Leafwise smoothing laminations
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Abstract We show that every topological surface lamination of a 3-manifold \( M \) is isotopic to one with smoothly immersed leaves. This carries out a project proposed by Gabai in [2]. Consequently any such lamination admits the structure of a Riemann surface lamination, and therefore useful structure theorems of Candel [1] and Ghys [3] apply.

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1 Basic notions

Definition 1.1 A lamination is a topological space which can be covered by open charts \( U_i \) with a local product structure \( i : U_i \rightarrow R^n \times X \) in such a way that the manifold-like factor is preserved in the overlaps. That is, for \( U_i \cap U_j \) nonempty,

\[ j_i^{-1} : R^n \times X \rightarrow R^n \times X \]

is of the form

\[ j_i^{-1}(t; x) = (f(t; x); g(x)) \]

The maximal continuations of the local manifold slices \( R^n \times x \) point are the leaves of the lamination. A surface lamination is a lamination locally modeled on \( R^2 \times X \). We usually assume that \( X \) is locally compact.

Definition 1.2 A lamination is leafwise \( C^n \) for \( n \geq 2 \) if the leafwise transition functions \( f(t; x) \) can be chosen in such a way that the mixed partial derivatives in \( t \) of orders less than or equal to \( n \) exist for each \( x \), and vary continuously as functions of \( x \).

A leafwise \( C^n \) structure on a lamination induces on each leaf of a \( C^n \) manifold structure, in the usual sense.
Definition 1.3 An embedding of a leafwise $C^n$ lamination $i: \mathcal{L} \rightarrow M$ into a manifold $M$ is an $C^n$ immersion if, for some $C^n$ structure on $M$, for each leaf of the embedding $\mathcal{L}$ in $M$ is $C^n$.

Note that if $i: \mathcal{L} \rightarrow M$ is an embedding with the property that the image of each leaf $i(\mathcal{L})$ is locally a $C^n$ submanifold, and these local submanifolds vary continuously in the $C^n$ topology, then there is a unique leafwise $C^n$ structure on which $i$ is a $C^n$ immersion.

A foliation of a manifold is an example of a lamination. For a foliation to be leafwise $C^n$ is a priori weaker than to ask for it to be $C^n$ immersed.

Example 1.4 Let $M$ be a manifold which is not stably smoothable, and $N$ a compact smooth manifold. Then $M \times N$ has the structure of a leafwise smooth foliation (by parallel copies of $N$), but there is no smooth structure on $M \times N$ for which the embedding of the foliation is a smooth immersion, since there is no smooth structure on $M \times N$ at all.

Remark 1.5 For readers unfamiliar with the notion, the "tangent bundle" of a topological manifold (i.e. a regular neighborhood of the diagonal in $M \times M$) is stably (in the sense of K{theory) classified by a homotopy class of maps $f: M \rightarrow BTOP$ for a certain topological space $BTOP$. There is a fibration $p: BO \rightarrow BTOP$, and the problem of lifting $f$ to $f^0: M \rightarrow BO$ such that $p^0f = f$ represents an obstruction to finding a smooth structure on $M$. For $N$ smooth as above, the composition $M \times M \rightarrow M \times N \rightarrow BTOP$ is homotopic to $f$, and therefore no lifting of the structure exists on $M \times N$ if none existed on $M$. For a reference, see [4], or the very readable [6].

With notation as above, the tangential quality of $F$ is controlled by the quality of $f(x; x)$ for each fixed $x$, for $f$ the first component of a transition function. For sufficiently large $k$ and $n-k$ questions of ambiently smoothing foliated manifolds come down to obstruction theory and classical surgery theory, as for example in [4]. But in low dimensions, the situation is more elementary and more hands-on.

2 Some 3-manifold topology

Let $M$ be a topological 3-manifold. It is a classical theorem of Moore (see [5]) that $M$ admits a PL or smooth structure, unique up to conjugacy.

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Lemma 2.1  Let \( \Sigma \) be a topological surface. Let \( S^1_j \) be a countable collection of circles, and let \( \Gamma_j : S^1_j \to \Sigma \) be a map with the following properties:

1. For each \( t \in I \), \( (\cdot ; t) : S^1_j \to \Sigma \) is an embedding.
2. For each \( t \in I \) and each pair \( j \neq k \) the intersection

\[ (S^1_j ; t) \setminus (S^1_k ; t) \]

is finite, and its cardinality is constant as a function of \( t \) away from finitely many values.
3. For every compact subset \( K \) the set of \( j \) for which \( (S^1_j ; t) \setminus K \) is nonempty for some \( t \) is finite.

Then there is a PL (resp. smooth) structure on \( \Sigma \) such that the graph of each map \( \Gamma_j (\cdot) : S^1_j \to \Sigma \) is PL (resp. smooth).

