UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXV
1985

The Central Limit Theorem for Transformations on the Real Line

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1. Introduction. The purpose of the present paper is to prove the central limit theorem for a family of mappings of the real line into itself. This family contains as a special case the functions of the form $T(x) = a\,\mathrm{tg}(bx+c)$.

There is a large class of chaotic functions for which the central limit theorem holds. This problem has been investigated in [4], [5], [6], [8], [10] for a transformation T: $[0, 1] \to [0, 1]$ or for T: $A \to A$, where A is an n-dimensional compact connected manifold.

In the proof of the central limit theorem we shall use the technique developed in [3], [9]. It is based on the fact that the Frobenius-Perron operator corresponding to the point transformation under consideration has such property which allows us to prove the proper approximation of the strong mixing coefficient some stationary sequence of random variables.

2. Preliminaries. Let (R, Σ, ν) be a measure space with σ-finite measure ν and let $L_1(R, \Sigma, \nu)$ be the space of all functions f defined on the real line for which $|f|$ is integrable. For a measurable nonsingular function T: $R \to R$ (i.e., if $A \in \Sigma$, $\nu(A) = 0$ implies $\nu(T^{-1}(A)) = 0$) we define the Frobenius-Perron operator P_T: $L_1(R, \Sigma, \nu) \to L_1(R, \Sigma, \nu)$ by the formula

$$\int_A P_T f d\nu = \int_{T^{-1}(A)} f d\nu$$

which is valid for each set $A \in \Sigma$. The operator P_T is linear, continuous and satisfies the following conditions:

(a) P_T is positive: $f \geqslant 0 \Rightarrow P_T f \geqslant 0$;

(b) P_T preserves integrals:

$$\int_{-\infty}^{\infty} P_T f d\nu = \int_{-\infty}^{\infty} f d\nu,\ f \in L_1(R, \Sigma, \nu);$$

(c) $P_{T^n} = P_T^n$, where T^n denotes the n-th iterate of T;

(d) $P_T f = f$ iff the measure $d\mu = f d\nu$ is invariant under T, i.e., $\mu(T^{-1}(A)) = \mu(A)$ for $A \in \Sigma$.

Let $\{I_k\}_{k=-\infty}^{\infty}$ be a countable partition of the real line R such that

(i) each I_k is an open set

(ii) $I_k \cap I_j = \emptyset$ for $k \neq j$

(iii) $R \setminus \bigcup_{k=-\infty}^{k=\infty} I_k$ is a countable set

(iv) $\sup |I_k| = L < \infty$

where $|I_k|$ denotes the length of I_k.

Let $T\colon \bigcup I_k \to R$ be a function satisfying the following conditions:

(v) for any k the restriction T_k of T to the interval I_k is differentiable and its derivative T_k' is locally Lipschitzean

(vi) $|T_k'(x)| \geqslant q > 1$ for $x \in I_k$

(vii) $T_k(I_k) = R$

(viii) $\dfrac{|T_k''(x)|}{(T_k'(x))^2} \leqslant M < \infty$

(ix) $\omega(x) = \sup\limits_k \dfrac{|\psi_k'(x)|}{|I_k|}$ is bounded and integrable on R, where $\psi_k = T_k^{-1}$.

It is shown in [1] that for a function T satisfying (i)—(ix) there exists a unique probabilistic, absolutely continuous (with respect to the Lebesgue measure m) measure μ on R, invariant under T.

In the sequel we shall denote by P_T and $\bar{P}_T$ the Frobenius-Perron operator defined on $L_1(R, \Sigma, m)$ and $L_1(R, \Sigma, \mu)$, respectively, where Σ is σ-field of Borel sets.

We state the following convergence theorem

THEOREM 1. *Let $T\colon \bigcup I_k \to R$ be a function satisfying* (i)—(ix). *Then there exist constants $K>0$, $C>0$ and $0<s<1$ such that for a nonnegative function f with bounded variation on R*

(1)
$$|P_T^n f - g_\mu \|f\|_{L_1(m)}| \leqslant s^n K\Big(\bigvee_{-\infty}^{\infty} f + C\|f\|_{L_1(m)}\Big)$$

and

(2)
$$\|P_T^n f - g_\mu \|f\|_{L_1(m)}\|_{L_1(m)} \leqslant s^n K\Big(\bigvee_{-\infty}^{\infty} f + C\|f\|_{L_1(m)}\Big),$$

where g_μ is the density of the probabilistic measure μ invariant under T and $\bigvee_{-\infty}^{\infty} f$ denotes the variation of f over R.

Proof of this Theorem is given in [3].

3. A central limit theorem. Before stating our theorem let us make some assumptions. Assume that there exists an increasing sequence of positive integers $\{a_n\}_1^{\infty}$ such that

(x) there are $M_1 > 0$, $p \in N$ such that

$$\operatorname{card}\{I_k\colon I_k \subset [-a_n, a_n]\} < M_1 n^p$$

(xi) there exists $\delta>0$ such that

$$\sum_{n=1}^{\infty}\Big(\int_{-\infty}^{\infty}\omega\,dm-\int_{-a_n}^{a_n}\omega\,dm\Big)^{\frac{\delta}{2+\delta}}<\infty ,$$

where $\omega(x)$ is from (ix), and

$$\sum_{n=1}^{\infty}\Big(s^n n^p\big(\min_{[-a_n,a_n]} g_\mu\big)^{-1}\Big)^{\frac{\delta}{2+\delta}}<\infty ,$$

where s, g_μ are from Theorem 1 and p is from (x).

