A note on the Lawrence–Krammer–Bigelow representation

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Abstract A very popular problem on braid groups has recently been solved by Bigelow and Krammer, namely, they have found a faithful linear representation for the braid group $B_n$. In their papers, Bigelow and Krammer suggested that their representation is the monodromy representation of a certain fibration. Our goal in this paper is to understand this monodromy representation using standard tools from the theory of hyperplane arrangements. In particular, we prove that the representation of Bigelow and Krammer is a sub-representation of the monodromy representation which we consider, but that it cannot be the whole representation.

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1 Introduction

Consider the ring $R = \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ of Laurent polynomials in two variables and the (abstract) free $R$-module:

$$V = \bigoplus_{1 \leq i < j \leq n} R e_{ij}$$

For $k \in \{1, \ldots, n-1\}$ define the $R$-homomorphism $\rho_k: V \to V$ by

$$\rho_k(e_{ij}) = \begin{cases} xe_{i-1,j} + (1-x)e_{ij} & \text{if } k = i-1 \\ e_{i+1,j} - xy(x-1)e_{k,k+1} & \text{if } k = i < j - 1 \\ -x^2 ye_{k,k+1} & \text{if } k = i = j - 1 \\ e_{ij} - y(x-1)^2e_{k,k+1} & \text{if } i < k < j - 1 \\ e_{i,j-1} - xy(x-1)e_{k,k+1} & \text{if } i < j - 1 = k \\ xe_{i,j+1} + (1-x)e_{ij} & \text{if } k = j \\ e_{ij} & \text{otherwise.} \end{cases}$$

The starting point of the present work is the following theorem due to Bigelow [1] and Krammer [5, 6].
Theorem 1.1 (Bigelow, [1]; Krammer, [5, 6]) Let \( B_n \) be the braid group on \( n \) strings, and let \( \sigma_1, \ldots, \sigma_{n-1} \) be the standard generators of \( B_n \). Then the mapping \( \sigma_k \mapsto \rho_k \) induces a well-defined faithful representation \( \rho: B_n \to \text{Aut}_R(V) \). In particular, the braid group \( B_n \) is linear.

Let \( V \) be an \( R \)-module. A representation of \( B_n \) on \( V \) is a homomorphism \( \rho: B_n \to \text{Aut}_R(V) \). By abuse of notation, we may identify the underlying module \( V \) with the representation if no confusion is possible. Two representations \( \rho_1 \) and \( \rho_2 \) on \( V_1 \) and \( V_2 \), respectively, are called equivalent if there exist an automorphism \( \nu: R \to R \) and an isomorphism \( f: V_1 \to V_2 \) of abelian groups such that:

- \( f(\rho_1(b)v) = \rho_2(b)f(v) \) for all \( b \in B_n \) and all \( v \in V_1 \);
- \( f(\nu(\kappa)v) = \nu(\kappa)f(v) \) for all \( \kappa \in R \) and all \( v \in V_1 \).

An LKB representation is a representation of \( B_n \) equivalent to the one of Theorem 1.1 (LKB stands for Lawrence–Krammer–Bigelow).

Let \( D \) be a disc embedded in \( \mathbb{C} \) such that \( 1, \ldots, n \) lie in the interior of \( D \) (say \( D = \{ z \in \mathbb{C} \mid |z - (n + 1)/2| \leq (n + 1)/2 \} \)), and choose a basepoint \( P_0 \) on the boundary of \( D \) (say \( P_0 = (n + 1)(1 - i)/2 \)). Define a fork to be a tree embedded in \( D \) with four vertices \( P_0, p, q, z \) and three edges, and such that \( T \cap \partial D = \{ P_0 \} \), \( T \cap \{1, \ldots, n\} = \{ p, q \} \), and all three edges have \( z \) as vertex. The LKB representation \( V \) defined in [5] is the quotient of the free \( R \)-module generated by the isotopy classes of forks by certain relations. One can easily verify that these relations are invariant by the action of \( B_n \), viewed as the mapping class group of \( D \setminus \{1, \ldots, n\} \), thus \( V \) is naturally endowed with a \( B_n \)-action. Krammer in [5] stated that a monodromy representation of \( B_n \) on a twisted homology, \( H_2(F_n; \Gamma_\pi) \), is an LKB representation and referred to Lawrence’s paper [8] for the proof. The object of the present paper is the study of this monodromy representation on \( H_2(F_n; \Gamma_\pi) \). Let \( R \to \mathbb{C} \) be an embedding. The representation considered by Lawrence [7, 8] is isomorphic to \( V \otimes \mathbb{C} \), but her geometric construction is slightly different from the construction suggested by Krammer [5]. In his proof of the linearity of braid groups, Bigelow [1] associated to each fork \( T \) an element \( S(T) \) of \( H_2(F_n; \Gamma_\pi) \), and used this correspondence to compute the action of \( B_n \) on \( H_2(F_n; \Gamma_\pi) \). A consequence of his calculation is that \( H_2(F_n; \Gamma_\pi) \otimes \mathbb{Q}(x, y) \) is isomorphic to \( V \otimes \mathbb{Q}(x, y) \).

In [3] and [2], Digne, Cohen and Wales introduced a new conceptual approach to the LKB representations based on the theory of root systems, and extended the results of [6] to all spherical type Artin groups. Using the same approach, linear representations have been defined for all Artin groups [9], but it is not
known whether the resulting representations are faithful in the non-spherical case.

The formulae in our definition of the LKB representations are those of [3]; the formulae of [5], [1] and [7] can be obtained by a change of basis which will be given in Section 5. We choose this basis because it is the most natural basis in our construction and, as pointed out before, it has an interpretation in terms of root systems which can be extended to all Artin groups.

Our goal in this paper is to understand the monodromy action on $H_2(F_n; \Gamma_\pi)$ using standard tools from the theory of hyperplane arrangements, essentially the so-called Salvetti complexes. These tools are especially interesting in the sense that they are less specific to the case “braid groups” than the tools of Lawrence, Krammer and Bigelow, and we hope they will be used in the future for constructing linear representations of other groups like Artin groups. The main result of the paper is the following:

**Theorem 1.2** There is a sub-representation $V$ of $H_2(F_n; \Gamma_\pi)$ such that:

(i) $V$ is an LKB representation;

(ii) $V \neq H_2(F_n; \Gamma_\pi)$ if $n \geq 3$;

(iii) if $V'$ is a sub-representation of $H_2(F_n; \Gamma_\pi)$ and $V'$ is an LKB representation, then $V' \subset V$, if $n \geq 4$;

(iv) $V \otimes \mathbb{Q}(x, y) = H_2(F_n; \Gamma_\pi) \otimes \mathbb{Q}(x, y)$.

We also prove that $H_2(F_n; \Gamma_\pi)$ is a free $R$-module of rank $n(n-1)/2$ and give a basis for $H_2(F_n; \Gamma_\pi)$. Note that (ii) and (iii) imply that $H_2(F_n; \Gamma_\pi)$ is not an LKB representation if $n \geq 4$. This fact is still true if $n = 3$ but, in this case, one has two minimal LKB representations in $H_2(F_n; \Gamma_\pi)$. The proof of this fact is left to the reader. Note also that the equality $V \otimes \mathbb{Q}(x, y) = H_2(F_n; \Gamma_\pi) \otimes \mathbb{Q}(x, y)$ is already known and can be found in [1].

We end this section with a detailed description of the monodromy representation $H_2(F_n; \Gamma_\pi)$.