Here the graph \( \Gamma_j (\cdot) \) of \( \cdot \) is the function \( \Gamma_j (\cdot) : S^1_j \to \Sigma \) defined by

\[ \Gamma_j (\cdot)(t) = (\cdot ; t) \]

Proof  The conditions imply that the image of \( j S^1_j \) in \( \Sigma \) for a fixed \( t \) is topologically a locally finite graph. Such a structure in a 2 manifold is locally flat, and the combinatorics of any finite subgraph is locally constant away from isolated values of \( t \). It is therefore straightforward to construct a PL (resp. smooth) structure on a collar neighborhood of the image of \( j S^1_j \) in \( \Sigma \). This can be extended canonically to a PL (resp. smooth) structure on \( \Sigma \), by the relative version of Moise's theorem (see [5]). \( \square \)

Lemma 2.2  Let \( \Psi : S^1 \to \Sigma \) satisfy the conditions of lemma 2.1. Let \( \Psi_0 : S^1 \to \Sigma \) and \( \Psi_1 : S^1 \to \Sigma \) be homotopic embeddings such that \( \Psi_0(S^1) \) intersects finitely many circles in \( (\cdot ; 0) \) in finitely many points, and similarly for \( \Psi_1(S^1) \). Then there is a map \( \Psi : S^1 \to \Sigma \) which is a homotopy between \( \Psi_0 \) and \( \Psi_1 \) so that

\[ \Psi : S^1 \to \Sigma \]

satisfies the conditions of lemma 2.1.

Proof  Since the combinatorics of the image of \( \Psi \) is locally finite, and since the image of \( \Psi \) is bounded, it suffices to treat the case when \( \Psi \) is constant as a function of \( t \).

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Choose a PL structure on $\Sigma$ for which the image of $(\cdot;0)$ and $\Psi_0$ are polygonal. Then produce a polygonal homotopy from $\Psi_0$ (with respect to this polygonal structure) to a new polygonal $\Psi^0_0$ such that $\Psi^0_0(S^1)$ and $\Psi(S^1)$ intersect the image of $(\cdot; t)$ in a finite set of points in the same combinatorial configuration. Then $\Psi^0_0$ is isotopic to $\Psi_1$ rel. its intersection with the image of $(\cdot; t)$.

3 Surface laminations of 3-manifolds

**Definition 3.1** Let $F$ be a codimension one foliation of a 3-manifold $M$. A snake in $M$ is an embedding $\delta : D^2 \times I \to M$ where $D^2$ denotes the open unit disk, and $I$ the open unit interval, which extends to an embedding of the closure of $D^2 \times I$, in such a way that each horizontal disk gets mapped into a leaf of $F$. That is, $\delta : D^2 \times t \to M$.

The terminology suggests that we are typically interested in snakes which are reasonably small and thin in the leafwise direction, and possibly large in the transverse direction.

A collection of snakes in a foliated manifold intersect a leaf of $F$ in a locally finite collection of open disks. For a snake $S$, let $@S$ denote the "vertical boundary" of the closed ball $\overline{S}$; this is topologically an embedded closed cylinder transverse to $F$, intersecting each leaf in an inessential circle.

We say that an open cover of $M$ by finitely many snakes $S_i$ is combinatorially tame if the embeddings $@S_i \to M$ are locally of the form described in lemma 2.1.

Note that the induced pattern on each leaf of $F$ of the circles $@S_i \setminus$ is topologically conjugate to the transverse intersection of a locally finite collection of polygons.

**Lemma 3.2** A codimension one foliation $F$ of a closed 3-manifold $M$ admits a combinatorially tame open cover by finitely many snakes.

**Proof** Since $M$ is compact, any cover by snakes contains a finite subcover; any such cover induces a locally finite cover of each leaf. We prove the lemma by induction.

Let $S_i$ be a collection of snakes in $M$ which is combinatorially tame. Let $C_i = @S_i$ be their vertical boundaries, and let $S$ be another snake with vertical
boundary C. We will show that there is a snake $S^0$ containing $S$ such that the collection $fS \circ g \circ fS^0 \circ g$ is combinatorially tame.

Let $t$ for $t \in I$ parameterize the foliation of $S$. Let $E_i(t)$ denote the pattern of circles $C_i \setminus t$ in a neighborhood of $E(t) = C \setminus t$. By hypothesis, the $C_i$ can be thought of as polygons with respect to a PL structure on $t$. Let $E_i(t)$ denote the pattern of circles $C_i \setminus t$ in a neighborhood of $E(t) = C \setminus t$. By hypothesis, the $C_i$ can be thought of as polygons with respect to a PL structure on $t$. Then $E_i(t)$ can be straightened to a polygon $E_i(t)^0$ in general position with respect to the $E_i(t)$ in a small neighborhood, where the interior of the region in $t$ bounded by $E_i(t)^0$ contains $E(t)$. If $t$ does not intersect the horizontal boundary of any $S_i$, then the combinatorial pattern of intersections of the $E_i(t)$ is locally generic, i.e., the pattern might change, but it changes by the graph of a generic PL isotopy, by lemma 2.1.

