Poincaré submersions

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Abstract We prove two kinds of fibering theorems for maps $X \to P$, where $X$ and $P$ are Poincaré spaces. The special case of $P = S^1$ yields a Poincaré duality analogue of the fibering theorem of Browder and Levine.

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1 Introduction

One of the early successes of surgery theory was the fibering theorem of Browder and Levine [B-L], which gives criteria for when a smooth map $f: M \to S^1$ is homotopic to a submersion. Here $M$ is assumed to be a connected closed, smooth manifold of dimension $\geq 6$, and we also require $f$ to induce an isomorphism of fundamental groups. The Browder-Levine fibering theorem then says that $f$ is homotopic to a submersion if and only if the homotopy groups of $M$ are finitely generated in each degree.

The purpose of the current note is to prove fibering results in the Poincaré duality category. Note that a submersion of closed manifolds is a smooth fiber bundle with closed manifold fibers. Replacing the closed manifolds with finitely dominated Poincaré spaces and the fiber bundle with a fibration yields the notion of Poincaré submersion: this is a map between Poincaré spaces whose homotopy fibers are Poincaré spaces.

Our first result concerns the case when the target is acyclic (this includes the Browder-Levine situation). Let $X$ be a connected, finitely dominated Poincaré duality space of (formal) dimension $d$ and fundamental group $\pi$. Let

$$f: X \to B\pi$$

be the classifying map for the universal cover of $X$. We will be assuming that the classifying space $B\pi$ is a finitely dominated Poincaré space of dimension $p$. 
**Theorem A** Let $F$ denote the homotopy fiber of $f$. Then $F$ is a homotopy finite Poincaré duality space of dimension $d - p$ if and only if the homotopy groups of $X$ are finitely generated in each degree.

For our second result, let $f : X \to P$ be a map of orientable, finitely dominated and connected Poincaré duality spaces. Assume $X$ has dimension $d$ and $P$ has dimension $p$. We will give criteria for deciding when the homotopy fiber $F$ of $f$ satisfies Poincaré duality.

Let $i : F \to X$ be the evident map. There is an umkehr homomorphism

$$i^! : H_*(X) \to H_{*+p}(F)$$

which is defined if $p \geq 3$ or if $P$ is 1-connected (cf. §4). The pushforward of a fundamental class $[X] \in H_d(X)$ for $X$ with respect to $i^!$ then gives a class

$$x_f := i^!([X]) \in H_{d-p}(F).$$

This will be our candidate for a fundamental class of $F$.

**Theorem B** Assume that $f$ is 2-connected. Then the following are equivalent:

1. $H_*(F) = 0$ in sufficiently large degrees.
2. $F$ is homotopy finite.
3. $F$ is a Poincaré duality space.

If in addition $X$ is 1-connected, then the above are equivalent to the assertion that

4. the homomorphism

$$\cap x_f : H^*(F) \to H_{d-p-\ast}(F)$$

is an isomorphism in all degrees.

**Remark** When $P = S^p$ is a sphere, (1) $\Rightarrow$ (3) overlaps with [C, lemma 1.1]. The implication (2) $\Rightarrow$ (3) is a consequence of [Kl1, theorem B].

We do not a priori assume that Poincaré duality spaces satisfy a finiteness condition, so the implication (3) $\Rightarrow$ (2) is non-trivial.

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1Correction added June, 2005: If $X$ is not 1-connected, one also requires the hypothesis that the homotopy groups of $X$ are finitely generated. I am indebted to Jonathan Hillman for pointing out that a hypothesis was missing here. Hillman also communicated to me the following counterexample: take $X$ to be the connected sum of $S^5 \times S^1$ with $S^3 \times S^3$ and let $f : X \to S^1$ classify the universal cover. Then $\pi_3(F)$ is infinitely generated.
Conventions A space is homotopy finite if has the homotopy type of a finite cell complex. A space is finitely dominated if it is the retract of a homotopy finite space.

