UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXXVIII
2000

SOME ELEMENTARY PROOFS OF PUISEUX'S THEOREMS

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Abstract. This paper presents a short elementary proof of the Newton–Puiseux theorem to the effect that the quotient field of the ring of Puiseux series with complex coefficients is algebraically closed. As a consequence, we deduce the classical Puiseux theorem on parametrization of one-dimensional analytic germs.

We begin with setting up the notation:

$\mathbb{C}[[z]]$ and $\mathbb{C}\{z\}$ denote the rings of formal and convergent power series, respectively;

$\mathbb{C}((z))$ and $\mathbb{C}(\{z\})$ are their quotient fields;

a formal (or convergent) Puiseux series is any series of the form $f(z^{1/r})$ with $f(z) \in \mathbb{C}[[z]]$ (or $f(z) \in \mathbb{C}\{z\}$) and $r \in \mathbb{N}$;

$\mathbb{C}[[z^*]]$ and $\mathbb{C}\{z^*\}$ denote the rings of formal and convergent Puiseux series, respectively;

$\mathbb{C}((z^*))$ and $\mathbb{C}(\{z^*\})$ are their quotient fields.

Any element $\phi(z) \in \mathbb{C}((z^*))$ can be written as $\sum_{k=n}^{\infty} a_k \cdot z^{k/r}$ with $r \in \mathbb{N}$, $n \in \mathbb{Z}$, $a_k \in \mathbb{C}$; when $a_n \neq 0$, we say that $\phi(z)$ is of order n/r, $\operatorname{ord} \phi(z) = n/r$. The units of the rings $\mathbb{C}[[z]]$, $\mathbb{C}\{z\}$, $\mathbb{C}[[z^*]]$ and $\mathbb{C}\{z^*\}$ are exactly the elements of order zero.

NEWTON–PUISEUX THEOREM. *(see e.g. [**4**], p. 61) The fields $\mathbb{C}((z^*))$ and $\mathbb{C}(\{z^*\})$ are algebraically closed.*

PROOF. It suffices to prove that any monic polynomial

$$P(z,T) = T^n + a_1(z)T^{n-1} + \cdots + a_n(z)$$

of degree $n > 1$ with coefficients in $\mathbb{C}((z^*))$ (or $\mathbb{C}(\{z^*\})$) is reducible. Making use of the Tschirnhausen transformation of variables $T' = T + 1/n \cdot a_1(z)$, we

1991 *Mathematics Subject Classification.* 30B10, 32B10.
Key words and phrases. Power series, Puiseux series, one-dimensional analytic germs.

can assume that $a_1(z) \equiv 0$. Put $r_k := \operatorname{ord} a_k(z) \in \mathbb{Q}$ unless $a_k(z) \equiv 0$, and $r := \min\{r_k/k\}$; obviously, $r_k/k - r \geq 0$ and we have equality for at least one k. Take a positive integer q so large that all the Puiseux series $a_k(z)$ are of the form $f_k(z^{1/q})$ with $f_k(z)$ in $\mathbb{C}[[z]]$ (or $\mathbb{C}\{z^*\}$), and let $r = p/q$ with $p \in \mathbb{Z}$. After the transformation of variables $z = w^q$, $T = U \cdot w^p$, we get $P(z,T) = w^{np} \cdot Q(w,U)$, where

$$Q(w,U) = U^n + b_2(w)U^{n-2} + \cdots + b_n(w)$$

with $b_k(w) = a_k(w^q)w^{-kp}$. Since $\operatorname{ord} b_k(z) \in \mathbb{Z}$ and

$$\operatorname{ord} b_k(w) = q \cdot r_k - p \cdot k = qk(r_k/k - r) \geq 0,$$

$Q(w,U)$ is a polynomial with coefficients in $\mathbb{C}[[z]]$ (or $\mathbb{C}\{z\}$); furthermore, $\operatorname{ord} b_k(z) = 0$ for at least one k, and thus $b_k(0) \neq 0$ for every such k. Therefore the complex polynomial