For $1 \leq i < j \leq n$, let $H_{ij}$ be the hyperplane of $\mathbb{C}^n$ with equation $z_i = z_j$, and let $M'_n = \mathbb{C}^n \setminus (\bigcup_{1 \leq i < j \leq n} H_{ij})$ denote the complement of these hyperplanes. The symmetric group $\Sigma_n$ acts freely on $M'_n$ and $B_n$ is the fundamental group of $M'_n/\Sigma_n = M_n$. By [4], the map $p': M'_{n+2} \to M'_n$ which sends $(z_1, \ldots, z_{n+2})$ to $(z_1, \ldots, z_n)$ is a locally trivial fibration. Let

$$L_{1t} = \{z \in \mathbb{C}^2 | z_1 = t\}, \quad L_{2t} = \{z \in \mathbb{C}^2 | z_2 = t\}, \quad t = 1, \ldots, n,$$
The fibre of \( p' \) at \((1, \ldots, n)\) is the complement of the above \(2n + 1\) complex lines:

\[
F'_n = \mathbb{C}^2 \setminus (\bigcup_{t=1}^{n} L_{1t} \cup \bigcup_{t=1}^{n} L_{2t} \cup L_3)
\]

Let \( N_n = M_{n+2}' / (\Sigma_n \times \Sigma_2) \). Then \( p': M_{n+2}' \rightarrow M_n' \) induces a locally trivial fibration \( p: N_n \rightarrow M_n \) whose fibre is \( F_n = F'_n / \Sigma_2 \).

Write \( \|z\| = \max\{|z_i| \mid i = 1, \ldots, n\} \) for \( z \in \mathbb{C}^n \). The map \( s': M'_n \rightarrow M'_{n+2} \) given by

\[
s'(z) = \begin{cases} (z, n + 1, n + 2) & \text{if } \|z\| \leq n \\ (z, \frac{n+1}{2}\|z\|, \frac{n+2}{2}\|z\|) & \text{if } \|z\| \geq n \end{cases}
\]

is a well-defined section of \( p' \) and moreover induces a section \( s: M_n \rightarrow N_n \) of \( p \). So, by the homotopy long exact sequence of \( p \), the group \( \pi_1(N_n) \) can be written as a semi-direct product \( \pi_1(F_n) \rtimes B_n \).

To construct the monodromy representation we need the following two propositions whose proofs will be given in Sections 3 and 4, respectively.

**Proposition 1.3** \( H_1(F_n) \) is a free \( \mathbb{Z} \)-module of rank \( n + 1 \).

In fact, we shall see that \( H_1(F_n) \) has a natural basis \( \{[a_1], \ldots, [a_n], [c_1]\} \). Let \( H \) be the free abelian group freely generated by \( \{x, y\} \), let \( \pi_0: H_1(F_n) \rightarrow H \) be the homomorphism which sends \([a_i]\) to \( x \) for \( i = 1, \ldots, n \), and \([c_1]\) to \( y \), and let \( \pi: \pi_1(F_n) \rightarrow H_1(F_n) \rightarrow H \) be the composition of the natural projection \( \pi_1(F_n) \rightarrow H_1(F_n) \) with \( \pi_0 \).

**Proposition 1.4**

(i) The kernel of \( \pi \) is invariant for the action of \( B_n \). In particular, the action of \( B_n \) on \( \pi_1(F_n) \) induces an action of \( B_n \) on \( H \).

(ii) The action of \( B_n \) on \( H \) is trivial.

Let \( \overline{F}_n \rightarrow F_n \) be the regular covering space associated to \( \pi \). One has \( \pi_1(\overline{F}_n) = \ker \pi \), \( H \) acts freely and discontinuously on \( \overline{F}_n \), and \( \overline{F}_n / H = F_n \). The action of \( H \) on \( \overline{F}_n \) endows \( H_*(\overline{F}_n) \) with a structure of \( \mathbb{Z}[H] \)-module. This homology group is called the homology of \( F_n \) with local coefficients associated to \( \pi \), and is denoted by \( H_*(F_n; \Gamma_\pi) \).

Now, Proposition 1.4 implies that the fibration \( p: N_n \rightarrow M_n \) induces a representation \( \rho_\pi: \pi_1(M_n) = B_n \rightarrow \text{Aut}_{\mathbb{Z}[H]}(H_*(F_n; \Gamma_\pi)) \), called monodromy representation on \( H_*(F_n; \Gamma_\pi) \). In this paper, we shall consider the monodromy representation \( \rho_\pi: \pi_1(M_n) = B_n \rightarrow \text{Aut}_{\mathbb{Z}[H]}(H_2(F_n; \Gamma_\pi)) \) which is the one referred to by Krammer and Bigelow.
The Salvetti complex

An arrangement of lines in $\mathbb{R}^2$ is a finite family $\mathcal{A}$ of affine lines in $\mathbb{R}^2$. The complexification of a line $L$ is the complex line $L_C$ in $\mathbb{C}^2$ with the same equation as $L$. The complement of the complexification of $\mathcal{A}$ is

$$M(\mathcal{A}) = \mathbb{C}^2 \setminus \bigcup_{L \in \mathcal{A}} L_C.$$ 

Let $\mathcal{A}$ be an arrangement of lines in $\mathbb{R}^2$. Then $\mathcal{A}$ subdivides $\mathbb{R}^2$ into facets. We denote by $\mathcal{F}(\mathcal{A})$ the set of facets and, for $h = 0, 1, 2$, we denote by $\mathcal{F}_h(\mathcal{A})$ the set of facets of dimension $h$. A vertex is a facet of dimension 0, an edge is a facet of dimension 1, and a chamber is a facet of dimension 2. We partially order $\mathcal{F}(\mathcal{A})$ with the relation $F < G$ if $F \subset \overline{G}$, where $\overline{G}$ denotes the closure of $G$.

We now define a CW-complex of dimension 2, called the Salvetti complex of $\mathcal{A}$, and denoted by $\text{Sal}(\mathcal{A})$. This complex has been introduced by Salvetti in [10] in the more general setting of hyperplane arrangements in $\mathbb{R}^n$, $n$ being any positive integer, and Theorem 2.1, stated below for the case $n = 2$, is proved in [10] for any $n$.

To every chamber $C \in \mathcal{F}_2(\mathcal{A})$ we associate a vertex $w_C$ of $\text{Sal}(\mathcal{A})$. The 0-skeleton of $\text{Sal}(\mathcal{A})$ is $\text{Sal}_0(\mathcal{A}) = \{w_C \mid C \in \mathcal{F}_2(\mathcal{A})\}$.

Let $F \in \mathcal{F}_1(\mathcal{A})$. There exist exactly two chambers $C, D \in \mathcal{F}_2(\mathcal{A})$ satisfying $C, D > F$. We associate to $F$ two oriented 1-cells of $\text{Sal}(\mathcal{A})$: $a(F, C)$ and $a(F, D)$. The source of $a(F, C)$ is $w_C$ and its target is $w_D$ while the source of $a(F, D)$ is $w_D$ and its target is $w_C$ (see Figure 1). The 1-skeleton of $\text{Sal}(\mathcal{A})$ is the union of the $a(F, C)$’s, where $F \in \mathcal{F}_1(\mathcal{A})$, $C \in \mathcal{F}_2(\mathcal{A})$ and $F < C$.

![Figure 1: Edges in Sal(\mathcal{A})](attachment:figure1.png)

Let $P \in \mathcal{F}_0(\mathcal{A})$ and let $\mathcal{F}_P(\mathcal{A})$ be the set of chambers $C \in \mathcal{F}_2(\mathcal{A})$ such that $P < C$. Fix some $C \in \mathcal{F}_P(\mathcal{A})$ and write $\mathcal{F}_P(\mathcal{A}) = \{C, C_1, \ldots, C_{n-1}, D, D_{n-1}, \ldots, D_1\}$ (see Figure 2). The set $\mathcal{F}_P(\mathcal{A})$ has a natural cyclic ordering induced by the
orientation of $\mathbb{R}^2$, so we shall assume the list given above to be cyclically ordered in this way. Write $C = C_0 = D_0$ and $D = C_n = D_n$. For all $i = 1, \ldots, n$, there is a unique edge $a_i$ of $Sal_1(A)$ with source $w_{C_{i-1}}$ and target $w_{C_i}$ and a unique edge $b_i$ of $Sal_1(A)$ with source $w_{D_{i-1}}$ and target $w_{D_i}$. We associate to the pair $(P, C)$ an oriented 2-cell $A(P, C)$ of $Sal(A)$ whose boundary is
\[ \partial A(P, C) = a_1 a_2 \ldots a_n b^{-1}_n \ldots b^{-1}_2 b^{-1}_1. \]

The 2-skeleton of $Sal(A)$ is the union of the $A(P, C)$’s, where $P \in \mathcal{F}_0(A)$ and $C \in \mathcal{F}_P(A)$.

