A lower bound to the action dimension of a group

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Abstract The action dimension of a discrete group $\Gamma$, $\text{actdim}(\Gamma)$, is defined to be the smallest integer $m$ such that $\Gamma$ admits a properly discontinuous action on a contractible manifold. If no such $m$ exists, we define $\text{actdim}(\Gamma) = 1$. Bestvina, Kapovich, and Kleiner used Van Kampen's theory of embedding obstruction to provide a lower bound to the action dimension of a group. In this article, another lower bound to the action dimension of a group is obtained by extending their work, and the action dimensions of the fundamental groups of certain manifolds are found by computing this new lower bound.

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1 Introduction

Van Kampen constructed an $m$-complex that cannot be embedded into $\mathbb{R}^{2m}$ [8]. A more modern approach to Van Kampen's theory of embedding obstruction uses co/homology theory. To see the main idea of this co/homology theoretic approach, let $K$ be a simplicial complex and $jK_j$ denote its geometric realization. Define the deleted product

$$jK_j \rightarrow (x, y) \rightarrow jK_j \times_{\mathbb{Z}_2} y$$

such that $\mathbb{Z}_2$ acts on $jK_j$ by exchanging factors. Observe that there exists a two-fold covering $jK_j \rightarrow \mathbb{S}^1$ with the following classifying map:

$$jK_j \rightarrow \mathbb{S}^1$$

$$y \rightarrow y$$

$$jK_j \rightarrow \mathbb{R}P^1$$
Now let $m^2 \mathbb{H}^m(\mathbb{R}P^1; \mathbb{Z}_2)$ be the nonzero class. If $(w^m) \not\in 0$ then $jK_j$ cannot be embedded into $\mathbb{R}^m$. That is, there is $m^2 \mathbb{H}_m((jK_j)_Z; \mathbb{Z}_2)$ such that $h(w^m); i \not\in 0$.

A similar idea was used to obtain a lower bound to the action dimension of a discrete group $\Gamma$ [2]. Specifically, the obstructor dimension of a discrete group $\Gamma$, $\text{obdim}(\Gamma)$, was defined by considering an $m\{\text{obstructor } K \text{ and a proper, Lipschitz, expanding map}

$$f : \text{cone}(K)^{(0)} \rightarrow \Gamma$$

And it was shown that

$$\text{obdim}(\Gamma) \leq \text{actdim}(\Gamma).$$

See [2] for details. An advantage of considering $\text{obdim}(\Gamma)$ becomes clear when $\Gamma$ has well-defined boundary $\partial \Gamma$, for example, when $\Gamma$ is $\text{CAT}(0)$ or torsion free hyperbolic. In these cases, if an $m\{\text{obstructor } K \text{ is contained in } \partial \Gamma$ then $m + 2 \text{ obdim}(\Gamma)$.

If $\Gamma$ acts on a contractible $m\{\text{manifold } W$ properly discontinuously and cocompactly, then it is easy to see that $\text{actdim}(\Gamma) = m$. For example, let $M$ be a Davis manifold. That is, $M$ is a closed, aspherical, four-dimensional manifold whose universal cover $M^*$ is not homeomorphic to $\mathbb{R}^4$. We know that $\text{actdim}(\text{1}(M)) = 4$. However, it is not easy to see that $\text{obdim}(\text{1}(M)) = 4$.

The goal of this article is to generalize the definitions of obstructor and obstructor dimension. To do so, we define proper obstructor (Definition 2.5) and proper obstructor dimension (Definition 5.2). The main result is the following.

**Main Theorem** The proper obstructor dimension of $\Gamma = \text{actdim}(\Gamma)$.

As applications we will answer the following problems:

Suppose $W$ is a closed aspherical manifold and $W$ is its universal cover so that $\text{1}(W)$ acts on $W$ properly discontinuously and cocompactly. We show that $W$ in this case is indeed an $m\{\text{proper obstructor and } \text{obdim}(\text{1}(W)) = m$.

Suppose $W_i$ is a compact aspherical $m\{\text{manifold with all boundary components aspherical and incompressible, } i = 1; \cdots; d$. (Recall that a boundary component $N$ of a manifold $W$ is called incompressible if $i : j(N) ! j(W)$ is injective for $j \neq 1$.) Also assume that for each $i, 1 \leq i \leq d$, there is a component of $\partial W_i$, call it $N_i$, so that $j_1(W_i) : j_1(N_i) \geq 2$. Let $G = \text{1}(W_1) \cdots \text{1}(W_d)$. Then

$$\text{actdim}(G) = m_1 + \cdots + m_d.$$
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The organization of this article is as follows. In Section 2, we define proper obstructor. The coarse Alexander duality theorem by Kapovich and Kleiner [5], is used to construct the first main example of proper obstructor in Section 3. Several examples of proper obstructors are constructed in Section 4. Finally, the main theorem is proved and the above problems are considered in Sections 5.

2 Proper obstructor

To work in the PL-category we define simplicial deleted product
\[ K \sim f \quad 2K \] 
\[ Kj \quad = \] 
\[ g \]
such that \( \mathbb{Z}_2 \) acts on \( K \sim \) by exchanging factors. It is known that \( jKj=\mathbb{Z}_2 \) \((jKj)\) is a deformation retract of \( K\sim=\mathbb{Z}_2 \) \((K\sim)\), see [7, Lemma 2.1]. Therefore, WLOG, we can use \( H_m(K\sim=\mathbb{Z}_2;\mathbb{Z}_2) \) instead of \( H_m(jKj=\mathbb{Z}_2;\mathbb{Z}_2) \).

Throughout the paper, all homology groups are taken with \( \mathbb{Z}_2 \) coefficients unless specified otherwise.

To define proper obstructor, we need to consider several definitions and preliminary facts.

**Definition 2.1** A proper map \( h: A \to B \) between proper metric spaces is uniformly proper if there is a proper function \( : [0; 1) \to [0; 1) \) such that
\[ d_B(h(x); h(y)) \geq \frac{1}{d_A(x; y)} \]
for all \( x; y \in A \). (Recall that a metric space is said to be proper if any closed metric ball is compact, and a map is said to be proper if the preimages of compact sets are compact.)

Let \( W \) be a contractible \( m \)-manifold and define
\[ W_0 \quad f(x; y) \quad 2W \quad Wj x \quad \mathcal{g}; y: \]
Consider a uniformly proper map \( : Y \to W \) where \( Y \) is a simplicial complex and \( W \) is a contractible manifold. Since \( Y \) is uniformly proper, we can choose \( r > 0 \) such that \( (a) \not\in (b) \) if \( d_Y (a; b) > r \). Note that \( \) induces an equivariant map:
\[ f(y; y) \quad 2Y \quad Yj d_Y (y; y) > r \mathcal{g} \]
As we work in the PL-category we make the following definition.
Definition 2.2 If $K \subseteq Y$ is a subcomplex and $r$ is a positive integer then we define the combinatorial $r$-tubular neighborhood of $K$, denoted by $N_r(K)$, to be $r$-fold iterated closed star neighborhood of $K$.

