A flat plane that is not the limit of periodic flat planes

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Abstract We construct a compact nonpositively curved squared 2-complex whose universal cover contains a flat plane that is not the limit of periodic flat planes.

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1 Introduction

Gromov raised the question of which "semi-hyperbolic spaces" have the property that their flats can be approximated by periodic flats [4, x6.B3]. In this note we construct an example of a compact nonpositively curved squared 2-complex \( Z \) whose universal cover \( \tilde{Z} \) contains an isometrically embedded flat plane that is not the limit of a sequence of periodic flat planes.

A flat plane \( E \rightarrow \tilde{Z} \) is periodic if the map \( E \rightarrow \tilde{Z} \) factors as \( E \rightarrow T \rightarrow \tilde{Z} \) where \( E \rightarrow T \) is a covering map of a torus \( T \). Equivalently, \( \pi_1 Z \) contains a subgroup isomorphic to \( \mathbb{Z} \times \mathbb{Z} \) which stabilizes \( E \) and acts cocompactly on it. A flat plane \( f : E \rightarrow \tilde{Z} \) is the limit of periodic flat planes if there is a sequence of periodic flat planes \( f_i : E_i \rightarrow \tilde{Z} \) which converge pointwise to \( f : E \rightarrow \tilde{Z} \). In our setting, \( \tilde{Z} \) is a 2-dimensional complex, and so \( E \rightarrow \tilde{Z} \) is the limit of periodic flat planes if and only if every compact subcomplex of \( E \) is contained in a periodic flat plane.

In Section 2 we describe a compact nonpositively curved 2-complex \( X \) whose universal cover contains a certain aperiodic plane called an "anti-torus". In Section 3 we construct \( Z \) from \( X \) by strategically gluing tori and cylinders to \( X \) so that \( \tilde{Z} \) contains a flat plane which is a mixture of the anti-torus and periodic planes. This flat plane is not approximable by periodic flats because it contains a square that does not lie in any periodic flat. Our example \( Z \) is a \( K(\pi,1) \) for a negatively curved square of groups, and in Section 4 we describe an interesting related triangle of groups.
2 The anti-torus in $X$

2.1 The 2-complex $X$

Let $X$ denote the complex consisting of the six squares indicated in Figure 1. The squares are glued together as indicated by the oriented labels on the edges. Note that $X$ has a unique 0-cell, and that the notion of vertical and horizontal are preserved by the edge identifications. Let $H$ denote the subcomplex consisting of the 2 horizontal edges, and let $V$ denote the subcomplex consisting of the 3 vertical edges.

The complex $X$, which was first studied in [8], has a number of interesting properties that we record here. The link of the unique 0-cell in $X$ is a complete bipartite graph. It follows that the universal cover $\tilde{X}$ is the product of two trees $\tilde{H} \times \tilde{V}$ where $\tilde{H}$ and $\tilde{V}$ are the universal covers of $H$ and $V$. In particular, the link contains no cycle of length $< 4$ and so $X$ satisfies the combinatorial nonpositive curvature condition for squared 2-complexes [3, 1] which is a special case of the $C(4)-T(4)$ small-cancellation condition [6].

The 2-complex $X$ was used in [8] to produce the first examples of non-residually finite groups which are fundamental groups of spaces with the above properties. The connection to finite index subgroups arises because while $\tilde{X}$ is isomorphic to the cartesian product of two trees, $X$ does not have a finite cover which is the product of two graphs.

2.2 The anti-torus

The exotic behavior of $X$ can be attributed to the existence of a strangely aperiodic plane in $\tilde{X}$ that we shall now describe. Let $x \in 2^0$ be the basepoint of $\tilde{X}$. Let $c^1$ denote the infinite periodic vertical line in $X$ which is the based component of the preimage of the loop labeled by $c$ in $X$. Denote $y^1$...
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Figure 2: The Anti-Torus: The plane above is the convex hull of two periodically labeled lines in \( X \). A small region of the northeast quadrant has been tiled by the squares of \( X \).

analogously. Let \( \) denote the convex hull in \( X^* \) of the infinite geodesics labeled by \( c^1 \) and \( y^1 \), so \( = y^1 \ c^1 \). The plane \( \) is tiled by the six orbits of squares in \( X^* \) as in Figure 2. The reader can extend \( c^1 \ \ y^1 \) to a flat plane by successively adding squares wherever there is a pair of vertical and horizontal edges meeting at a vertex. From a combinatorial point of view, the existence and uniqueness of this extension is guaranteed by the fact that the link of \( X \) is a complete bipartite graph.