Figure 2: A 2-cell in $Sal(A)$

**Theorem 2.1** (Salvetti, [10]) Let $A$ be an arrangement of lines in $\mathbb{R}^2$. There exists an embedding $\delta : Sal(A) \to M(A)$ which is a homotopy equivalence.

Let $A$ be an arrangement of lines in $\mathbb{R}^2$, and let $G$ be a finite subgroup of $\text{Aff} (\mathbb{R}^2)$ which satisfies:

- $g(A) = A$ for all $g \in G$;
- $G$ acts freely on $\mathbb{R}^2 \setminus (\bigcup_{L \in \mathcal{A}} L)$.

Then $G$ acts freely on $Sal(A)$ and acts freely on $M(A)$, and the embedding $\delta : Sal(A) \to M(A)$ can be chosen to be equivariant with respect to these actions. Such an equivariant construction can be found in [11] for the particular case where $G$ is a Coxeter group, and can be carried out in the same way for any group $G$ which satisfies the above two conditions. So, $\delta : Sal(A) \to M(A)$ induces a homotopy equivalence $\bar{\delta} : Sal(A) / G \to M(A) / G$.

Recall now the spaces $F_n$ and $F'_n$ defined in Section 1. Let
\[ L_{1t} = \{ x \in \mathbb{R}^2 \mid x_1 = t \}, \quad L_{2t} = \{ x \in \mathbb{R}^2 \mid x_2 = t \}, \quad t = 1, \ldots, n, \]
\[ L_3 = \{ x \in \mathbb{R}^2 \mid x_1 = x_2 \}, \]
which are used to study the topology of the space $Sal(A) / G$. This allows us to apply results from geometric group theory, such as Theorem 2.1, to understand the homotopy type of $Sal(A) / G$. The diagram illustrates a 2-cell in $Sal(A)$ with its boundary labeled accordingly.
\[ A_n = \{L_{11}, \ldots, L_{1n}, L_{21}, \ldots, L_{2n}, L_3\}. \]

Then \( F'_n = M(A_n) \) and \( F_n = M(A_n)/\Sigma_2 \). The action of \( \Sigma_2 \) on \( \mathbb{R}^2 \) satisfies:

- \( g(A_n) = A_n \) for all \( g \in \Sigma_2 \);
- \( \Sigma_2 \) acts freely on \( \mathbb{R}^2 \setminus (\bigcup_{L\in A_n} L) \).

It follows that the embedding \( \delta : Sal(A_n) \to M(A_n) \) induces a homotopy equivalence \( \delta : Sal(A_n)/\Sigma_2 \to M(A_n)/\Sigma_2 = F_n \).

We now define a new CW-complex, denoted by \( Sal(F_n) \), obtained from the complex \( Sal(A_n)/\Sigma_2 \) by collapsing cells, and having the same homotopy type as \( F_n \). Most of our calculations in Sections 3 and 4 will be based on the description of this complex.

The complex \( Sal(A_n)/\Sigma_2 \) can be formally described as follows (see Figure 3):

The set of vertices of \( Sal(A_n)/\Sigma_2 \) is

\[ \{P_{ij} \mid 1 \leq i \leq j \leq n + 1\} \]

The set of edges of \( Sal(A_n)/\Sigma_2 \) is

\[ \{c_i \mid 1 \leq i \leq n + 1\} \cup \{a_{ij}, \bar{a}_{ij} \mid 1 \leq i \leq j \leq n\} \cup \{b_{ij}, \bar{b}_{ij} \mid 1 \leq i \leq j \leq n\}. \]

One has:

source\((a_{ij}) = \text{target}(\bar{a}_{ij}) = P_{ij}\) \hspace{1cm} \text{source}(b_{ij}) = \text{target}(\bar{b}_{ij}) = P_{i+1,j+1}

source\((\bar{a}_{ij}) = \text{target}(a_{ij}) = P_{ij+1}\) \hspace{1cm} \text{source}(\bar{b}_{ij}) = \text{target}(b_{ij}) = P_{i,j+1}

source\((c_i) = \text{target}(c_i) = P_{ii}\)

The set of 2-cells of \( Sal(A_n)/\Sigma_2 \) is

\[ \{A_{ijr} \mid 1 \leq i < j \leq n \text{ and } 1 \leq r \leq 4\} \cup \{B_{ir} \mid 1 \leq i \leq n \text{ and } 1 \leq r \leq 3\}. \]

One has:

\[
\begin{align*}
\partial A_{ij1} &= (b_{i,j-1}a_{ij})(a_{i+1,j}b_{ij})^{-1} & \partial B_{i1} &= (a_{ii}\bar{b}_{ii}c_{i+1})(c_{i}a_{ii}\bar{b}_{ii})^{-1} \\
\partial A_{ij2} &= (\bar{a}_{i+1,j}b_{ij})(b_{ij}\bar{a}_{ij})^{-1} & \partial B_{i2} &= (\bar{b}_{ii}c_{i+1}b_{ii})(\bar{a}_{ii}c_{ii}\bar{a}_{ii})^{-1} \\
\partial A_{ij3} &= (a_{ij}b_{ij})(\bar{b}_{i,j-1}a_{i+1,j})^{-1} & \partial B_{i3} &= (c_{i+1}b_{ii}\bar{a}_{ii})(\bar{b}_{ii}a_{ii}c_{ii})^{-1} \\
\partial A_{ij4} &= (b_{ij}\bar{a}_{i+1,j})(\bar{a}_{ij}b_{ij-1})^{-1}
\end{align*}
\]

Let \( K \) be the union of all the \( A_{ij4} \)’s. The set \( K \) is a subcomplex of \( Sal(A_n)/\Sigma_2 \) which contains all the vertices and all the edges of \( \{\bar{a}_{ij}, \bar{b}_{ij} \mid 1 \leq i \leq j \leq n\} \), and which is homeomorphic to a disc. Collapsing \( K \) to a single point, we obtain a new CW-complex denoted by \( Sal'(F_n) \). The complex \( Sal'(F_n) \) has a unique vertex, its set of edges is

\[ \{c_i \mid 1 \leq i \leq n + 1\} \cup \{a_{ij}, b_{ij} \mid 1 \leq i \leq j \leq n\}. \]
and its set of 2-cells is
\[ \{A_{ijr} \mid 1 \leq i < j \leq n \text{ and } 1 \leq r \leq 3\} \cup \{B_{ir} \mid 1 \leq i \leq n \text{ and } 1 \leq r \leq 3\}. \]

Note that, in \( Sal'(F_n) \), the cell \( A_{ij2} \) is a bigon with boundary \( \partial A_{ij2} = b_{ij-1}b_{ij}^{-1} \), and \( A_{ij3} \) is a bigon with boundary \( \partial A_{ij3} = a_{ij}a_{i+1j}^{-1} \). The complex \( Sal(F_n) \) is obtained from \( Sal'(F_n) \) by collapsing all the \( A_{ij2} \)'s for \( j = i + 1, \ldots, n \) to a single edge, \( b_i = b_{ii} \), and by collapsing all the \( A_{ij3} \)'s for \( i = 1, \ldots, j - 1 \) to a single edge, \( a_j = a_{jj} \). The complex \( Sal(F_n) \) has a unique vertex, its set of edges is
\[ \{c_i \mid 1 \leq i \leq n + 1\} \cup \{a_i, b_i \mid 1 \leq i \leq n\}, \]
and its set of 2-cells is
\[ \{A_{ij} = A_{ij1} \mid 1 \leq i < j \leq n\} \cup \{B_{ir} \mid 1 \leq i \leq n \text{ and } 1 \leq r \leq 3\}. \]

One has:
\[
\begin{align*}
\partial A_{ij} &= (b_ia_j)(a_jb_i)^{-1} \\
\partial B_{i1} &= (a_ic_{i+1})(c_i a_i)^{-1} \\
\partial B_{i2} &= (c_{i+1}b_i)(c_i a_i)^{-1} \\
\partial B_{i3} &= (c_{i+1}b_i)(b_i c_i)^{-1}
\end{align*}
\]
3 Computing the homology

For a loop \( \alpha \) in \( \text{Sal}_1(F_n) \), we denote by \([\alpha]\) the element of \( H_1(\text{Sal}(F_n)) \) represented by \( \alpha \). Now, standard methods in homology of CW-complexes immediately show:

**Proposition 3.1** \( H_1(F_n) = H_1(\text{Sal}(F_n)) \) is the free abelian group with basis \{\([a_1], \ldots, [a_n], [c_1]\)\}.