Recall that when $Y$ is a simplicial complex, $j_Y: j_Y$ can be triangulated so that each cell is a subcomplex. Let $d: Y \rightarrow Y^2$ be the diagonal map, $d(\cdot) = (\cdot \; \cdot)$, where $Y^2$ is triangulated so that $d(Y)$ is a subcomplex. Define

$$Y_r = \text{Cl}(Y^2 - N_r(d(Y))).$$

Note that a uniformly proper map $\mathcal{Y} \rightarrow W$ induces an equivariant map $\mathcal{Y} \rightarrow W_0 \rightarrow S^{m-1}$ for some $r > 0$.

Definition 2.3 (Essential $\mathbb{Z}_2 - m$-cycle) An essential $\mathbb{Z}_2 - m$-cycle is a pair $(\sim^m, a)$ satisfying the following conditions:

(i) $\sim^m$ is a finite simplicial complex such that $j_{\sim^m}$ is a union of $m$-simplices and every $(m-1)$-simplex is the face of an even number of $m$-simplices.

(ii) $a: \sim^m \rightarrow \sim^m$ is a free involution.

(iii) There is an equivariant map $\mathcal{Y} \rightarrow S^m$ with $\deg(\cdot) = 1 (\mod 2)$.

Some remarks are in order.

1. We recall how to find $\deg(\cdot)$. Choose a simplex $s$ of $S^m$ and let $f$ be a simplicial approximation to $\cdot$. Then $\deg(\cdot)$ is the number of $m$-simplices of $\sim^m$ mapped into $s$ by $f$.

2. Let $\sim$ be the sum of all $m$-simplices of $\sim^m$. Condition (i) of Definition 2.3 implies that $[-] \in H_m(\sim^m)$. We call $[-]$ the fundamental class of $\sim^m$.

3. Let $\sim^m = \mathbb{Z}_2$ and consider a two-fold covering $q: \sim^m \rightarrow m$. As $\cdot$ is equivariant it induces $\cdot: m \rightarrow \mathbb{R}^m$. Let $\deg(\cdot)$ denote $\deg(\cdot)(\mod 2)$.

Note that $\deg(\cdot) = h(\mathbb{R}^m; q)$ where $w^m: H^m(\mathbb{R}^m; \mathbb{Z}_2)$ is the nonzero element. If $\cdot: \sim^m \rightarrow S^m$ is an equivariant map then $\deg(\cdot) = 1$. To see this, we prove the following proposition.

Proposition 2.4 Suppose a map $\cdot: \sim^m \rightarrow S^m$ is equivariant. Then $\deg(\cdot) = 1$.
Proof Consider the classifying map and the commutative diagram for a two-fold covering \( q: \sim^m \to m \):

\[
\begin{array}{ccc}
\sim^m & \to & S^1 \\
\downarrow & & \downarrow \\
m & \to & \mathbb{R}P^1 \\
\end{array}
\]

We also have:

\[
\begin{array}{ccc}
\sim^m & \to & S^m \\
\downarrow & & \downarrow \\
m & \to & \mathbb{R}P^m \\
\end{array}
\]

Because \( S^1 \to \mathbb{R}P^1 \) is the classifying covering, \( i \) and \( i' \).

Observe that

\[
\deg_2(i') = \text{ht} w_i^m; \quad q-i = \text{ht}(i') w_i^m; \quad q-i
\]

where \( 0 \leq w_i^m \leq H^m(\mathbb{R}P^1) \). But, since \( i \) and \( i' \),

\[
\text{ht}(i') w_i^m; \quad q-i = \text{ht} w_i^m; \quad q-i = \deg_2(i').
\]

Now we modify the definition of obstructor.

Definition 2.5 (Proper obstructor) Let \( T \) be a contractible\(^1\) simplicial complex. Recall that \( T_r \overset{\text{Cl}(T^2 - N_r(d(T)))}{\to} \) where \( N_r(d(T)) \) denotes the \( r \) tubular neighborhood of the image of the diagonal map \( d: T \to T^2 \). Let \( m \) be the largest integer such that for any \( r > 0 \), there exists an essential \( \mathbb{Z}_2 \{m\text{-cycle} (\sim^m, a) \) and a \( \mathbb{Z}_2 \{\text{equivariant map} f: \sim^m \to T_r \). If such \( m \) exists then \( T \) is called an \( m \{\text{proper obstructor} \).

The rst example of proper obstructor is given by the following proposition.

Proposition 2.6 Suppose that \( M \) is a \( k \{\text{dimensional closed aspherical manifold where} k > 1 \) and \( X \) is the universal cover of \( M \). Suppose also that \( X \) has a triangulation so that \( X \) is a metric simplicial complex and a group \( G = \pi_1(M) \) acts on \( X \) properly discontinuously, cocompactly, simplicially, and freely by isometries. Then \( X^k \) is a \( (k-1) \{\text{proper obstructor} \).

We prove Proposition 2.6 in Section 3. The key ideas are the following:

\(^1\)Contractibility is necessary for Proposition 5.2

(1) Since $G$ acts on $X$ properly discontinuously, cocompactly, simplicially, and freely by isometries, $X$ is uniformly contractible. Recall that a metric space $Y$ is uniformly contractible if for any $r > 0$, there exists $R > r$ such that $B_r(y)$ is contractible in $B_R(y)$ for any $y \in Y$.

(2) For any $R > 0$, there exists $R_0 > R$ so that the inclusion induced map
\[ i : H_j(X_{R_0}) \to H_j(X_R) \]
is trivial for $j \neq k - 1$ and $Z_2 = i(H_{k-1}(X_{R_0})) \to H_{k-1}(X_R)$. (See Lemma 3.6.)

(3) We recall the definition of complex and use it to complete the proof as sketched below.

**Definition 2.7** A complex is a quotient space of a collection of disjoint simplices of various dimensions, obtained by identifying some of their faces by the canonical linear homeomorphisms that preserve the ordering of vertices.

Suppose $(\ classified; a)$ is an essential $Z_2 - m$-cycle with a $Z_2$-equivariant map $f : (\ classified; a) \to T_r$. Let $j_m = \left[ \bigcup_{i=1}^n m \right]$ (union of $n$ copies of $m$-simplices, use subscripts to denote different copies of $m$-simplices) and $f_i$ for $i = 1, \ldots, n$.

Then condition (i) of Definition 2.3 implies that $\bigcup_{i=1}^n f_i$ is an $m$-cycle of $T_r$ (over $Z_2$). That is, an essential $Z_2 \{m\}-cycle (~m; a)$ with a $Z_2 \{equivariant\} map f : (~m; a) \to T_r$ can be considered as an $m\{cycle\}$ of $T_r$ (over $Z_2$). Next suppose that $g = \bigcup_{i=1}^n g_i$ is an $m\{chain\}$ of $T_r$ (over $Z_2$) where $g_i \in T_r$ are singular $m\{simplices\}$. Take an $m\{simplex\}$ for each $i$ and index them as $m_i$. Let $m^{-1}$ denote a codimension 1 face of $m$. Construct a complex as follows:

For each $i$, we identify $m$ with $m_i$ along $m^{-1}$ and $m_i^{-1}$ whenever $g_j$ of $m_i^{-1}$ whenever $g_j$. Subdivide if necessary so that becomes a simplicial complex. Consider when $g$ is an $m\{cycle\}$ and an $m\{boundary\}$.