The \"axes\" \( c^1 \) and \( y^1 \) of \( \) are obviously periodic, and using that \( X \) is compact, it is easy to verify that for any \( n \in \mathbb{N} \), the infinite strips \([−n; n] \ \ \mathbb{R} \) and \( \mathbb{R} \ \ [−n; n] \) are periodic. However, the period of these infinite strips increases exponentially with \( n \). Thus, the entire plane \( \) is aperiodic. Note that to say that \([−n; n] \ \ \mathbb{R} \) is periodic means that the immersion \([−n; n] \ \ \mathbb{R} \rightarrow X \) factors as \([−n; n] \ \ \mathbb{R} \rightarrow C \rightarrow X \) where \([−n; n] \ \ \mathbb{R} \rightarrow C \) is the universal covering map of a cylinder. The map \( \rightarrow X \) is aperiodic in the sense that it does not factor through an immersed torus.

We conclude this section by giving a brief explanation of the aperiodicity of \( \). A complete proof that \( \) is aperiodic is given in [8]. Let \( W_n(m) \) denote the word corresponding to the length \( n \) horizontal positive path in \( \) beginning at the endpoint of the vertical path \( c^m \). Thus, \( W_n(m) \) is the label of the side opposite \( y^n \) in the rectangle which is the combinatorial convex hull of \( y^n \) and \( c^m \). Equivalently, \( W_n(m) \) occupies the interval \( [f g \ 0 \ \ m \ 2^n − 1] \). For each \( n \), the words \( f W_n(m) \) for \( m \in \mathbb{Z} \) are all distinct! Consequently every positive length \( n \) word in \( x \) and \( y \) is \( W_n(m) \) for some \( m \). This implies that the infinite
Figure 3: The complex \( Y \) is formed by gluing four cylinders to a square.

strip \([0; n]\) \( \mathbb{R} \) has period \( 2^n \), and in particular cannot be periodic.

We refer to \( Z \) as an anti-torus because the aperiodicity of \( R \) implies that \( c \) and \( y \) do not have nonzero powers which commute. Indeed, if \( c^p \) and \( y^q \) commuted for \( p, q \neq 0 \) then the flat torus theorem (see [1]) would imply that \( c^1 \) and \( y^1 \) meet in a periodic flat plane, which would contradict that \( Z \) is aperiodic.

3 The 2-complex \( Z \) with a nonapproximable flat

We first construct a new complex \( Y \) as follows: Start with a square \( s \), and then attach four cylinders each of which is isomorphic to \( S^1 \times I \). One such cylinder is attached along each side of \( s \). The resulting complex \( Y \) containing exactly five squares is illustrated in Figure 3.

Let \( T^2 \) denote the torus \( S^1 \times S^1 \) with the usual product cell structure consisting of one 0-cell, two 1-cells, and a single square 2-cell. We let \( \mathbb{R}^2 \) denote the universal cover and we shall identify \( \mathbb{R}^2 \) with \( \mathbb{R}^2 \).

At each corner of \( s \), \( Y \), there is a pair of intersecting circles in \( Y^1 \), which are boundary circles of distinct cylinders. Note that they meet at an angle of \( \frac{3\pi}{4} \) in \( Y \). At each of three (NW, SW, & SE) corners of \( s \), \( Y \) we attach a copy of \( T^2 \) by identifying the pair of circles in the 1-skeleton of \( T^2 \) with the pair of intersecting circles noted above at the respective corner of \( s \). At the fourth (NE) corner of \( s \), we attach a copy of the complex \( X \). Here we identify the pair of circles meeting at the corner of \( s \) with the pair of perpendicular circles \( c \) and \( y \) of \( X \). We denote the resulting complex by \( Z \). Thus, \( Z = T^2 \cup T^2 \cup T^2 \cup Y \cup X \). See Figure 4 for a depiction of the 8 squares of \( Z - X \) and their gluing patterns.

Definition 3.1 In nite cross An in nite cross is a squared 2-complex isomorphic to the subcomplex of \( \mathbb{R}^2 \) consisting of \([0; 1] \times \mathbb{R} \cup \mathbb{R} \times [0; 1] \). The base square of the in nite cross is the square \([0; 1] \times [0; 1] \).

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Figure 4: $Z - X$ and $Z$: The eight squares of the figure on the left are glued together following the gluing pattern to form $Z - X$. To form $Z$, we add a copy of $X$ at the NE corner, identifying the loops in $X$ labeled by $c$ and $y$, with the black single and double arrows of the diagram. The figure on the right represents an infinite cross whose convex hull in $Z$ is not approximable by any periodic plane. Note that while the NW, SW, and SE quarters of this plane are periodic, the NE quarter is an aperiodic quarter of $Z$.

