follows from Sard’s lemma.) Then also \( c_\tau \) meets \( l \) transversely for \( \tau \) sufficiently close to \( t_0 \). Applying the Gauss–Bonnet theorem to the two curves bounded by the bounded segment of \( l \) cut out by \( c_\tau \) and the two remaining pieces of \( c_\tau \) we find \( \kappa(c_\tau) > 2\pi \).

References


Department of mathematics, USC
Los Angeles CA 90803, USA

tekholm@usc.edu

Received: 8 February 2005 Revised: 22 February 2006