First, suppose $g$ is an $m\{cycle\}$. Then for any codimension 1 face $m^{-1}$ of $m$ there are an even number of $j$'s (including i itself) between 1 and $n$ such
that $g_{j m}^{-1} = g_{j m}^{-1}$. So satisfies condition (i) of Definition 2.3 and we can consider $g$ as a map

$$g: T_r$$

by setting $g_{j m} = g$.

Second, suppose $g$ is an $m$-boundary. Then there is an $(m + 1)$-chain $G = \sum_{i=1}^{N} G_i$ where $G_i: T_r \to \mathbb{R}$ is singular $(m + 1)$-simplices such that $\partial G = g$. As before one can construct a simplicial complex $\Omega$ and consider $G$ as a map

$$G: \Omega \to T_r$$

Let $@\Omega$ be the homology of $\Omega$ which are the faces of an odd number of $(m+1)$-simplices $g$. Note that $@\Omega = \Omega$ where $\cong$ denotes combinatorial equivalence. This observation will be used to construct an essential cycle in the proof of Proposition 2.6.

### 3 Coarse Alexander duality

We first review the terminology of [5]. Some terminology already defined is modified in the PL category. Let $X$ be (the geometric realization of) a locally finite simplicial complex. We equip the 1-skeleton $X^{(1)}$ with path metric by de ning each edge to have unit length. We call such an $X$ with the metric on $X^{(1)}$ a metric simplicial complex. We say that $X$ has bounded geometry if all links have a uniformly bounded number of simplices. Recall that $X_r C(C(L(\mathbb{R}^{X^2} - N_r(d(X)))), see Definition 2.2. Also denote:

$$B_r(c) = \text{face of } X \quad jd(c, x) \quad \Delta B_r(c) = \text{face of } X \quad jd(c, x) = \text{rg}$$

If $C(X)$ is the simplicial chain complex and $A = C(X)$ then the support of $A$, denoted by $\text{Support}(A)$, is the smallest subcomplex of $K$ of $X$ such that $A = C(K)$. We say that a homomorphism

$$h: C(X) \to C(X)$$

is coarse Lipschitz if for each simplex $X$, $\text{Support}(h(C(X)))$ has uniformly bounded diameter. We call a coarse Lipschitz map with

$$D = \max \text{diam}(\text{Support}(h(C(X))))$$

$D$-Lipschitz. We call a homomorphism $h$ uniformly proper, if it is coarse Lipschitz and there exists a proper function $: \mathbb{R}^+ \to \mathbb{R}$ so that for each subcomplex $K$ of $X$ of diameter $r$, $\text{Support}(h(C(X)))$ has diameter $(r)$.
We say that a homomorphism $h$ has displacement $D$ if for every simplex $X$, $\text{Support}(h(C(X))) \subset N_D(X)$. A metric simplicial complex is uniformly acyclic if for every $R_1$ there is an $R_2$ such that for each subcomplex $K$ of diameter $R_1$ the inclusion $K \rightarrow N_{R_2}(K)$ induces zero on reduced homology groups.

**Definition 3.1** (PD group) A group $\Gamma$ is called an $n$-dimensional Poincare duality group ($PD(n)$ group in short) if the following conditions are satisfied:

(i) $\Gamma$ is of type $FP$ and $n = \dim(\Gamma)$.

(ii) $H^j(\Gamma; \mathbb{Z}_\Gamma) = 0$ for $j \neq n$.

$\mathbb{Z}_j = n$.

**Example 3.2** The fundamental group of a closed aspherical manifold is a PD($k$) group. See [3] for details.

**Definition 3.3** (Coarse Poincare duality space [5]) A coarse Poincare duality space of formal dimension $k$, PD($k$) space in short, is a bounded geometry metric simplicial complex $X$ so that $C(X)$ is uniformly acyclic, and there is a constant $D_0$ and chain mappings

$$C(X) \xrightarrow{P} C^{k-1}(X) \xrightarrow{P} C(X)$$

so that

(i) $P$ and $P$ have displacement $D_0$.

(ii) $P$, $P$ and $P$ are chain homotopic to the identity by $D_0$-Lipschitz chain homotopies $: C(X) \rightarrow C^{k+1}(X); C(X) \rightarrow C^{k-1}(X)$. We call coarse Poincare duality spaces of formal dimension $k$ a coarse PD($k$) spaces.

**Example 3.4** An acyclic metric simplicial complex that admits a free, simplicial cocompact action by a PD($k$) group is a coarse PD($k$) space.

For the rest of the paper, let $X$ denote the universal cover of a $k$-dimensional closed aspherical manifold where $k > 1$.

Assume also that $X$ has a triangulation so that $X$ is a metric simplicial complex with bounded geometry, and $G = \pi_1(M)$ acts on $X$ properly discontinuously, cocompactly, simplicially, and freely by isometries. In particular, $G$ is a PD($k$) group and $X$ is a coarse PD($k$) space. The following theorem was proved in [5].

Pro-Category theory is reviewed in Appendix A.
Theorem 3.5 (Coarse Alexander duality [5]) Suppose \( Y \) is a coarse PD\((n)\) space, \( Y^0 \) is a bounded geometry, uniformly acyclic metric simplicial complex, and \( f : C(Y^0) \rightarrow C(Y) \) is a uniformly proper chain map. Let \( K \) \( \text{Supp}(f(C(Y^0))); Y_R \text{Cl}(Y - N_R(K)) \). Then we can choose \( 0 < r_1 < r_2 < r_3 < \ldots \) and define the inverse system \( \text{pro} \mathcal{H}_j(Y_{r}) ; f \mathcal{H}_j(Y_{r}); i ; \mathbb{N}g \) so that
\[
\text{pro} \mathcal{H}_{n-j-1}(Y_{r}) = H_{c}(Y^0);
\]
We rephrase the coarse Alexander duality theorem.

Lemma 3.6 Recall that \( X \) is a metric simplicial complex with bounded geometry and a group \( G \) acts on \( X \) properly discontinuously, cocompactly, simplicially, and freely by isometries. Also recall that \( X_r \text{Cl}(X^2 - N_r(d(X))) \). One can choose \( 0 < r_1 < r_2 < r_3 < \ldots \) and define the inverse system \( \text{pro} \mathcal{H}_j(X_{r}) ; f \mathcal{H}_j(X_{r}); i ; \mathbb{N}g \) so that:
\[
\text{pro} \mathcal{H}_j(X_{r}) = \begin{cases} 0 & \text{ if } j \neq k - 1 \\ \mathbb{Z}_2 & \text{ if } j = k - 1 \end{cases}
\]

Proof Consider the diagonal map
\[
d : X \rightarrow X^2; x \mapsto (x; x)
\]
and note that \( d \) is uniformly proper and \( X^2 \) is a PD\((2k)\) space. Theorem 3.5 implies that
\[
\text{pro} \mathcal{H}_{2k-1}(X_{r}) = H_{c}(X);
\]
Finally observe that \( H_{c}(X) = H_{k-}(\mathbb{R}^k) = H_{c}(\mathbb{R}^k) \).