A Poincaré space of formal dimension \(d\) is a space \(X\) for which there exists a pair \((\mathcal{L}, [X])\) consisting of a rank one abelian system of local coefficients \(\mathcal{L}\) on \(X\) and a (fundamental) class \([X] \in H_d(X; \mathcal{L})\) such that the cap product homomorphism
\[
\cap[X]: H^*(X; A) \rightarrow H_{d-*}(X; \mathcal{L} \otimes A)
\]
is an isomorphism, for all local coefficient modules \(A\) on \(X\) (cf. [W1], [Kl2]). If \(X\) is connected, then it is enough to establish the isomorphism when \(A\) is the integral group ring of the fundamental group of \(X\). When the local system \(\mathcal{L}\) is constant, we say that \(X\) is orientable. We do not at assume any finiteness conditions in the definition of Poincaré space appearing here. However, in the 1-connected case, homotopy finiteness is actually a consequence of Poincaré duality (see 3.2 below).

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2 Proof of Theorem A

We first prove the ‘only if’ part. Assume that \(F\) is a homotopy finite Poincaré space. Since \(F\) is 1-connected and homotopy finite, we infer that its homology is finitely generated. Apply the mod \(\mathbb{C}\) Hurewicz theorem (with \(\mathbb{C}\) = the Serre class of finitely generated abelian groups) to see that the homotopy groups of \(F\) are finitely generated [S, corollary 9.6.16].

We now prove the ‘if’ part. Note that \(F\) has the homotopy type of the universal cover of \(X\), so \(F\) is homotopy finite dimensional because \(X\) is. By the long exact homotopy sequence and the fact that \(\pi_*(X)\) is degreewise finitely generated, we infer that \(\pi_*(F)\) is degreewise finitely generated. Since \(F\) is simply connected, the mod \(\mathbb{C}\) Hurewicz theorem shows that the homology groups of \(F\) are finitely generated. By a result of Wall [W2], we see that \(F\) is homotopy finite.

We now know that each space in the homotopy fiber sequence
\[
F \rightarrow X \rightarrow B\pi
\]
is finitely dominated. It follows directly from [Kl1, theorem B] (see also [G]) that $F$ satisfies Poincaré duality and has formal dimension $d - p$. This completes the proof of Theorem A.

\[\square\]

3 Duality and finiteness

A chain complex $C$ of abelian groups is said to be dualizable if there is chain complex $D$ and a map

\[d: Z \to C \otimes D\]

($\otimes =$ derived tensor product, and $d$ is allowed to be degree shifting) such that, for all $P$, we get that the induced map of complexes

\[\text{hom}(C, P) \to \text{hom}(Z, P \otimes D)\]

(derived hom) given by $f \mapsto (f \otimes 1_D) \circ d$ induces an isomorphism on homology, where $1_D$ denote the identity map of $D$.

A chain map $C \to D$ is said to be a weak equivalence if it induces an isomorphism in homology. More generally $C$ and $D$ are said to be weak equivalent if there is a finite sequence of weak equivalences starting at $C$ and ending at $D$.

A chain complex is (chain) homotopy finite if it is weak equivalent to a finite chain complex, i.e., a complex of finite rank free abelian groups with finitely many non-trivial degrees. A chain complex is finitely dominated if is a retract up to homotopy of a finite chain complex. It is well-known chain complex over $Z$ is homotopy finite if and only if it is finitely dominated (see [W2]).

Lemma 3.1 If $C$ is dualizable, then it is homotopy finite over $Z$.

Proof Since $Z$ is “compact,” there exists a finite chain complex $C_0$, a map $i: C_0 \to C$ and a map $d_0: Z \to C_0 \otimes D$ such that

\[\begin{array}{c}
Z \\
\downarrow d_0
\end{array} \to \begin{array}{c}
C_0 \\
\downarrow i
\end{array} \otimes D \to \begin{array}{c}
C \\
\downarrow i
\end{array} \otimes D
\]

is homotopic to $d$. Consider the homotopy commutative diagram

\[\begin{array}{ccc}
\text{hom}(C, C) & \xrightarrow{\text{(-\otimes 1_C)od}} & \text{hom}(Z, C \otimes D) \\
\uparrow i_* & & \uparrow i_* \\
\text{hom}(C, C_0) & \xrightarrow{\text{(-\otimes 1_C)od}} & \text{hom}(Z, C_0 \otimes D)
\end{array}\]
The map \( d_0 \) lives in the lower right corner and maps to \( d \) under the right vertical map. The map \( 1_C \) maps to \( d \) under the top horizontal map. Since the lower horizontal map is an equivalence, we get a map \( j: C \to C_0 \) such that \( i_* (j) = j \circ i \) is homotopic to \( 1_C \). We conclude that the identity map of \( C \) factors up to homotopy through the finite object \( C_0 \).