Under the assumptions (i)—(xi) about the point transformation $T\colon R\to R$ we shall prove the following

THEOREM 2. *If either*

(a) f is a function of bounded variation on R, or

(b) f is Hölder continuous,

then

$$\sigma^2=E_\mu(f-E_\mu f)^2+2\sum_{j=1}E_\mu[(f-E_\mu f)(f\circ T^j-E_\mu f)]<\infty \tag{3}$$

$$\lim_{n\to\infty}\mu\left\{\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}(f\circ T^j-E_\mu f)<z\right\}=\Phi_\sigma(z), \tag{4}$$

where $E_\mu f=\displaystyle\int_{-\infty}^{\infty}f\,d\mu$, $\Phi_\sigma(z)=\dfrac{1}{\sqrt{2\pi\sigma}}\displaystyle\int_{-\infty}^{z}\exp\left(\frac{-t^2}{2\sigma^2}\right)dt$ if $\sigma>0$ *and* $\Phi_0(z)=\begin{cases}1 & (z>0)\\ 0 & (z\leqslant 0)\end{cases}$.

In order to prove this theorem we need the following notations and one lemma.

Let ξ_n be a process on the probability space (R,Σ,μ), (Σ-σ-field of the Borel sets) given by the formula

$$\xi_n=\chi(T^n)$$

where $\chi=\sum_{i=-\infty}^{\infty}\alpha_i\chi_{I_i}$ with $\alpha_i\neq\alpha_j$ for $i\neq j$.

Denote by $\mathfrak{M}_k^l$ the σ-field generated by the sets of the form

$$\{x\in R\colon (\xi_k(x),\dots,\xi_l(x))\in A\},$$

where $A\subset R^{l-k+1}$ is a $(l-k+1)$-dimensional cube.

It is easy to see that $\mathfrak{M}_0^0$ is generated by the set of intervals $\{I_i\}_{i=-\infty}^{\infty}$.

We have the following

LEMMA. *If T satisfies the assumptions* (i)—(xi), *then ξ_n is a stationary process with the strong mixing coefficients*

$$\alpha(k)=\sup_{A\in\mathfrak{M}_0^0}\ \sup_{B\in\mathfrak{M}_k^\infty}|\mu(A\cap B)-\mu(A)\mu(B)|$$

satisfying the following inequalities

(5) $$\alpha(n) \leqslant s^n(cn^p+d)(\min_{[-a_n, a_n]} g_\mu)^{-1} + 2(\int_{-\infty}^{\infty} \omega dm - \int_{-a_n}^{a_n} \omega dm)$$

where c, d are some constants, ω is from (ix) *and* g_μ *is the density of the invariant measure* μ.

Proof. Let B be a Borel set and $A = \bigcup_{i \in J} I_i$, where J is a subset of the integer set. Define $A_n' = A \cap [-a_n, a_n]$ and $A_n'' = A - A_n'$. We have

$$\begin{aligned}|\mu(T^{-n}(B) \cap A) - \mu(B)\mu(A)| &= |\mu(T^{-n}(B) \cap (A_n' \cup A_n'')) - \mu(B)\mu(A_n' \cup A_n'')| \\ &= \mu(T^{-n}(B) \cap A_n') - \mu(A_n')\mu(B) + \mu(T^{-n}(B) \cap A_n'') - \mu(A_n'')\mu(B)| \\ &\leqslant |\mu(T^{-n}(B) \cap A_n') - \mu(A_n')\mu(B)| + 2\mu(A_n'').\end{aligned}$$

Since $(\bar{P}_T f) g_\mu = P_T(f g_\mu)$ for $f \in L_1(R, \Sigma, \mu)$ and $\mu(T^{-n}(B) \cap A_n') = \int_{-\infty}^{\infty} (\bar{P}^n \chi_{A_n'}) \chi_B d\mu$, by (1) and (x) we have

$$\begin{aligned}|\mu(T^{-n}(B) \cap A_n') - \mu(A_n')\mu(B)| &= \int_{-\infty}^{\infty} (\bar{P}^n \chi_{A_n'} - \mu(A_n')) \chi_B d\mu| \\ &\leqslant \mu(B) \max_{[-a_n, a_n]} |\bar{P}^a \chi_{A_n'} - \mu(A_n')| \leqslant \max_{[-a_n, a_n]} \left| \frac{P^n(\chi_{A_n'} g_\mu) - \|\chi_{A_n'} g_\mu\|_{L_1(m)} g_\mu}{g_\mu} \right| \\ &\leqslant s^n K (\bigvee_{-\infty}^{\infty} (\chi_{A_n'} g_\mu) + C \|\chi_{A_n'} g_\mu\|_{L_1(m)}) (\min_{[-a_n, a_n]} g_\mu)^{-1} \\ &\leqslant s^n K (2M_1 n^p \max g_\mu + \bigvee_{-\infty}^{+\infty} g_\mu + C \|g_\mu\|_{L_1(m)}) (\min_{[-a_n, a_n]} g_\mu)^{-1} \\ &\leqslant s^n (cn^p + d)(\min_{[-a_n, a_n]} g_\mu)^{-1},\end{aligned}$$

where $c = 2M_1 K \max g_\mu + \bigvee_{-\infty}^{+\infty} g_\mu$ and $d = CK\|g_\mu\|_{L_1(m)}$.