Now we prove Proposition 2.6.

Proof of Proposition 2.6 Let \( r > 0 \) be given. First use Lemma 3.6 to choose \( r = r_1 < r_2 < \cdots < r_{k-1} < r_k \) so that
\[
i : H_j(X_{r_{m+1}}) \rightarrow H_j(X_{r_m})
\]
is trivial for \( j \neq k - 1 \). In particular, \( i : X_{r_k} \rightarrow X_{r_{k-1}} \) is trivial in \( 0 \). Let \( S^0 \) \( f; w, g \) and define an involution \( a_0 \) on \( S^0 \) by \( a_0(e) = w \) and \( a_0(w) = e \). Let \( s : (S^0; a_0) \rightarrow (X_{r_k}; s) \) be an equivariant map where \( s \) is the obvious involution on \( X_{r_1} \). Now let
\[
: \stackrel{1}{\rightarrow} X_{r_{k-1}}
\]
be a path so that \( (0) = (e) \) and \( (1) = (w) \). Define
\[
: \stackrel{0}{\rightarrow} X_{r_{k-1}}
\]
by \( q(t) = s(t) \). Observe that \( 1 + 0 \) is an 1-cycle in \( X_{r_k - 1} \). See Figure 1.

Let \( a_1 \) be the obvious involution on \( S^1 \) and consider \( 1 \) as an equivariant map

\[
1: (S^1; a_1) \to (X_{r_k - 1}; s).
\]

Since \( i : H_1(X_{r_k - 1}) \to H_1(X_{r_k - 2}) \) is trivial, \( 1 \) is the boundary of a 2-chain in \( X_{r_k - 2} \). Call this 2-chain \( \sum_{i=1}^{m} g_i \) where \( g_i \) are singular 2-simplices. Following Remark (3) after Proposition 2.6, construct a simplicial complex \( \sim_{+}^{2} \) such that

\[
\sim_{+}^{2}: \sim_{+}^{2}! \to X_{r_k - 2} \quad \text{and} \quad \partial_{+}^{2} = 1.
\]

See Figure 2. Define the boundary of \( \sim_{+}^{2} \), \( \partial_{+}^{2} \), to be the union of 1-simplices, which are the faces of an odd number of 2-simplices. Recall also from Remark (3) that \( \sim_{+}^{2} \cdot \sim_{-}^{2} \), where \( \cdot \) denotes combinatorial equivalence.

Next, let \( \sim_{-}^{2} = s \sim_{+}^{2} = \sum_{i=1}^{m} s g_i \). Take a copy of \( \sim_{-}^{2} \), denoted by \( \sim_{-}^{2} \), such that

\[
\sim_{-}^{2}: \sim_{-}^{2}! \to X_{r_k - 2} \quad \text{and} \quad \partial_{-}^{2} = 1.
\]

Construct \( \sim_{-}^{2} \) by attaching \( \sim_{+}^{2} \) and \( \sim_{-}^{2} \) along \( S^1 = \sim_{+}^{2} = \sim_{-}^{2} \) by identifying \( x \to a_1(x) \). That is, \( \sim_{-}^{2} \sim_{+}^{2} \mid S^1 \sim_{-}^{2} \). See Figure 3. Define an involution \( a_2 \).
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Figure 3: Constructing ~

on ~ by setting

\[
a_2(x) = \begin{cases} 
8 & \text{if } x \sim_2 \frac{\mathbb{R}^2}{\mathbb{Z}} - S^1 \\
2 & \text{if } x \sim_2 \frac{\mathbb{R}^2}{\mathbb{Z}} - S^1 \\
a_1(x) & \text{if } x \in S^1
\end{cases}
\]

Observe that \( x \sim_2 \frac{\mathbb{R}^2}{\mathbb{Z}} + \frac{\mathbb{R}^2}{\mathbb{Z}} \) is a 2-cycle in \( X \), and we can consider \( \sim_2 \) as an equivariant map

\( 2: (\sim_2; a_2)! (X_{r_k}; s) \):

Continue inductively and construct a \((k-1)\)-cycle

\[ a_{k-1}: (\sim^{k-1}; a_{k-1})! (X_{r_1} = X_{r_k-1}; s) \]

Simply write \( a \) instead of \( a_{k-1} \), and note that \( X_{r_1} \sim X_r \). So \((\sim^{k-1}; a)\) satisfies conditions (i)–(ii) of Definition 2.3 and we only need to show that it satisfies condition (iii).

It was proved in [2] that there exists a \(\mathbb{Z}_2\)-equivariant homotopy equivalence \( \mathbb{H}: X_0! S^{k-1} \). So \( \mathbb{H} \) induces a homotopy equivalence

\[ h: X_0= ! \mathbb{R}P^{k-1}; \]

Let \( g \) be \( h_k^{-1}: \sim_{k-1} \uparrow^{k-1} X_r \uparrow \downarrow^{k-1} X_0 \uparrow h S^{k-1} \). Note that \( g \) is equivariant. We shall prove that \( \deg(g) = 1(\mod 2) \) by constructing another map

\[ f_{k-1}: \sim_k ! S^{k-1} \]

with odd degree and applying Proposition 2.4.

Observe that

\[ S^1 \sim_2 \sim_3 \cdots \sim_2 \sim_{k-2} \sim_{k-1} \]

and for each \( i, 2 < i < k-1 \):

\[ \sim_i = \sim_{i-1} [ \sim_{i-1} \sim_{i-1} ] \]

Now construct a map \( f_{k-1} : \sim^{k-1}! S^{k-1} \) as follows: First let \( f_1 : S^1 \to S^1 \) be the identity and extend \( f_1 \) to \( f_2 : \sim^2! B^2 \) by Tietze Extension theorem. Without loss of generality assume that \((f_2)^{-1}S^1\sim^2 \to S^2\). Note that \((f_2)^{-1}(B^2_+\sim^2, f_2^{-1}(B^2_-\sim^2), and \( f_2^{-1}(S^1)\to S^1 \).

Continue inductively and construct an equivariant map

\[
 f_{k-1} : \sim^{k-1}! S^{k-1}.
\]

By construction, we know that

\[
 f_{k-1}^{-1}(B^1_+) \sim^1 \sim^1; f_{k-1}^{-1}(B^1_-) \sim^1; \text{ and } f_{k-1}^{-1}(S^{j-1}) \sim^1; 2 \leq j \leq k-1.
\]

Observe that \( \deg(f_{k-1}) = \deg(f_{k-2}) = \cdots = \deg(f_2) = \deg(f_1) \). (Recall that \( \deg(f_m) \) the number of \( m \{-\text{simplices of} \sim^m \text{mapped into a simplex} s \text{ of} S^m \text{ by} f. \}) \) But \( \deg(f_1) = 1 \) so \( f_{k-1} : \sim^{k-1}! S^{k-1} \) has nonzero degree.