**Remark 3.2** We also have the equalities:

\[
[c_i] = [c_1] \quad \text{for} \quad 1 \leq i \leq n + 1 \\
[b_i] = [a_i] \quad \text{for} \quad 1 \leq i \leq n
\]

Recall that \( H \) denotes the free abelian group generated by \{\(x, y\)\}. Define \( \pi_0: H_1(F_n) \to H \) to be the homomorphism which sends \([a_i]\) to \(x\) for \( i = 1, \ldots, n \), and sends \([c_1]\) to \(y\), and let \( \pi: \pi_1(F_n) \to H_1(F_n) \to H \) be the composition.

In the following we shall describe a chain complex \( C_*(F_n; \Gamma_\pi) \) whose homology is \( H_*(F_n; \Gamma_\pi) \), define a family \( \{E_{ij} \mid 1 \leq i < j \leq n\} \) in \( H_2(F_n; \Gamma_\pi) \), and prove that \( \{E_{ij} \mid 1 \leq i < j \leq n\} \) is a basis for \( H_2(F_n; \Gamma_\pi) \otimes \mathbb{Q}(x, y) \). The sub-module generated by this family will be the LKB representation \( V \) of the statement of 1.2. We shall end Section 3 by showing that \( H_2(F_n; \Gamma_\pi) \) is a free \( \mathbb{Z}[H] \)-module.

For \( h = 0, 1, 2 \), let \( C_h \) be the set of \( h \)-cells in \( \text{Sal}(F_n) \), and let \( C_h(F_n; \Gamma_\pi) \) be the free \( \mathbb{Z}[H] \)-module with basis \( C_h \). Define the differential \( d: C_2(F_n; \Gamma_\pi) \to C_1(F_n; \Gamma_\pi) \) as follows. Let \( D \in C_2 \). Write \( \partial D = \alpha_1^{\varepsilon_1} \cdots \alpha_l^{\varepsilon_l} \), where \( \alpha_i \) is an (oriented) 1-cell and \( \varepsilon_i \in \{\pm 1\} \). Set

\[
\pi^{(i)}(D) = \begin{cases} 
\pi(\alpha_1^{\varepsilon_1} \cdots \alpha_{i-1}^{\varepsilon_{i-1}}) & \text{if} \ \varepsilon_i = 1 \\
\pi(\alpha_1^{\varepsilon_1} \cdots \alpha_{i-1}^{\varepsilon_{i-1}} \alpha_i^{-1}) & \text{if} \ \varepsilon_i = -1.
\end{cases}
\]

Then

\[
dD = \sum_{i=1}^l \varepsilon_i \pi^{(i)}(D) \alpha_i.
\]

The following lemma is a straightforward consequence of this construction.

**Lemma 3.3**

(i) \( \ker d = H_2(F_n; \Gamma_\pi) \).

(ii) Let \( d_\mathbb{Q} = d \otimes \mathbb{Q}(x, y): C_2(F_n; \Gamma_\pi) \otimes \mathbb{Q}(x, y) \to C_1(F_n; \Gamma_\pi) \otimes \mathbb{Q}(x, y) \). Then \( \ker d_\mathbb{Q} = H_2(F_n; \Gamma_\pi) \otimes \mathbb{Q}(x, y) \).
It is easy to obtain the following formulae:

\[ dA_{ij} = (x - 1)(a_j - b_i) \]
\[ dB_{i1} = (1 - y)a_i - c_i + xc_{i+1} \]
\[ dB_{i2} = -ya_i + yb_i - c_i + c_{i+1} \]
\[ dB_{i3} = (y - 1)b_i - xc_i + c_{i+1} \]

Now, we define the family \(\{E_{ij} \mid 1 \leq i < j \leq n\}\). For \(1 \leq i \leq n\) set:

\[ V_{ib} = -xyB_{i1} + x(y - 1)B_{i2} + B_{i3} \]
\[ V_{ia} = B_{i1} + x(y - 1)B_{i2} - xyB_{i3} \]
\[ V_{i0} = -yB_{i1} + (y - 1)B_{i2} - yB_{i3} \]

For \(1 \leq i < j \leq n\) set

\[ E_{ij} = (y - 1)(xy + 1)A_{ij} + (x - 1)V_{ib} + (x - 1)V_{ja} + \sum_{k=i+1}^{j-1} (x - 1)^2 V_{k0}. \]

The chains \(V_{ib}, V_{ia}\) and \(V_{i0}\) have been found with algebraic manipulations. Their interest lies in the fact that the support of each of them is \(\{B_{i1}, B_{i2}, B_{i3}\}\), the boundary of \(V_{ib}\) is a multiple of \(c_{i+1}\) minus a multiple of \(b_i\), the boundary of \(V_{ia}\) is a multiple of \(c_i\) minus a multiple of \(a_i\), and the boundary of \(V_{i0}\) is a multiple of \(c_i - c_{i+1}\). More precisely, one has:

\[ dV_{ib} = (y - 1)(xy + 1)b_i - (x - 1)(xy + 1)c_{i+1} \]
\[ dV_{ia} = -(y - 1)(xy + 1)a_i + (x - 1)(xy + 1)c_i \]
\[ dV_{i0} = (xy + 1)(c_i - c_{i+1}) \]

Another fact which will be of importance in our calculations is that all the \(A_{ls}\)-coordinates of \(E_{ij}\) are zero except the \(A_{ij}\)-one.

**Proposition 3.4** The set \(\{E_{ij} \mid 1 \leq i < j \leq n\}\) is a basis for \(\ker d_Q = H_2(F_n; \Gamma_\pi) \otimes \mathbb{Q}(x,y)\).

**Proof** It is easy to see that \(dE_{ij} = 0\) for all \(1 \leq i < j \leq n\). Moreover, since \(E_{ij}\) is the only element of \(\{E_{ls} \mid 1 \leq l < s \leq n\}\) such that the \(A_{ij}\)-coordinate is nonzero, the set \(\{E_{ij} \mid 1 \leq i < j \leq n\}\) is linearly independent.