Note now if \( X^d \) is a 1-connected space which is equipped with a chain level fundamental class \([X]\) for which Poincaré duality holds, then \( C(X) = \text{the singular chains on } X \) is dualizable using the maps

\[
\mathbb{Z} \xrightarrow{[X]} C(X) \xrightarrow{\text{diagonal}} C(X \times X) \simeq C(X) \otimes C(X),
\]

where the first map is the homomorphism (of degree \( d \)) induced by a choice of fundamental class. By the above lemma, we infer that \( C(X) \) is homotopy finite.

A result of Wall says that a 1-connected space is homotopy finite if and only if its chain complex is (chain) homotopy finite (see [W3]). Hence,

**Corollary 3.2** Let \( X \) be a 1-connected space which satisfies Poincaré duality. Then \( X \) is also homotopy finite.

### 4 The umkehr homomorphism

According to [W1, theorem 2.4], if \( \dim P \geq 3 \) is a Poincaré duality space, then there is a homotopy equivalence

\[
P \simeq P_0 \cup_{\alpha} D^p,
\]

in which \( P_0 \) is a CW complex of dimension \( \leq p-1 \). If \( P \) is 1-connected, then \( P_0 \) has the homotopy type of a CW complex of dimension \( \leq p-2 \). If \( P \) has dimension \( \leq 2 \), then \( P \simeq S^p \), and the above decomposition is also available.

Furthermore, once an orientation for \( P \) has been chosen, the above cell decomposition is unique up to oriented homotopy equivalence. From now on, we fix an identification \( P := P_0 \cup D^p \), where \( \dim P_0 \leq p-1 \).

Without loss in generality, let us assume that \( f: X \to P \) has been converted into a Hurewicz fibration. Let \( X_0 = f^{-1}(P_0) \). Then we obtain a pushout square

\[
\begin{array}{ccc}
 f^{-1}(S^{p-1}) & \to & X_0 \\
 \downarrow & & \downarrow \\
 f^{-1}(D^p) & \to & X.
\end{array}
\]
Using the homotopy lifting property, we see that the pair \((f^{-1}(D^p), f^{-1}(S^{p-1}))\) has the homotopy type of the pair \((F \times D^p, F \times S^{p-1})\). Taking vertical cofibers in the diagram, we get an umkehr map

\[
i^!_1: \ X \longrightarrow X/X_0 = f^{-1}(D^p)/f^{-1}(S^{p-1}) \simeq F_+ \wedge S^p
\]

The \textit{umkehr homomorphism}

\[
i^*_s: H_*(X) \to H_{*-p}(F)
\]

is the effect of applying singular homology to \(i^!_1\), and using the suspension isomorphism to perform the degree shift.

5 Proof of Theorem B

(1) \(\Rightarrow\) (2) By the long exact homotopy sequence of the fibration, we see that \(\pi_*(F)\) is degreewise finitely generated. By the mod \(C\) Hurewicz theorem, we infer that \(H_*(F)\) is finitely generated. Then \(F\) is homotopy finite by [W2].

(2) \(\Rightarrow\) (3) Follows from [KI1, theorem B].

(3) \(\Rightarrow\) (1) This follows from 3.2.

For the remainder of the proof of the theorem, we suppose that \(X\) is 1-connected. Then so are \(F\) and \(P\).

(3) \(\Rightarrow\) (4) It will be enough to show that the class \(x_f\) is a generator of \(H_{d-p}(F) \cong \mathbb{Z}\). By definition of \(x_f\), this is equivalent to knowing that the homomorphism

\[
i^!_s: H_d(X) \to H_{d-p}(F)
\]

is of degree \(\pm 1\).

This can be seen as follows: the space \(X_0\) is the pullback of the fibration \(f: X \to P\) along a CW complex \(P_0\) of dimension \(\leq p-2\) (this uses the fact that \(P\) is 1-connected, cf. §4). As \(F\) has formal dimension \(\leq d-p\), it is straightforward to check that \(X_0\) has the homotopy type of a CW complex of dimension \(\leq d-2\). Using the homotopy cofiber sequence

\[
X_0 \longrightarrow X \xrightarrow{i^!} F_+ \wedge S^p
\]

and the fact that the homology of \(X_0\) vanishes above degree \(d-2\), we see that \(i^!\) induces an isomorphism in homology in degree \(d\).

(4) \(\Rightarrow\) (3) Trivial. \(\square\)
References


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