Now Proposition 2.4 implies that \( \deg(g) = 1 \) (mod 2). Therefore \((\sim^{k-1};a)\) with equivariant map

\[
 f_{k-1} : \sim^{k-1}! X_r;
\]

satisfies conditions (i),(ii), and (iii) of Definition 2.3. Now the proof of Proposition 2.6 is complete.

4 New proper obstructors out of old

In this Section, we construct a \( k \)\{-proper obstructor from a \( (k-1) \)\{-proper obstructor \( X \).

Definition 4.1 Let \((Y;d_Y)\) be a proper metric space and \((A;d_A)\) be a metric space isometric to \([0;1]\). Let \(t : [0;1]! \to \) be an isometry and denote \((t)\) by \(t\). Define a metric space \((Y_\sim^1;d),\) called \(Y\) union a ray, as follows:

(i) As a set \(Y_\sim^1\) is the wedge sum. That is, \(Y_\sim^1 = Y| with Y\sim^1 = f_0g\)

(ii) The metric \(d\) of \(Y_\sim^1\) is defined by

\[
\begin{align*}
&d(v,w) = d_Y(v,w); \quad \text{if } v,w \in Y \\sim^1 \\
&d(v,w) = d_A(v,w); \quad \text{if } v,w \not\in Y \\
&d(v,w) = d_Y(v_0) + d(0;w); \quad \text{if } v \not\in Y, w \not\in Y \\
&d(v,w) = d_A(v_0, w_0); \quad \text{if } v,w \not\in Y
\end{align*}
\]
Proposition 4.2 Let $X$ be a $k$-dimensional contractible manifold without boundary and $k > 1$. Suppose also that $X$ has a triangulation so that $X$ is a metric simplicial complex and a group $G$ acts on $X$ properly discontinuously, cocompactly, simplicially, and freely by isometries. In particular, $X$ is a $(k-1)$-{proper obstructor. Then $X_\sim$ is a $k$-{proper obstructor.

Proof Recall that by Lemma 3.6, we can choose $0 < r_1 < r_2 < r_3 : :$ and define $\text{pro}_k(X_\sim)$ so that $\text{pro}_k(X_\sim) = \mathbb{Z}_2$. This means that for any $r > 0$ we can choose $R > r$ so that $r_0(R) = \mathbb{Z}_2 = \mathbb{Z}_2 \otimes \mathbb{Z}_2$.

Now let $r > 0$ be given and choose $R > r$ as above. Let $(\sim^{k-1};a)$ be an essential $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$-{cycle with a $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$-{equivariant map $f : \sim^{k-1} \rightarrow X_R$.

Next consider composition $i \circ f : \sim^{k-1} \rightarrow X_R \rightarrow X_R$. If $f : \sim^{k-1} \rightarrow X_R$ is the boundary of a $k$-{chain then we can construct an essential $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$-{cycle with $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$-{equivariant map into $X_R$ using the method used in the proof of Proposition 2.6. But this implies $X$ is a $k$-{proper obstructor. (Recall that $X^k$ is a $(k-1)$-{proper obstructor.) So we can assume $f : \sim^{k-1} \rightarrow X_R$.

Let $p_i : X_R \rightarrow X$ denote the projection to the $i$-th factor, $i = 1; 2$.

We need the following lemma.

Lemma 4.3 Define $j : X - B_R \rightarrow X_R \times \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$. Then the composition

$$i \circ j : H_{k-1}(X - B_R(0)) \rightarrow H_{k-1}(X_R) \rightarrow H_{k-1}(X_R) \rightarrow \mathbb{Z}_2$$

is nontrivial.

The proof of Lemma 4.3 Consider a map $f : H_{k-1}(X_0) \rightarrow \mathbb{Z}_2$ given by

$$[f] \otimes \begin{pmatrix} \text{Lk}(f; \ ) \mod 2 \end{pmatrix}$$

where $\text{Lk}(f; \ )$ denote the linking number of $f$ with the diagonal $2$. Now consider the composition:

$$H_{k-1}(X - B_R(0)) \rightarrow H_{k-1}(X_R) \rightarrow H_{k-1}(X_R) \rightarrow \mathbb{Z}_2$$

We can compute $\text{Lk}(f; \ )$ by letting $f$ bound a chain $f{^{-1}}$-transverse to ...
We shall show that $[f]$ is nontrivial. Choose $f_1 \in H_{k-1}(X - B_R)$ so that $Lk(f_1; 0) \neq 0$ where $[0] \in H_0(X)$. Then $Lk(i \circ (f_1); i) \neq 0$. (We can choose the same chain transverse to $i$.) Hence $[i]$ is nontrivial. In particular, $i$ and $j$ are nontrivial. □

Since $j : H_{k-1}(X - B_R) ! H_{k-1}(X_R)$ is nontrivial, we can choose $h \in Z_{k-1}(X - B_R) - B_{k-1}(X - B_R)$ with $g \cdot j \cdot h \in Z_{k-1}(X_R) - B_{k-1}(X_R)$. That is, $0 \notin [g] \in H_{k-1}(X_R)$. We can consider $g$ as a map $g : ! X_R$ where $X_R$ is a $(k-1)$-dimensional simplicial complex satisfying condition (i) of Definition 2.3 such that

$$0 \notin [g] \in H_{k-1}(X_R)$$

$$i \circ g : !^g X_R \overset{i}{\to} X_R \text{ with } p_2(i \circ g(\cdot)) = f \circ g = X \setminus \{0\}.$$ 

Next define $G^0 = sg$, that is,

$$g^0 : !^g X_R \overset{!}{\to} X_R.$$ 

Note that $i \circ g^0$ is a cycle in $X_R$ and $p_2(i \circ g^0(\cdot)) = 0$. Also $[f] ; [g] \in H_{k-1}(X_R)$ and $i \circ [f] ; i \circ [g] \in H_{k-1}(X_R)$ are nonzero. Observe that $i \circ f$ and $i \circ g$ must be homologous in $X_R$ since $Z_2 = i \circ (H_{k-1}(X_R)) = H_{k-1}(X_R)$. We simply write $f$, $g$, and $g^0$ instead of $i \circ f$, $i \circ g$, and $i \circ g^0$. There exists a $k$-chain $G \in C_k(X_R)$ such that $G = f + g$.

Again consider $G$ as a map $G : \Omega ! X_R$ where $\Omega$ is a simplicial complex so that

$$\theta \Omega = \theta^{k-1} \Omega :$$

See Figure 4.

Next define $G^0 = sG$, that is,

$$G^0 : \Omega !^G X_R \overset{!}{\to} X_R.$$ 

Note that

$$G^0 = f + g^0.$$ 

Now take two copies of $\Omega$ and index them as $\Omega_1$ and $\Omega_2$. Similarly $\Omega_1$ and $\Omega_2$. Hence

$$\theta \Omega_i = \theta^{k-1} \Omega_i : i = 1; 2.$$
Denote $\text{id}(x) = x^0$ for $x \in \Omega_1 \sim^k \Omega_2$. Construct a $k\{ \sim^k \}$ dimensional simplicial complex $\Omega$ by attaching $\Omega_1$ and $\Omega_2$ along $\sim^{k-1}$ by $a$: $\sim^{k-1} \sim^{k-1}$. That is,

$$\Omega = (\Omega_1 \mid \Omega_2) \Rightarrow ax; x \in \sim^{k-1}.$$ 

See Figure 5.