So, to prove Proposition 3.4, it suffices to show that \(\dim(\ker d_Q) \leq n(n - 1)/2\). To do so, we exhibit a linear subspace \(W\) of \(C_2(F_n; \Gamma_\pi) \otimes \mathbb{Q}(x,y)\) of codimension \(n(n - 1)/2\) and prove that \(d_Q|_W\) is injective.
Let $\mathcal{B} = \{B_{ir} \mid 1 \leq i \leq n \text{ and } 1 \leq r \leq 3\}$, and let $W$ be the linear subspace of $C_2(F_n; \Gamma_x) \otimes \mathbb{Q}(x, y)$ generated by $\mathcal{B}$. The codimension of $W$ is clearly $n(n-1)/2$. Let $\eta: C_1(F_n; \Gamma_x) \otimes \mathbb{Q}(x, y) \to W$ be the linear map defined by

$$
\eta(a_i) = -(xy - y + 1)B_{1i1} - (y - 1)B_{i2} + yB_{i3} \\
\eta(b_i) = -xyB_{i1} + x(y - 1)B_{i2} + B_{i3} \\
\eta(c_i) = \begin{cases} -y(y - 1)B_{i1} + (y - 1)^2B_{i2} - y(y - 1)B_{i3} & \text{if } 1 \leq i \leq n \\
0 & \text{if } i = n + 1
\end{cases}
$$

Choose a linear ordering of $\mathcal{B}$ which satisfies $B_{ir} > B_{i+1r}$ for $1 \leq r, s \leq 3$. A straightforward calculation shows that the matrix of $(\eta \circ d_Q|_W)$ with respect to the ordered basis $\mathcal{B}$ is a triangular matrix with nonzero entries on the diagonal, thus $(\eta \circ d_Q|_W)$ is invertible and, therefore, $d_Q|_W$ is injective.

**Remark 3.5** Let $U \in H_2(F_n; \Gamma_x) \otimes \mathbb{Q}(x, y)$. As pointed out before, $E_{ij}$ is the only element of $\{E_{ls} \mid 1 \leq l < s \leq n\}$ such that the $A_{ij}$-coordinate is nonzero. So, if $a_{ij}$ is the $E_{ij}$-coordinate of $U$, then $a_{ij}(y - 1)(xy + 1)$ is the $A_{ij}$-coordinate of $U$.

**Proposition 3.6** $H_2(F_n; \Gamma_x)$ is a free $\mathbb{Z}[H]$-module of rank $n(n - 1)/2$.

**Proof** Let

$$
X_{ij} = \begin{cases} E_{ij} & \text{if } j = i + 1 \\
(E_{12} + E_{23} - E_{13})/(y - 1) & \text{if } i = 1 \text{ and } j = 3 \\
(xyE_{i-1i} + (x - 1)yE_{i+1i}) & \text{if } i \geq 2 \text{ and } j = i + 2 \\
- E_{ii+1} - xyE_{i-1i+1} + E_{i+2})/(y - 1)(xy + 1) & \text{if } j > i + 2 \\
(E_{i+1j-1} - E_{ij-1} - E_{i+1j}) & \text{if } j > i + 2 \\
+ E_{ij})/(y - 1)(xy + 1) & \text{if } j > i + 2
\end{cases}
$$

and let $\mathcal{X} = \{X_{ij} \mid 1 \leq i < j \leq n\}$. We shall prove that $\mathcal{X}$ is a $\mathbb{Z}[H]$-basis for $H_2(F_n; \Gamma_x)$.

Since $X_{ij}$ is a linear combination (with coefficients in $\mathbb{Q}(x, y)$) of $\{E_{ls} \mid 1 \leq l < s \leq n\}$, one has $d_Q X_{ij} = 0$. Moreover, one can easily verify

$$
X_{ij} = \begin{cases} (xy + 1)A_{12} + (xy + 1)A_{23} - (xy + 1)A_{13} & \text{if } i = 1 \text{ and } j = 3 \\
- (x - 1)B_{21} + (x^2 - 1)B_{22} - (x - 1)B_{23} & \text{if } i = 1 \text{ and } j = 3 \\
x y A_{i-1i} + y(x - 1)A_{i+1i} - A_{i+1i+2} & \text{if } i \geq 2 \text{ and } j = i + 2 \\
x y A_{i-1i+1} + A_{i+2} + x(x - 1)B_{i+1i} - (x - 1)B_{i+3} & \text{if } i \geq 2 \text{ and } j = i + 2 \\
(x - 1)B_{i+1i+1} + (x - 1)B_{i+1i+3} & \text{if } j > i + 2 \\
A_{ij} - A_{ij-1} - A_{i+1j} + A_{i+1j-1} & \text{if } j > i + 2.
\end{cases}
$$
thus $X_{ij} \in C_2(F_n; \Gamma_n)$. So, $X_{ij} \in H_2(F_n; \Gamma_n)$.

Let $<$ be the linear ordering on $\{A_{ij} \mid 1 \leq i < j \leq n\}$ defined by $A_{ij} < A_{ls}$ if either $j - i < s - l$, or $j - i = s - l$ and $i < l$. The $A_{ij}$-coordinate of $X_{ij}$ is nonzero and, for $A_{ls} > A_{ij}$, the $A_{ls}$-coordinate of $X_{ij}$ is zero, thus $X$ is linearly independent.

It remains to show that any element of $H_2(F_n; \Gamma_n)$ can be written as a linear combination of $X$ with coefficients in $\mathbb{Z}[H]$. Suppose that there exists $U \in H_2(F_n; \Gamma_n)$ which cannot be written as a linear combination of $X$ with coefficients in $\mathbb{Z}[H]$. Write

$$U = \sum \alpha_{ij}A_{ij} + \sum \beta_{ir}B_{ir},$$

where $\alpha_{ij}, \beta_{ir} \in \mathbb{Z}[H]$. By Remark 3.5, we have

$$U = \sum \frac{\alpha_{ij}}{(y-1)(xy+1)}E_{ij}.$$

Moreover, $U \neq 0$. Let $A_{ij}$ be such that $\alpha_{ij} \neq 0$ and $\alpha_{ls} = 0$ for $A_{ls} > A_{ij}$. We choose $U$ so that $A_{ij}$ is minimal (with respect to the ordering defined above).

Suppose $j > i + 1$. The $A_{ij}$-coordinate of $X_{ij}$ is 1 and, for $A_{ls} > A_{ij}$, the $A_{ls}$-coordinate of $X_{ij}$ is 0, thus $U - \alpha_{ij}X_{ij}$ would contradict the minimality of $A_{ij}$.

Suppose $i \geq 2$ and $j = i + 2$. Again, the $A_{ij}$-coordinate of $X_{ij}$ is 1 and, for $A_{ls} > A_{ij}$, the $A_{ls}$-coordinate of $X_{ij}$ is 0, thus $U - \alpha_{ij}X_{ij}$ would contradict the minimality of $A_{ij}$.

Suppose $i = 1$ and $j = 3$. Recall the equality $U = \sum \alpha_{ls}/((y-1)(xy+1))E_{ls}$. The $B_{n1}$-coordinate of $U$ is $\beta_{n1} = (x-1)\alpha_{n-1,1}/((y-1)(xy+1))$, thus $xy + 1$ divides $\alpha_{n-1,1}$. For $k = 4, \ldots, n - 1$, the $B_{k1}$-coordinate of $U$ is $\beta_{k1} = (x-1)(\alpha_{k-1,k} - xy\alpha_{k,k+1})/((y-1)(xy+1))$. It successively follows, for $k = n - 1, n - 2, \ldots, 4$, that $xy + 1$ divides $\alpha_{k-1,k}$. The $B_{3,1}$-coordinate, $B_{2,1}$-coordinate, and $B_{1,1}$-coordinate of $U$ are respectively:

$$\beta_{31} = (x-1)(\alpha_{32} + \alpha_{13} - xy\alpha_{34})/((y-1)(xy+1))$$
$$\beta_{21} = (x-1)(\alpha_{12} + y\alpha_{13} - xy\alpha_{23})/((y-1)(xy+1))$$
$$\beta_{11} = -xy(x-1)(\alpha_{12} + \alpha_{13})/((y-1)(xy+1))$$

Thus

$$\alpha_{23} + \alpha_{13} \equiv 0 \pmod{xy + 1}$$
$$\alpha_{12} + \alpha_{23} + (y+1)\alpha_{13} \equiv 0 \pmod{xy + 1}$$
$$\alpha_{12} + \alpha_{13} \equiv 0 \pmod{xy + 1}.$$
Hence $xy + 1$ divides $a_{13}$. Let $a'_{13} \in \mathbb{Z}[H]$ be such that $a_{13} = (xy + 1)a'_{13}$. The $A_{13}$-coordinate of $X_{13}$ is $-(xy + 1)$, and, for $A_{t_k} > A_{13}$, the $A_{t_k}$-coordinate of $X_{13}$ is 0, thus $U + a'_{13}X_{13}$ would contradict the minimality of $A_{t_k} = A_{13}$.