The planes containing $s$: Observe that $Y$ contains various immersions of an infinite cross whose base square maps to $s$. In particular, there are exactly 16 distinct immersed infinite crosses $C \rightarrow Y$ that pass through $s$ exactly once. Each of these infinite crosses extends uniquely to an immersed flat plane in $Z$. Each such flat plane fails to be periodic because its four quarters map to distinct parts of $Z$. Our main result is that these immersed flat planes are not approximable by periodic flat planes because of the following:

Theorem 3.2 (No periodic approximation) There is no immersion of a torus $\mathbb{T}^2$ in $Z$ which contains $s$. Equivalently, there is no periodic plane in $\mathbb{Z}^+$ containing $s$.

Proof We argue by contradiction. Suppose that there is an immersed periodic plane $\Omega$ containing $s$. We shall now produce a rectangle as in Figure 5 that will yield a contradiction. We may assume that a copy of $s$ in $\Omega$ is oriented as in Figure 4. We begin at this copy of $s$ and travel north inside the northern cylinder until we reach another copy $s_n$ of $s$. The existence of $s_n$ is guaranteed by our assumption that $\Omega$ is periodic. Similarly, we travel east from $s$ to reach a square $s_e$. Travelling north from $s_e$ and east from $s_n$, we trace out the boundary of a rectangle whose NE corner is a square $s_{ne}$ (see Figure 5).
Figure 5: The figure above illustrates one of the four possible contradictions which explain why no periodic plane contains the square $s$.

This yields a contradiction because the inside of this rectangle is tiled by squares in $X$, yet the boundary of this rectangle is a commutator $c^n y^m$. As explained in Section 2, such a word cannot be trivial in $\tilde{\gamma}X$ because of the anti-torus.

**Remark 3.3** Using an argument similar to the above proof, one can show that these sixteen planes are the only flat planes in $\tilde{\gamma}X$ containing $\tilde{s}$. One considers the pair of "axes" intersecting at $s$ in a plane containing $s$. If this plane is different from each of the 16 mentioned above, then some translate of $s$ must appear along one of these "axes". The infinite strip in the plane whose corners are these two $s$ squares yields a contradiction similar to the one obtained above.

**Remark 3.4** While $X$ is a rather pathological complex, we note that every flat plane in $X^\infty$ is the limit of periodic flat planes. Indeed this holds for any compact 2-complex $X$ whose universal cover is isomorphic to the product of two trees [8].

4 Polygons of groups

4.1 The algebraic angle versus the geometric angle

Since the elements $c$ and $y$ have axes which intersect perpendicularly in a plane in $X^\infty$, the natural geometric angle between the subgroups $h_c$ and $h_y$
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of \( 1_X \) is \( \pi \). However, the algebraic Gersten-Stallings angle (see [7]) between these subgroups is \( \pi \). To see this, we must show that there is no non-trivial relation of the form \( c_1^y c^n y^n = 1 \).

Since \( X \) is isomorphic to the cartesian product \( \forall \times H \), of two trees and \( c \) and \( y \) correspond to distinct factors, it follows that the only relations that must be checked are rectangular (i.e., \( jk = jmj \) and \( jlj = jnj \)). However, these are easily ruled out by the anti-torus and the fact that \( X \) is nonpositively curved.

4.2 Square of groups and triangle of groups

The complex \( Z \) can be thought of in a natural way as a \( K(\pi;1) \) for a negatively curved square of groups (see [7, 5, 2]) with cyclic edge groups and trivial face group.

Because the algebraic angle between \( hci \) and \( hyi \) in \( 1_X \) is \( \pi \), it is tempting to form an analogous nonpositively curved triangle of groups \( D \). The face group of \( D \) is trivial, the edge groups of \( D \) are cyclic, the vertex groups of \( D \) are isomorphic to \( 1_X \), and each edge group of \( D \) is embedded on one (clockwise) side as \( hci \) and on the other (counter-clockwise) side as \( hyi \). This can be done so that the resulting triangle of groups \( D \) has \( \mathbb{Z}_3 \) symmetry. The tension between the algebraic and geometric angles should endow \( 1_D \) with some interesting properties. For instance, I suspect that \( 1_D \) fails to be the fundamental group of a compact nonpositively curved space, but it fails for reasons different from the usual types of problems.

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References


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