We can define an involution $a$ on $\Omega$ by

$$a(x) = a(x); x \in \sim^{k-1};$$

$$a(x^0) = x; x \in \sim^{k-1};$$

Also we can define a $\mathbb{Z}_2$ equivariant map $j: \Omega \rightarrow X_r$ by:

$$j_{\Omega_1} = G$$

$$j_{\Omega_2} = G^0$$

We define

$$\sim^k = (1 \ [0,1] \otimes (1;1)) \ [1 \ \Omega \ [2 \ (2 [0,1] \otimes (2;1)):$$

See Figure 6. Now extend \( \sim^k \) over \( \tilde{k} \), and denote \( \sim^k = \alpha(x) \) by \( k \).

\[ \sim_k \]

\[ \sim_{k-1} \]

\[ \Omega_1 \]

\[ \Omega_2 \]

\[ \Omega \]

Figure 6: Constructing \( \sim^k \)

Suppose that \( k \) classifies into \( \mathbb{RP}^m \) where \( m < k \). Let

\[ h: k \to \mathbb{RP}^m \]

be the classifying map and

\[ \mathcal{H}: \sim^k \to S^m \]

be the equivariant map covering \( h \). Observe that

\[ \deg \mathcal{H} \mid_{\sim^k - 1} = \deg \mathcal{H} = 0 \pmod{2} \]

This is a contradiction since there already exists a \( \mathbb{Z}_2 \) equivariant map

\[ \mathcal{F}: \sim^k \to S^{k-1} \]

of odd degree. Hence \( (\sim^k; \alpha) \) is an essential \( \mathbb{Z}_2 \) cycle.

Finally, we need to define an \( \mathbb{Z}_2 \) equivariant map:

\[ F: k \to (X, r) \]

Recall that \( p_1g_1(1) = 0 \) and let

\[ c: p_2g(1) \to X \]

be a contraction to \( 0 \). Similarly \( p_2g_0(1) = 0 \) and let

\[ c_0: p_2g_0(1) \to X \]

be a contraction to \( 0 \). Define a \( \mathbb{Z}_2 \) equivariant map

\[ F: k \to (X, r) \]

A lower bound to the action dimension of a group

as follows: Recall that \((t) = t\) in Definition 4.1, so \(d(0; s) = s\) and \(d(s; x) = s\) for any \(x \in X\).

\[
\begin{align*}
F_{m_2} &= \begin{cases} 
2 & 1: t \in [0, \frac{1}{2}] \\
2 & 1: t \in [\frac{1}{2}, 1]
\end{cases} \\
F(x; t) &= (r; c(2t-1)(p_2g(x))); x \in 2; t \in [0, -\frac{1}{2}]
\end{align*}
\]

The proof of Proposition 4.2 is now complete.  

If \(Y\) and \(Z\) are metric spaces we use the sup metric on \(Y\) and \(Z\) where

\[
d_{\text{sup}}((y_1; z_1); (y_2; z_2)) = \max d_r (y_1; y_2); d_z (z_1; z_2) g;
\]

**Proposition 4.4** Suppose \(X_1; X_2\) are \(m_1; m_2\) proper obstructors, respectively. Then \(X_1 X_2\) is an \((m_1 + m_2 + 1)\) proper obstructor.

**Proof** Let \(r > 0\) be given and let

\[
\begin{align*}
f_1 &= \begin{cases} -m_1 \uparrow \uparrow (X_1)_r \\
-1 \\
-2m_2 \uparrow \uparrow (X_2)_r
\end{cases} \\
&= \begin{cases} -m_1 \uparrow \uparrow (X_1)_r \\
-1 \\
-2m_2 \uparrow \uparrow (X_2)_r
\end{cases}
\end{align*}
\]

be \(\mathbb{Z}_2\) equivariant maps for essential \(\mathbb{Z}_2\) cycles. Note that

\[(X_1 X_2)_r = ((X_1)_r (X_2)_r) [(X_1)_r (X_2)_r] ((X_1)_r (X_2)_r); \]

Let \(a_1\) be the involution on \((X_1)_r\) and \(a_2\) be the involution on \((X_2)_r\). Recall that the join \(\sim_{m_1} \sim_{m_2}\) is obtained from \(\sim_{m_1} \sim_{m_2} [-1; 1]\) by identifying \(\sim_{m_1} f y g f y g\) to a point for every \(y \in \sim_{m_2}\) and identifying \(f y g \sim_{m_2} f -1 g\) to a point for every \(x \in \sim_{m_1}\). Define an involution \(a\) on \(\sim_{m_1} \sim_{m_2}\) by

\[
a(v; w; t) = (a_1(v); a_2(w); t);
\]

Let

\[
\begin{align*}
g_1 &= \begin{cases} -m_1 \uparrow \uparrow S^{m_1} \\
-2m_2 \uparrow \uparrow S^{m_2}
\end{cases} \\
g_2 &= \begin{cases} -m_1 \uparrow \uparrow S^{m_1} \\
-2m_2 \uparrow \uparrow S^{m_2}
\end{cases}
\end{align*}
\]

be equivariant maps of odd degree. Then:

\[
(\sim_{m_1} \sim_{m_2}; a) = \begin{cases} \sim_{m_1} \sim_{m_2} & (v; w; t) \neq (g_1(v); g_2(w); t)
\end{cases}
\]

is also an equivariant map of an odd degree. Hence \((\sim_{m_1} \sim_{m_2}; a)\) is an essential \(\mathbb{Z}_2\) \((m_1 + m_2 + 1)\) cycle.

Now let
\[ c: f_1(\mathbb{Z}_1) \to [1;1]! X_1 \]
be a $\mathbb{Z}_2$-equivariant contraction to a point such that $c_t = \text{id}$ for $t \in [1;0]$.
Similarly let
\[ d: f_2(\mathbb{Z}_2) \to [1;1]! X_2 \]
be a $\mathbb{Z}_2$-equivariant contraction to a point such that $d_t = \text{id}$ for $t \in [0;1]$.
Finally define
\[ f: \mathbb{Z}_1 \to \mathbb{Z}_2 \]
by
\[ f(v;w;t) = (c_t(f_1(v));d_t(f_2(w))) \]
We note that $f$ is $\mathbb{Z}_2$-equivariant.

5 Proper obstructor dimension

We review one more notion from [2].

Definition 5.1 The uniformly proper dimension, $\text{updim}(G)$, of a discrete
group $G$ is the smallest integer $m$ such that there is a contractible $m$-manifold
$W$ equipped with a proper metric $d_W$, and there is a $g: \Gamma \to W$ with the
following properties:
- $g$ is Lipschitz and uniformly proper.
- There is a function $h: (0;1) \to (0;1)$ such that any ball of radius $r$
centered at a point of the image of $h$ is contractible in the ball of radius
$(r)$ centered at the same point.
If no such $n$ exists, we define $\text{updim}(G) = 1$.