$$Q(0,U) = U^n + b_2(0)U^{n-2} + \cdots + b_n(0) \not\equiv (U-c)^n$$

for any $c \in \mathbb{C}$, and consequently, $Q(0,U)$ is the product of two relatively prime complex polynomials. Hence and by Hensel's lemma (see e.g. [**1**], Chap. I, §5.6), $Q(w,U)$ is the product of two polynomials $Q_1(w,U) \cdot Q_2(w,U)$ with coefficients in $\mathbb{C}[[z]]$ (or $\mathbb{C}\{z\}$). Then

$$P(z,T) = z^{nr} \cdot Q_1(z^{1/q}, z^{-r}T) \cdot Q_2(z^{1/q}, z^{-r}T),$$

and the theorem follows. □

In the sequel, ϵ_n shall denote an n-th primitive root of unity.

LEMMA. *If $f(z)$ is an element of $\mathbb{C}[[z]]$ (or $\mathbb{C}\{z\}$) and $r \in \mathbb{N}$, then*

$$Q(z,T) := (T - f(z)) \cdot (T - f(\epsilon_r z)) \cdots (T - f(\epsilon_r^{r-1} z))$$

is a monic polynomial in T with coefficients in $\mathbb{C}[[z^r]]$ (or $\mathbb{C}\{z^r\}$).

PROOF. For a proof, consider the elementary symmetric polynomials $s_j(U_1, \ldots, U_r)$ $(j = 1, 2, \ldots, r)$ in variables $U_1, \ldots, U_r$; let $S_j : \mathbb{C}[[z]] \longrightarrow \mathbb{C}[[z]]$ be defined by

$$S_j(f(z)) := s_j(f(z), f(\epsilon_r z), \ldots, f(\epsilon_r^{r-1} z)).$$

It is to be shown that $S_j(f(z)) \in \mathbb{C}[[z^r]]$ for all $f(z) \in \mathbb{C}[[z]]$. Since the mappings S_j are continuous in the maximal-adic topology of $\mathbb{C}[[z]]$, it is sufficient to prove the above assertion only for polynomials $f(z) \in \mathbb{C}[z]$. But this follows from the fact that

$$\sigma_i : \mathbb{C}(z) \longrightarrow \mathbb{C}(z), \ \ \sigma_i(z) = \epsilon_r^i \cdot z \quad (i = 0, 1, \ldots, r-1)$$

form the Galois group G of the field $\mathbb{C}(z)$ over $\mathbb{C}(z^r)$. Indeed, if $f(z) \in \mathbb{C}[z]$, then $S_j(f(z))$ is, of course, an invariant of G whence

$$S_j(f(z)) \in \mathbb{C}(z^r) \cap \mathbb{C}[z] = \mathbb{C}[z^r],$$

as desired. □

PROPOSITION. *The rings $\mathbb{C}[[z^*]]$ and $\mathbb{C}\{z^*\}$ are integral over the rings $\mathbb{C}[[z]]$ and $\mathbb{C}\{z\}$, respectively. If a Puiseux series $\phi(z)$ from $\mathbb{C}[[z^*]]$ (or from $\mathbb{C}\{z^*\}$) is a root of an irreducible monic polynomial $P(z,T)$ of degree n with coefficients in $\mathbb{C}[[z]]$ (or $\mathbb{C}\{z\}$), then $\phi(z)$ is of the form $g(z^{1/n})$ where $g(z)$ belongs to $\mathbb{C}[[z]]$ (or $\mathbb{C}\{z\}$). Moreover, the elements conjugate to $\phi(z)$ are exactly $g(\epsilon_n^i z^{1/n})$, $i = 0, 1, \ldots, n-1$.*

PROOF. The Puiseux series $\phi(z)$ is of the form $f(z^{1/r})$ where $f(z)$ belongs to $\mathbb{C}[[z]]$ (or $\mathbb{C}\{z\}$). It follows immediately from the above lemma that

$$Q(z,T) := \prod_{i=0}^{r-1} (T - f(\epsilon_r^i z^{1/r}))$$

is a monic polynomial in T with coefficients in $\mathbb{C}[[z]]$ (or $\mathbb{C}\{z\}$). Therefore the polynomial $Q(z,T)$ is divisible by $P(z,T)$ whence every root of $P(z,T)$ is of the form $f(\epsilon_r^i z^{1/r})$.