Suppose $j = i + 1$. The $B_{i+1}$-coordinate of $U$ is $\beta_{i+1} = (x - 1)a_{i+1}/((y - 1)(xy + 1))$, thus $(y - 1)(xy + 1)$ divides $a_{i+1}$. Let $a'_{i+1} \in \mathbb{Z}[H]$ such that $a_{i+1} = (y - 1)(xy + 1)a'_{i+1}$. The $A_{i+1}$-coordinate of $X_{i+1}$ is $(y - 1)(xy + 1)$ and, for $A_{t_k} > A_{i+1}$, the $A_{t_k}$-coordinate of $X_{i+1}$ is 0, thus $U - a'_{i+1}X_{i+1}$ would contradict the minimality of $A_{t_k} = A_{i+1}$.

**4 Computing the action**

We shall see in the next section how to interpret the “forks” of Krammer and Bigelow in our terminology, and, from this interpretation, how to use Bigelow’s calculations [1, Sec.4] to recover the action of $B_n$ on $H_2(F_n; \Gamma \pi)$. In this section, we shall apply our techniques for calculating the action of $B_n$ on $H_2(F_n; \Gamma \pi)$. Since most of the results of the section are well-known, some technical details will be left to the reader.

Let $k \in \{1, \ldots, n - 1\}$. Choose some small $\varepsilon > 0$ (say $\varepsilon < 1/4$) and an embedding $\mathcal{V}: \mathbb{S}^1 \times [0, 1] \to \mathbb{C}$ which satisfies:

- $\text{im} \mathcal{V} = \{z \in \mathbb{C} : 1/2 - \varepsilon \leq |z - k - 1/2| \leq 1/2 + \varepsilon\}$;
- $\mathcal{V}(\zeta, 1/2) = k + 1/2 + \zeta/2$, for all $\zeta \in \mathbb{S}^1$.

Consider the Dehn twist $T_k^0: \mathbb{C} \to \mathbb{C}$ defined by

$$(T_k^0 \circ \mathcal{V})(\zeta, t) = \mathcal{V}(e^{-2i\pi t}\zeta, t)$$

for all $(\zeta, t) \in \mathbb{S}^1 \times [0, 1]$, and $T_k^0$ is the identity outside the image of $\mathcal{V}$. Note that $T_k^0$ interchanges $k$ and $k + 1$ and fixes the other points of $\{1, \ldots, n\}$.

Consider now the diagonal homeomorphism $(T_k^0 \times T_k^0): \mathbb{C}^2 \to \mathbb{C}^2$. One has $(T_k^0 \times T_k^0)(F_n') = F_n'$, and $(T_k^0 \times T_k^0)$ commutes with the action of $\Sigma_2$ (namely, $(T_k^0 \times T_k^0) \circ g = g \circ (T_k^0 \times T_k^0)$ for all $g \in \Sigma_2$), thus $(T_k^0 \times T_k^0)$ induces a homeomorphism $T_k: F_n'/\Sigma_2 = F_n \to F_n$. Recall that $\sigma_1, \ldots, \sigma_{n-1}$ denote the standard generators of the braid group $B_n$. Then $T_k$ represents $\sigma_k$, namely $(T_k)_* = \sigma_k: \pi_1(F_n) \to \pi_1(F_n)$.

We assume that $P_b = (n + 1, 0)$ is the basepoint of $F_n$, we denote by $\ast$ the unique vertex of $Sal(F_n)$, and choose a homotopy equivalence $\delta: Sal(F_n) \to F_n$ which sends $\ast$ to $P_b$.

Let $\gamma \sim_h \beta$ denote two loops in $F_n$ based at $P_b$ that are homotopic.
Lemma 4.1  Let $k \in \{1, \ldots, n-1\}$. Then

$$T_k(\delta(a_i)) \sim_h \begin{cases} \delta(a_{i-1}) & \text{if } k = i-1 \\ \delta(a_i a_{i+1} a_i^{-1}) & \text{if } k = i \\ \delta(a_i) & \text{otherwise} \end{cases}$$

$$T_k(\delta(b_i)) \sim_h \begin{cases} \delta(b_i b_{i-1} b_i^{-1}) & \text{if } k = i-1 \\ \delta(b_{i+1}) & \text{if } k = i \\ \delta(b_i) & \text{otherwise} \end{cases}$$

$$T_k(\delta(c_i)) \sim_h \begin{cases} \delta(a_{i-1} b_i c_i b_i^{-1} a_i^{-1}) & \text{if } k = i-1 \\ \delta(c_i) & \text{otherwise} \end{cases}$$

Proof The homotopy relations for $\delta(a_i)$ follow from the fact that $\delta(a_i)$ can be drawn in the plane $C \times \{0\}$ as shown in Figure 4, $C \times \{0\}$ is invariant by $T_k$, and $T_k$ acts on $C \times \{0\}$ as the Dehn twist $T_k^0$. The other homotopy relations can be proved in the same way.

![Figure 4: The curve $\delta(a_i)$ in $C \times \{0\}$](image)

A straightforward consequence of Lemma 4.1 is:

Corollary 4.2  The action of $B_n$ on $H_1(F_n)$ is given by:

$$\sigma_k([a_i]) = \begin{cases} [a_{i-1}] & \text{if } k = i-1 \\ [a_{i+1}] & \text{if } k = i \\ [a_i] & \text{otherwise} \end{cases}$$

$$\sigma_k([c_i]) = [c_i]$$

We now consider the homomorphism $\pi: \pi_1(F_n) \to H$ defined in Section 3.

Corollary 4.3

(i) $\ker \pi$ is invariant by the action of $B_n$. In particular, the action of $B_n$ on $\pi_1(F_n)$ induces an action of $B_n$ on $H$.

(ii) The action of $B_n$ on $H$ is trivial.
So, as pointed out in Section 1, this implies:

**Corollary 4.4**  The locally trivial fibration $p: N_n \to M_n$ induces a representation $\rho_\pi: B_n = \pi_1(M_n) \to \text{Aut}_\mathbb{Z}[H](H_n(F_n; \Gamma_\pi))$.

We turn now to compute the action of $B_n$ on $H_2(F_n; \Gamma_\pi)$.

Let $k \in \{1, \ldots, n-1\}$. Define the map $S_k: \text{Sal}_1(F_n) \to \text{Sal}(F_n)$ by:

$$S_k(\ast) = a_{i-1} \quad \text{if } k = i - 1$$

$$S_k(a_i) = \begin{cases} a_i & \text{if } k = i \\ a_{i+1}^{-1}a_i & \text{otherwise} \end{cases}$$

$$S_k(b_i) = \begin{cases} b_i b_{i-1}^{-1} & \text{if } k = i - 1 \\ b_{i+1} & \text{if } k = i \\ b_i & \text{otherwise} \end{cases}$$

$$S_k(c_i) = \begin{cases} a_{i-1}b_i c_i^{-1}a_i^{-1} & \text{if } k = i - 1 \\ c_i & \text{otherwise} \end{cases}$$

By Lemma 4.1, $S_k$ induces a homomorphism

$$(S_k)_*: \pi_1(\text{Sal}_1(F_n)) \to \pi_1(\text{Sal}(F_n))$$

which is equal to $\sigma_k$. Moreover, by [4], $\text{Sal}(F_n)$ is aspherical, thus $S_k$ extends to a map $S_k: \text{Sal}(F_n) \to \text{Sal}(F_n)$ which is unique up to homotopy.

Let $K$ and $K'$ be two CW-complexes. Call a map $f: K \to K'$ a **combinatorial map** if:

- the image of any cell $C$ of $K$ is a cell of $K'$;
- if $\dim C = \dim f(C)$, then $f|_C: C \to f(C)$ is a homeomorphism.