It follows that we can extend the straightening of $E_i(t)$ to $E_i(t)^0$ for some collar neighborhood of $t = 0$. In general, a straightening of $E_i(t)$ to $E_i(t)^0$ can be extended in the positive direction until a $t_0$ which contains some lower horizontal boundary of an $S_i$. The straightening can be extended past an upper horizontal boundary of an $S_i$ without any problems, since the combinatorial pattern of intersections becomes simpler: circles disappear.

The straightening of $E(t)$ over all $t$ can be done by welding straightenings centered at the finitely many values of $t$ which contain horizontal boundary of some $S_i$. Call these critical values $t_j$. So we can produce a finite collection of straightenings $E(t)$ to $E(t)^0$ each valid on the open interval $t \in (t_{j-1}, t_{j+1})$. To weld these straightenings together at intermediate values $s_j$ where $t_j < s_j < t_{j+1}$, we insert a PL isotopy from $E(s_j)^0$ to $E(s_j)^0$ in a little collar neighborhood of $s_j$, by appealing to lemma 2.2. So these welded straightenings give a straightening of $E(t)$ for all $t \in I$, and they bound a snake $S^0$ with the requisite properties.

To prove the lemma, cover $M$ with finitely many snakes $S_i$, and apply the induction step to straighten $S_j$ while fixing $S_k$ with $k < j$. Since snakes can be straightened by an arbitrarily small (in the $C^0$ topology) homotopy, the union of straightened snakes can also be made to cover $M$, and we are done.

**Lemma 3.3** Let $M$ be a 3-manifold, and $F$ a foliation of $M$ by surfaces. Then $F$ is isotopic to a foliation such that all leaves are PL or smoothly immersed, and the images of leaves vary locally continuously in the $C^1$ topology.

**Proof** If $S_i$ is a combinatorially tame cover of $F$ by snakes, the image of the union $\bigcup S_i$ can be taken to be a PL or smooth 2 complex in $M$, whose complementary regions are polyhedral 3 manifolds. Each complementary region

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is foliated as a product by $F$. We can straighten $F$ cellwise inductively on its intersection with the skeleta of $F$. First, we keep $F \setminus 1$ constant. Then the foliation of $F \setminus (n^{-1})$ by lines can be straightened to be PL or smooth, and this straightened foliation extended in a PL or smooth manner over the product complementary regions in $M - \cdot$.

**Theorem 3.4** Let $\mathcal{F}$ be a surface lamination in a 3-manifold $M$. Then $\mathcal{F}$ is isotopic to a lamination such that all leaves are PL or smoothly immersed, and the images of leaves vary locally continuously in the $C^1$ topology.

**Proof** By the definition of a lamination, there is an open cover of $M$ by balls $B_i$ such that $\setminus B_i$ is a product lamination, which can be extended to a product foliation. It is straightforward to produce an open submanifold $N$ with $N \setminus M$ such that $N$ can be foliated by a foliation $F$ which contains $\mathcal{F}$ as a sublamination. Then the open manifold $N$ can be given a PL or smooth structure in which $F$, and hence $\mathcal{F}$, is PL or smoothly immersed, by lemma 3.3. This PL or smooth structure can be extended compatibly over $M - N$ by Moise's theorem.

**Corollary 3.5** Let $\mathcal{F}$ be a surface lamination in a 3-manifold $M$. Then $\mathcal{F}$ admits a leafwise PL or smooth structure.

In particular, such a lamination admits the structure of a Riemannian surface lamination. In Gabai's problem list [2], he lists theorem 3.4 as a "project". The corollary allows us to apply the technology of complex analysis and algebraic geometry to such laminations; in particular, the following theorems of Candel and Ghys from [1] and [3] apply:

**Theorem 3.6** (Candel) Let $F$ be an essential Riemann surface lamination of an atoroidal 3-manifold. Then there exists a continuously varying path metric on $F$ for which the leaves of $F$ are locally isometric to $\mathbb{H}^2$.

**Theorem 3.7** (Ghys) Let $F$ be a taut foliation of a 3-manifold $M$ with Riemann surface leaves. Then there is an embedding $e : M \to \mathbb{C}P^n$ for some $n$ which is leafwise holomorphic. That is, $e : \cdot \to \mathbb{C}P^n$ is holomorphic for each leaf $\cdot$. 

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