Conversely, each Puiseux series $f(\epsilon_r^i z^{1/r})$ is a root of $P(z,T)$. Indeed, $f(z) = f((z^r)^{1/r})$ is a root of the polynomial $P(z^r,T)$, and thus $f(\epsilon_r^i z)$ is a root of $P((\epsilon_r^i z)^r, T) = P(z^r, T)$. Hence $f(\epsilon_r^i z^{1/r})$ is a root of $P(z,T)$, as asserted.

Summing up, the set X of Puiseux series

$$f(\epsilon_r^i z^{1/r}) \quad (i = 0, 1, \ldots, r-1)$$

consists of precisely n roots of the polynomial $P(z,T)$. Consider now an action of the group $\mathbb{Z}_r$ on the set X defined by the formula

$$(j \bmod r, f(\epsilon_r^i z^{1/r})) \longmapsto f(\epsilon_r^{i+j} z^{1/r})).$$

As the set X is the orbit of the element $f(z^{1/r})$, the stabilizer of $f(z^{1/r})$ is a subgroup of $\mathbb{Z}_r$ of index n, and thus it is the subgroup $\mathbb{Z}_s \subset \mathbb{Z}_r$ where $r = n \cdot s$. This yields

$$f(\epsilon_s^i z^{1/r}) = f(z^{1/r}) \quad (i = 0, 1, \ldots, s-1).$$

Hence and by the lemma,

$$s \cdot f(z^{1/s}) = f((z^n)^{1/r}) + f(\epsilon_s (z^n)^{1/r}) + \cdots + f(\epsilon_s^{s-1} (z^n)^{1/r}) =$$
$$= f(z^{1/s}) + f(\epsilon_s z^{1/s}) + \cdots + f(\epsilon_s^{s-1} z^{1/s})$$

belongs to $\mathbb{C}[[z]]$ (or $\mathbb{C}\{z\}$). Therefore, $f(z^{1/s}) = g(z)$ with $g(z)$ in $\mathbb{C}[[z]]$ (or $\mathbb{C}\{z\}$). Consequently,

$$\phi(z) = f(z^{1/r}) = f((z^{1/n})^{1/s}) = g(z^{1/n}),$$

and the proof is complete. □

We conclude this paper with a corollary concerning parametrization of a one-dimensional analytic germ (cf. [**3**] or [**2**], Chap. II, §6).

PUISEUX THEOREM. *If $P(z,T) \in \mathbb{C}\{z\}[T]$ is an irreducible monic polynomial in T of degree n, then there exists a convergent power series $g(z) \in \mathbb{C}\{z\}$ such that*

$$P(z^n, T) = \prod_{i=0}^{n-1} (T - g(\epsilon_n^i z)).$$

PROOF. Indeed, according to the Newton–Puiseux theorem, the polynomial $P(z,T)$ has a root $\phi(z)$ in $\mathbb{C}(\{z^*\})$; $\phi(z)$ is, of course, a convergent Puiseux series. Now it follows from the proposition that $\phi(z) = g(z^{1/n})$ for some $g(z) \in \mathbb{C}\{z\}$, and that

$$P(z, T) = \prod_{i=0}^{n-1} (T - g(\epsilon_n^i z^{1/n})).$$

This finishes the proof. □

REMARK. The above assertion can be interpreted geometrically as follows. If an irreducible analytic germ V at $0 \in \mathbb{C}^2$ is determined by the polynomial $P(z,T) \in \mathbb{C}\{z\}[T]$, then

$$(\mathbb{C}, 0) \ni z \longmapsto (z^n, g(z)) \in (V, 0)$$

is a parametrization of V near zero.

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Received April 29, 1999

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