We can, and will, suppose that every cell $D$ of $\text{Sal}(F_n)$ is endowed with a cellular decomposition such that $S_k|_D: D \to \text{Sal}(F_n)$ is a combinatorial map. Under this assumption, the map $S_k$ determines a $\mathbb{Z}[H]$-homomorphism $(S_k)_*: C_2(F_n; \Gamma_\pi) \to C_2(F_n; \Gamma_\pi)$ as follows.

Let $D \in C_2$ be a 2-cell of $\text{Sal}(F_n)$. Recall that $D$ is endowed with the cellular decomposition such that $S_k|_D: D \to \text{Sal}(F_n)$ is a combinatorial map. Let $C_2^0(D)$ denote the set of 2-cells $R$ in $D$ such that $S_k(R)$ is a 2-cell of $\text{Sal}(F_n)$.

Let $D \in C_2$ be a 2-cell of $\text{Sal}(F_n)$. Recall that $D$ is endowed with the cellular decomposition such that $S_k|_D: D \to \text{Sal}(F_n)$ is a combinatorial map. Let $C_2^0(D)$ denote the set of 2-cells $R$ in $D$ such that $S_k(R)$ is a 2-cell of $\text{Sal}(F_n)$.
the interior of $D$, and $\partial \phi_D = \alpha_i^{a1} \ldots \alpha_i^{a1}$, where $\partial \phi_D: [0,1] \to D$ is defined by $\partial \phi_D(t) = \phi_D(e^{2\pi i t})$. Now, every 2-cell $R \in C^2_2(D)$ is also endowed with a cellular map $\phi_R: D^2 \to R$ defined by $S_k \circ \phi_R = \phi_{S_k(R)}$. For $R \in C^0_2(D)$, we set $Q_R = \phi_R(1)$. This point should be understood as the starting point of the reading of the boundary of $R$.

For every $R \in C^0_2(D)$ we set $\epsilon(R) = 1$ if $S_k: R \to S_k(R)$ preserves the orientation and $\epsilon(R) = -1$ otherwise, and we choose a path $\gamma_R$ from $*$ to $Q_R$ in the 1-skeleton of $D$. Then

$$(S_k)_*(D) = \sum_{R \in C^0_2(D)} \epsilon(R)(\pi \circ S_k)(\gamma_R) S_k(R)$$

The sub-module $\ker d = H_2(F_n; \Gamma_\pi)$ is invariant by $(S_k)_*$ and the restriction of $(S_k)_*$ to $H_2(F_n; \Gamma_\pi)$ is equal to the action of $\sigma_k$ on $H_2(F_n; \Gamma_\pi)$.

**Lemma 4.5** One can choose $S_k$ such that:

$$(S_k)_*(A_{ij}) = \begin{cases} (1 - x)A_{ij} + xA_{i-1,j} & \text{if } k = i - 1 \\ A_{i+1,j} & \text{if } k = i < j - 1 \\ U_i & \text{if } i = j - 1 = k \\ A_{i,j-1} & \text{if } i < j - 1 = k \\ (1 - x)A_{ij} + xA_{i,j+1} & \text{if } k = j \\ A_{ij} & \text{otherwise} \end{cases}$$

$$(S_k)_*(B_{i1}) = \begin{cases} xB_{i3} & \text{if } k = i - 1 \\ B_{i1} + xB_{i+1,1} - x^2B_{i+1,3} & \text{if } k = i \\ B_{i1} & \text{otherwise} \end{cases}$$

$$(S_k)_*(B_{i2}) = \begin{cases} U_{i-1} + B_{i,3} - xB_{i-1,1} + xB_{i-2,1} & \text{if } k = i - 1 \\ B_{i1} + xB_{i+1,2} - xB_{i+1,3} + yU_i & \text{if } k = i \\ B_{i2} & \text{otherwise} \end{cases}$$

$$(S_k)_*(B_{i3}) = \begin{cases} B_{i,3} + xB_{i-1,3} - x^2B_{i-1,1} - x(y-1)U_i - x & \text{if } k = i - 1 \\ (y - 1)U_i + xB_{i1} & \text{if } k = i \\ B_{i3} & \text{otherwise} \end{cases}$$

where $U_i = (x - 1)(B_{i1} - B_{i2} - B_{i+1,2} + B_{i+1,3}) - yA_{i,i+1}$.

**Proof** The method for constructing the extension of $S_k$: $Sal_1(F_n) \to Sal(F_n)$ is as follows. For every $D \in C_2$, we compute $S_k(\partial D)$, and, from this result, we construct a cellular decomposition of $D$ and a combinatorial map $S_{k,D}: D \to Sal(F_n)$ which extends the restriction of $S_k$ to $\partial D$. This can be done case by case without any difficulty. The maps $S_{k,D}, D \in C_2$, determine the required
extension $S_k: \text{Sal}(F_n) \to \text{Sal}(F_n)$. With this construction, it is easy to compute $(S_k)_*(D)$ from the definition given above.

Now, from Lemma 4.5, one can easily compute the action of $B_n$ on $H_2(F_n; \Gamma_\pi)$ and obtain the following formulae.

**Proposition 4.6** Let $k \in \{1, \ldots, n-1\}$ and $1 \leq i < j \leq n$. Then

$$
\sigma_k(E_{ij}) = \begin{cases} 
xE_{i-1j} + (1-x)E_{ij} & \text{if } k = i-1 \\
nx^2yE_{k+1} & \text{if } k = i = j-1 \\
nE_{ij} - y(x-1)^2E_{k+1} & \text{if } k = i < j-1 \\
nE_{ij} - xy(x-1)E_{k+1} & \text{if } i < j-1 = k \\
nx^2E_{ij} + (1-x)E_{ij} & \text{if } k = j \\
nE_{ij} & \text{otherwise.}
\end{cases}
$$

Now we can prove our main theorem.

**Proof of Theorem 1.2** Let $V$ be the $\mathbb{Z}[H]$-submodule generated by \{ $E_{ij}$ | $1 \leq i < j \leq n$ \}.

**Proof of (i)** By Proposition 3.4, $V$ is a free $\mathbb{Z}[H]$-module with basis \{ $E_{ij}$ | $1 \leq i < j \leq n$ \}, and, by Proposition 4.6, $V$ is a sub-representation of $H_2(F_n; \Gamma_\pi)$, and is an LKB representation.

**Proof of (ii)** The element $X_{13}$ of the proof of Proposition 3.6 lies in $H_2(F_n; \Gamma_\pi)$ but does not lie in $V$.

**Proof of (iii)** Let $V'$ be a sub-representation of $H_2(F_n; \Gamma_\pi)$ such that $V'$ is an LKB representation. By definition, $V'$ is a free $\mathbb{Z}[H]$-module, and there exist a basis \{ $E'_{ij}$ | $1 \leq i < j \leq n$ \} for $V'$ and an automorphism $\nu: \mathbb{Z}[H] \to \mathbb{Z}[H]$ such that

$$
\sigma_k(E'_{ij}) = \begin{cases} 
\nu(x)E'_{i-1j} + \nu(1-x)E'_{ij} & \text{if } k = i-1 \\
n\nu(x^2y)E'_{k+1} & \text{if } k = i = j-1 \\
n\nu(x^2y)E'_{ij} & \text{if } i < j < 1 \\
n\nu(x^2y)E'_{k+1} & \text{if } i < j < 1 = k \\
n\nu(x)E'_{ij} + \nu(1-x)E'_{ij} & \text{if } k = j \\
nE'_{ij} & \text{otherwise.}
\end{cases}
$$

Note that $V'$ is generated as a $\mathbb{Z}[H]$-module by the $B_n$-orbit of $E'_{12}$, thus, in order to prove that $V' \subset V$, it suffices to show that $E'_{12} \in V$.