It was proved in [2] that
\[ \text{updim}(G) = \text{actdim}(G) \]
Now we generalize the obstructor dimension of a group.

Definition 5.2 The proper obstructor dimension of $G$, $\text{pobdim}(G)$, is defined
to be 0 for finite groups, 1 for 2-ended groups, and otherwise $m+1$ where $m$
is the largest integer such that for some $m$-proper obstructor $Y$, there exists a
uniformly proper map
\[ : Y \to T_G \]
where $T_G$ is a proper metric space with a quasi-isometry $q: T_G \to G$. 
**Lemma 5.3** Let $Y$ be an $m$-proper obstructor. If there is a uniformly proper map $f: Y \to W^d$ where $W$ is a contractible $d$-manifold then $d > m$.

**Proof** Assume $d \leq m$. Observe that if $f$ is uniformly proper then it induces an equivariant map $Y \to W^d$ where $W$ is a contractible $d$-manifold. Let $h: W \to S^{d-1}$ be an equivariant homotopy equivalence. We have an equivariant map

$$g = ih: Y \to T_G \to G \to W^d \to S^{d-1} \to S^{m-1} \to S^m$$

where $i: S^{d-1} \to S^{m-1} \to S^m$ is the inclusion. Note that $g$ is equivariant but $\deg(g) = 0 (\mod 2)$. This is a contradiction by Proposition 2.4.

Suppose that $G$ is finite so that $\pobdim(G) = 0$ by definition. Clearly, $\actdim(G) = 0$ if $G$ is finite. Hence $\pobdim(G) = \actdim(G) = 0$ in this case. Next suppose that $G$ has two ends so that $\pobdim(G) = 1$. Note that there exists $Z = H \subset G$ with $jG: Hj < 1$. And this implies that

$$\actdim(G) = \actdim(H) = \actdim(Z) = 1.$$ Therefore, $\pobdim(G) = \actdim(G) = 1$ when $G$ has two ends. Now we prove the main theorem for the general case.

**Main Theorem** $\pobdim(G) = \updim(G) = \actdim(G)$

**Proof** We only need to show the first inequality. Let $\pobdim(G) = m + 1$ for some $m > 0$. That is, there exists an $m$-proper obstructor $Y$, a proper metric space $T_G$, a uniformly proper map $f: Y \to T_G$, and a quasi-isometry $q: T_G \to G$. Let $\updim(G)$ be such that there exists a uniformly proper map $g: G \to W^d$ where $W$ is a contractible $d$-manifold. But the composition

$$q \circ Y \to T_G \to G \to W^d$$

is uniformly proper. Therefore

$$m + 1 = \pobdim(G) = \updim(G)$$

by the previous lemma.

Before we consider some applications, we make the following observation about compact aspherical manifolds with incompressible boundary.
Lemma 5.4 Assume that $W$ is a compact aspherical $m$-manifold with all boundary components incompressible. Let $\nu : \tilde{W} \to W$ denote the universal cover of $W$. Suppose that there is a component of $\partial \tilde{W}$, call it $N$, so that $j_1(W) : \pi_1(N) \to \pi_1(W)$ has index $j > 1$. Then $j_1(W) : \pi_1(N)$ is infinite.

Proof Observe that $N$ is aspherical also. First, we show that if

$$1 < j_1(W) : \pi_1(N) < 1$$

then $M^e \subset \tilde{W} = \pi_1(N)$ has two boundary components and $W$ has one boundary component. We claim that $M^e$ has a boundary component homeomorphic to $N$ which is still denoted by $N$. To see this consider the long exact sequence:

$$\cdots \to H_1(M^e) \to H_1(M) \to H_1(M; \mathbb{Z}) \to H_0(M^e) \to \mathbb{Z} \to \cdots$$

Since $\pi_1(M^e; \mathbb{Z})$ is infinite, $M^e$ is compact. Now $H_1(M; \mathbb{Z}) = H^{m-1}(M^e)$ by duality. But $H^{m-1}(M) = H^{m-1}(N)$ and $H^{m-1}(N) = \mathbb{Z}$ since $N$ is a closed $(m-1)$-manifold. That is, $H_0(\mathbb{Z}; \mathbb{Z}) = \mathbb{Z}$ so $M^e$ has two boundary components. Next let $N$ and $N^0$ denote two boundary components of $\mathbb{Z}$ both of which are mapped to $\pi_1(W)$ by $p : M^e \to W$. Hence $\partial \tilde{W}$ has one component.

Now assume that $m > j_1(W) : \pi_1(N) > 1$. Suppose $m$ is finite. Note that $p^N : \pi_1(M^e; \mathbb{Z}) \to \pi_1(W)$ has index $1$, and $p^N : \pi_1(N^0; \mathbb{Z}) \to \pi_1(W)$ has index $m - 1$. This means that $j_1(M^e; \mathbb{Z}) : \pi_1(N^0) : \pi_1(N) = m - 1$ since $j_1(M^e) = 1$. There are two alternative arguments:

If $m > 1$ then $M^e$ is an aspherical manifold with two boundary components $N$ and $N^0$ with $j_1(M^e) : \pi_1(N^0) = m - 1 > 1$. Consider $W = \pi_1(N^0)$. The same argument applied to $\tilde{W} = \pi_1(N^0)$ shows that $M^e$ has one boundary component, which is a contradiction. Therefore $j_1(W) : \pi_1(N)$ is infinite.

Suppose $m > 1$. Choose a point $x \in 2N \subset \partial \tilde{W}$ and let $x \in 2N \subset \mathbb{Z}$ so that $p(x) = x$. Next choose two loops in $\tilde{W}$ based at $x$ so that $p_{\gamma_1} : \pi_1(N) : \pi_1(N)$ are distinct cosets. (We are assuming $j_1(W) : \pi_1(N) > 1$.) Let $\gamma$ be the loopings of and respectively so that $\gamma = \gamma_1$. Note that $\gamma_1 \sim \gamma_2$ in $\pi_1(\mathbb{Z})$. Let $\gamma_1 \sim \gamma_2$ in $\pi_1(\mathbb{Z})$. Now consider a path $\gamma$ in $\pi_1(\mathbb{Z})$ from $\gamma_1$ to $\gamma_2$. Observe that $\gamma_{\gamma_1} \gamma_{\gamma_2}$ is a loop based at $x$, and $[\gamma] \pi_1(\mathbb{Z}) \sim \pi_1(\mathbb{Z})$. But $[\gamma] \pi_1(\mathbb{Z}) = 1$ and this implies that $[\gamma]^{-1} [\gamma] \sim [\gamma] \pi_1(\mathbb{Z})$ contrary to $[\gamma] \pi_1(\mathbb{Z})$. □
Corollary 5.5 (Application) Suppose that $W$ is a compact aspherical $m\{manifold with incompressible boundary. Also assume that there is a component of $\partial W$, call it $N$, so that $j_1(W): 1(N)j > 2$.

Then $\text{actdim}(1(W)) = m$.