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For \( j = 3, \ldots, n \), let
\[
F_j = (xy - 1)E_{1j} - (xy - 1)E_{2j} + y(1 - x)E_{12}
\]
\[
G_j = x(x^2y + 1)E_{1j} + (x^2y + 1)E_{2j} + x^2y(1 - x)E_{12}.
\]

It is easy to see that:
\[
\sigma_1(E_{12}) = -x^2yE_{12}
\]
\[
\sigma_1(F_j) = -xF_j\quad \text{for } 3 \leq j \leq n
\]
\[
\sigma_1(G_j) = G_j\quad \text{for } 3 \leq j \leq n
\]
\[
\sigma_1(E_{ij}) = E_{ij}\quad \text{for } 3 \leq i < j \leq n
\]

The set \( \{E_{12}\} \cup \{F_j, G_j \mid 3 \leq j \leq n\} \cup \{E_{ij} \mid 3 \leq i < j \leq n\} \) is a basis for \( H_2(F_n;\Gamma_{\pi}) \otimes \mathbb{Q}(x,y) \), thus the eigenvalues of \( \sigma_1 \) are \(-x^2y\) of multiplicity 1, \(-x\) of multiplicity \((n - 2)\), and 1 of multiplicity \((n - 1)(n - 2)/2\). The same argument shows that \(-\nu(x^2y)\) is an eigenvalue of \( \sigma_1 \) of multiplicity 1 and \( E'_{12} \) is an eigenvector associated to this eigenvalue. Since \( n \geq 4 \), it follows that \( \nu(-x^2y) = -x^2y \) and \( E'_{12} \) is a multiple of \( E_{12} \).

Write \( E'_{12} = \lambda E_{12} \), where \( \lambda \in \mathbb{Q}(x,y) \). The \( A_{12} \)-coordinate of \( E'_{12} \), and \( B_{13} \)-coordinate of \( E'_{12} \) are \( \lambda(y - 1)(xy + 1) \) and \( \lambda(x - 1) \), respectively, and both coordinates lie in \( \mathbb{Z}[H] \), thus \( \lambda \in \mathbb{Z}[H] \) and \( E'_{12} = \lambda E_{12} \in V \).

**Proof of (iv)** This follows directly from Proposition 3.4. \( \square \)

### 5 Computing the action with the forks

Recall that \( \widetilde{F}_n \to F_n \) denotes the regular covering associated to \( \pi; \pi_1(F_n) \to H \). Choose some \( \varepsilon > 0 \) (say \( \varepsilon < 1/4 \)), and, for \( p = 1, \ldots, n \), write \( v(p) = \{z \in \mathbb{C} \mid |z - p| < \varepsilon\} \). Let \( T \) be a fork with vertices \( F_p, p, q, z \). Let \( U(T) \) be the set of pairs \( \{x, y\} \) in \( F_n \) such that either \( x \) or \( y \) lies in \( v(p) \cup v(q) \), and let \( \tilde{U}(T) \) be the pre-image of \( U(T) \) in \( \widetilde{F}_n \). Bigelow [1] associated to any fork \( T \) a disc \( S^{(1)}(T) \) embedded in \( F_n \), whose boundary lies in \( \tilde{U}(T) \), and proved that \((x-1)^2(xy+1)\partial S^{(1)}(T)\) is a boundary in \( \tilde{U}(T) \). Thus, there exists an immersed surface \( S^{(2)}(T) \) whose boundary is equal to \((x-1)^2(xy+1)\partial S^{(1)}(T)\). Note that the surface \( S^{(2)}(T) \) is unique since \( H_2(\tilde{U}(T),\mathbb{Z}) = 0 \). So, the element \( S(T) = S^{(2)}(T) - (x-1)^2(xy+1)S^{(1)}(T) \) is a well-defined 2-cycle which represents a non-trivial element of \( H_2(\widetilde{F}_n;\mathbb{Z}) = H_2(F_n;\Gamma_{\pi}) \). Moreover, the mapping \( T \mapsto S(T) \) is equivariant by the action of \( B_n \).

In [5], Krammer defined a family \( T = \{T_{pq} \mid 1 \leq p < q \leq n\} \) of forks, called **standard forks**, proved that \( T \) is a basis for the LKB representation \( V \), defined
as a quotient of the free $\mathbb{Z}[H]$-module generated by the isotopy classes of forks, and explicitly computed the action of $B_n$ on $T$. Let $\text{Sal}(F_n) \to \text{Sal}(F_n)$ be the regular covering associated to $\pi: \pi_1(\text{Sal}(F_n)) \to H$. Let $T_{pq}$ be a standard fork with vertices $F_{b,p,q,z}$, and assume $p + 1 < q$. Let

$$\text{Sal}(T_{pq}) = (\cup_{r=1}^{3} B_{pr}) \cup (\cup_{r=1}^{3} B_{qr}) \cup A_{pq} \cup (\cup_{k=p+1}^{q-1} A_{bk}) \cup (\cup_{k=p+1}^{q-1} A_{pk}),$$

and let $\overline{\text{Sal}(T_{pq})}$ be the pre-image of $\text{Sal}(T_{pq})$ in $\overline{\text{Sal}(F_n)}$. Then the pair $(\overline{\text{Sal}(F_n)}, \overline{\text{Sal}(T_{pq})})$ is homotopy equivalent to $(\overline{F_n}, \overline{U}(T_{pq}))$. Let

$$X_{pq}^{(1)} = \cup_{k=p+1}^{q-1} x^{k-1} B_{k1}.$$ 

The set $X_{pq}^{(1)}$ is a disc embedded in $\overline{\text{Sal}(F_n)}$ whose boundary lies in $\overline{\text{Sal}(T_{pq})}$, and one can choose the homotopy equivalence

$$(\overline{\text{Sal}(F_n)}, \overline{\text{Sal}(T_{pq})}) \to (\overline{F_n}, \overline{U}(T_{pq}))$$

such that $X_{pq}^{(1)}$ is sent to $S^{(1)}(T_{pq})$. Let

$$X_{pq}^{(2)} = x^{p}(x-1)V_{pb} + x^{q-1}(x-1)V_{qa} + x^{q-1}(y-1)(xy+1)A_{pq} + \sum_{k=p+1}^{q-1} x^{k-1}(x-1)(y-1)(xy+1)A_{pk}.$$ 

$X_{pq}^{(2)}$ is a 2-chain in $C_2(\overline{\text{Sal}(T_{pq})}; \mathbb{Z})$ and one has

$$dX_{pq}^{(2)} = (x-1)^2(xy+1) dX_{pq}^{(1)} =$$

$$= (x-1)^2(xy+1) \left( - \sum_{k=p+1}^{q-1} x^{k-1}(y-1)a_k - x^{p}c_{p+1} + x^{q-1}c_{q} \right).$$

(Here, $X_{pq}^{(1)} = \sum_{k=p+1}^{q-1} x^{k-1} B_{k1}$ is viewed as a 2-chain). In particular, one has the equality $S(T_{pq}) = X_{pq}^{(2)} - (x-1)^2(xy+1)X_{pq}^{(1)}$ in $H_2(F_n; \Gamma_{x}) = H_2(\text{Sal}(F_n); \Gamma_{x})$.

A similar argument shows that $S(T_{pq}) = x^{p}E_{pq}$ if $q = p + 1$.

Let

$$X_{pq} = \begin{cases} 
  x^{p}E_{pq} & \text{if } q = p + 1 \\
  X_{pq}^{(2)} - (x-1)^2(xy+1)X_{pq}^{(1)} & \text{if } q > p + 1.
\end{cases}$$

Note that, by Remark 3.5, one has

$$X_{pq} = x^{q-1}E_{pq} - \sum_{k=p+1}^{q-1} x^{k-1}(x-1)E_{pk}.$$ 

The set $\{X_{pq} \mid 1 \leq p < q \leq n\}$ is a $\mathbb{Z}[H]$-basis for the LKB representation $V$ of the statement of 1.2 and, by the above remarks, the action of $B_n$ on the $X_{pq}$'s is given by the formulae of [1, Thm. 4.1].
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References