Proof Let $p: W \to W$ be the universal cover of $W$. It is obvious that $\text{actdim}(1(W)) = m$ as $1(W)$ acts cocompactly and properly discontinuously on $W$. Denote $G = 1(W)$ and $H = 1(N)$. Let $N^\ast$ be a component of $p^{-1}(N)$. Therefore $N^\ast$ is the contractible universal cover of $N^{(m-1)}$. Note that $N^\ast$ is an $(m-2)$-proper obstructor by Proposition 2.6. Now $W = H$ has a boundary component homeomorphic to $N$. Call this component $N$ also. $jG: H$ is infinite by the previous lemma, and this implies that $W = H$ is not compact. In particular, there exists a map $0: [0, 1) \to W = H$ with the following property: For each $D > 0$ there exists $T 2 [0, 1)$ such that for any $x \in 2 N$, $d(0(t); x) > D$ for $t > T$, and $0(0) \notin 2 N$. Let $N = [0, 1) \to W$ be a lifting of $0$ such that $N = [0, 1) \to W$. Now we define a uniformly proper map:

$\phi: N^\ast \to W$

$\phi = \text{inclusion}$

$\phi(0) = -0$

Observe that $\phi$ is a uniformly proper map. Since $N^\ast$ is an $(m-1)$-proper obstructor and $W$ is quasi-isometric to $G$, $\text{pobdim}(G) = m$. But $\text{pobdim}(G) = \text{updim}(G) = \text{actdim}(G) = m$.

The last inequality follows from the fact that $G$ acts on $W$ properly discontinuously. Therefore $\text{pobdim}(G) = m$.

The following corollary answers Question 2 found in [2].

Corollary 5.6 (Application) Suppose that $W_i$ is a compact aspherical $m_i\{manifold with incompressible boundary for $i = 1, \ldots, d$. Also assume that for each $i$, $1 \leq i \leq d$, there is a component of $\partial W_i$, call it $N_i$, so that $j_1(W_i): 1(N_i)j > 2$. Let $G = 1(W_1) \cdots 1(W_d)$. Then:

$\text{actdim}(G) = m_1 + \cdots + m_d$

Proof It is easy to see that $\text{actdim}(G) = m_1 + \cdots + m_d$.
as $G$ acts cocompactly and properly discontinuously on $W_1 \sim W_d$. Denote $1(W_i) \sim G_1$ and $1(N_1) \sim H_1$. Let
\[ p: \tilde{W}_i \to W_i \]
be the contractible universal cover and let $N_1$ be a component of $p^{-1}(N_1)$. Since $N_1$ is incompressible, $N_1$ is the contractible universal cover of $N_1^{(m_1-1)}$.

By the previous Corollary, there are uniformly proper maps:
\[ 1: \tilde{N}_1 \sim W_1 \]
\[ 2: \tilde{N}_2 \sim W_2 \]
So there exists a uniformly proper map:
\[ 1 \sim 2: (\tilde{N}_1) (\tilde{N}_2) \sim W_1 \sim W_2 \]
Recall that $(\tilde{N}_1)$ $(\tilde{N}_2)$ is an $(m_1 + m_2 - 1)\{\text{proper obstructor by Proposition 4.4.}$. Since $W_1 \sim W_2$ is quasi-isometric to $G_1 \sim G_2$:
\[ \text{pobdim}(G_1 \sim G_2) = m_1 + m_2 \]
But $G_1 \sim G_2$ acts on $W_1 \sim W_2$ properly discontinuously, and this implies that:
\[ \text{pobdim}(G_1 \sim G_2) = m_1 + m_2 \]
Therefore, $\text{pobdim}(G_1 \sim G_2) = m_1 + m_2$.
Continue inductively and we conclude that:
\[ \text{pobdim}(G) = \text{pobdim}(G_1 \sim G_d) = m_1 + \cdots + m_d \]
Finally we see that
\[ \text{pobdim}(G) \leq \text{updim}(G) \leq \text{actdim}(G) = m_1 + \cdots + m_d \]

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A Pro-Category of Abelian Groups

With every category $K$ we can associate a new category $\text{pro}(K)$. We briefly review the definitions, see [1] or [6] for details. Recall that a partially ordered set $(\sim)$ is directed if, for $i; j \sim$, there exists $k \sim$ so that $k \sim i; j$.
Definition A.1 (Inverse system) Let \( (\ldots) \) be a directed set. The system 
\[ A = fA; p^0; g \] 
is called an inverse system over \( (\ldots) \) in the category \( K \), if
the following conditions are true.

(i) \( A \in \text{Ob}_K \) for every \( 0 \)
(ii) \( p^0 2 \text{Mor}_K (A; A) \) for \( 0 \)
(iii) \( 0^0 = p^0 \cdot p^0 = p^0 \)

Definition A.2 (A map of systems) Given two inverse systems in \( K \),
\[ A = fA; p^0; g \quad \text{and} \quad B = fB; q^0; M g \]
the system
\[ f = (f; f) : A \rightarrow B \]
is called a map of systems if the following conditions are true.

(i) \( f : M \rightarrow M \) is an increasing function
(ii) \( f(M) \) is co-final with
(iii) \( f : 2 \text{Mor}_K (A_f; B) \)
(iv) For \( 0^0 \) there exists \( f(0); f(0) \) so that:
\[
\begin{array}{c}
A_{f(0)} \\
A_f(0)
\end{array}
\]
\[
\begin{array}{c}
f(0) \\
q(0)
\end{array}
\]
\[ f = (f; f) : A \rightarrow B \]

\[ f \text{ is called a special map of systems if } 0 = M, f = \text{id}, \text{ and } f p^0 = q^0 f 0. \]

Two maps of systems \( f; g : A \rightarrow B \) are considered equivalent, \( f' \text{ g} \), if for every \( 0 \) there is a \( 0; f(0); g(0) \), such that \( f p(0) = g p(0) \).
This is an equivalence relation.

Definition A.3 (Pro-category) \( \text{pro}(K) \) is a category whose objects are inverse systems in \( K \) and morphisms are equivalence classes of maps of systems. The class containing \( f \) will be denoted by \( f \).

Our main interest is the following pro-category.

Example A.4 Pro-category of abelian groups Let \( A \) be the category of abelian groups and homomorphisms. Then corresponding \( \text{pro}(A) \) is called the category of pro-abelian groups.
Example A.5  Homology pro-groups  Suppose $f(X;X_0);(p^0_i);N_g$ is an object in the pro-homotopy category of pairs of spaces having the homotopy type of a simplicial pair. Then $fH_j((X;X_0)_i);(p^0_i);N_g$ is an object of $pro(A)$. Denote $fH_j((X;X_0)_i);(p^0_i);N_g$ by $proH_j(X;X_0)$.

We list useful properties of $pro(A)$:

1. A system $0$ consisting of a single trivial group is a zero object in $pro(A)$.
2. A pro-abelian group $fG_i;p^0_i;N_g = 0$ every $i$ admits a $i^0$ such that $p^0_i = 0$.
3. Let $A$ denote a constant pro-abelian group $fA;id_A;N_g$. If a pro-abelian group $fG_i;p^0_i;N_g = A$ then

$$\lim G_i = A$$

See [4, Lemma 4.1].

References


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