On the other hand

$$|\mu(A_n'')| = |\int_{A_n''} d\mu| = |\int_{A_n''} g_\mu dm| \leqslant \int_{A_n''} \omega dm \leqslant \int_{-\infty}^{\infty} \omega dm - \int_{-a_n}^{a_n} \omega dm.$$

Therefore

$$|\mu(T^{-n}(B) \cap A) - \mu(A) \cdot \mu(B)| \leqslant s^n(cn^p+d)(\min_{[-a_n, a_n]} g_\mu)^{-1} + 2(\int_{-\infty}^{\infty} \omega dm - \int_{-a_n}^{a_n} \omega dm).$$

Since

$$\alpha(n) \leqslant \sup_{A \in \mathfrak{M}_0^0} \sup_{B \in \Sigma} |\mu(T^{-n}(B) \cap A) - \mu(A)\mu(B)|,$$

we have

$$\alpha(n) \leqslant s^n(cn^p+d)(\min_{[-a_n, a_n]} g_\mu)^{-1} + 2(\int_{-\infty}^{\infty} \omega dm - \int_{-a_n}^{a_n} \omega dm).$$

This completes the proof of the lemma.

Proof of Theorem 2. The estimation of the strong mixing coefficients for a stationary process ξ_n allows us to invoke Theorem 18.6.2 of [1] directly in order to obtain a central limit theorem. Thus, we need only to prove

$$E_\mu|f|^{2+\delta}<\infty \tag{6}$$

$$\sum_{k=1}^{\infty}[E_\mu|f-E_\mu\{f|\mathfrak{M}_0^k\}|^{\frac{2+\delta}{1+\delta}}]^{\frac{1+\delta}{2+\delta}}<\infty, \tag{7}$$

where $E_\mu\{f|\mathfrak{M}_0^k\}$ is the conditional expectation of f given a σ-field $\mathfrak{M}_0^k$, and

$$\sum_{n=1}^{\infty}(\alpha(n))^{\frac{\delta}{2+\delta}}<\infty \tag{8}$$

for some $\delta>0$ and any f satisfying the assumptions of Theorem 2.

Let f be of bounded variation. The inequality (6) is obvious. The condition (8) is a simple consequence of the inequality (5) and the assumption (ix). Therefore, it remains only to prove (7). The σ-field $\mathfrak{M}_0^k$ is generated by the intervals of the form

$$\bigcap_{i=1}^{k} T^{-i}(I_{j_i}). \tag{9}$$

Denote by Q_k the set of intervals given by (9). It is obvious that the length of each interval from Q_k is not greater then $L(\inf|T'|)^{-k}$, that is

$$m(A)\leqslant L(\inf|T'|)^{-k}\leqslant Lq^{-k} \quad \text{for every } A\in Q_k. \tag{10}$$

By (10) for any f of bounded variation we have

$$E_\mu|f-E_\mu\{f|\mathfrak{M}_0^k\}|^2\leqslant\sum_{A\in Q_k}\int_A\Big[f-\frac{1}{\mu(A)}\int_A f\,d\mu\Big]^2 d\mu$$

$$\leqslant\sum_{A\in Q_k}\int_A\Big(\bigvee_A f\Big)^2 d\mu\leqslant\bigvee_{-\infty}^{\infty} f\sum_{A\in Q_k}\int_A\Big(\bigvee_A f\Big)d\mu$$

$$\leqslant\bigvee_{-\infty}^{\infty} f\sum_{A\in Q_k}\Big(\bigvee_A f\Big)\sup_{A\in Q_k}\mu(A)\leqslant\Big(\bigvee_{-\infty}^{\infty} f\Big)^2 L\sup\omega\cdot(\inf|T'|)^{-k}.$$

Hence since $L^\theta(\mu)\supset L^2(\mu)$ for $\theta=\dfrac{2+\delta}{1+\delta}<2$, we obtain (7). Now, let f be Hölder continuous. We have

$$\sup_R|f-E_\mu\{f|\mathfrak{M}_0^k\}|\leqslant\sup_{A\in Q_k}\sup_A\Big|f-\frac{1}{\mu(A)}\int_A f\,d\mu\Big|$$

$$\leqslant\sup_{A\in Q_k}\sup_{x,y\in A}|f(x)-f(y)|\leqslant\bar{K}(m(A))^\beta\leqslant\bar{K}((\inf|T'|)^\beta)^{-k}$$

for some $\bar{K}>0$ and $\beta>0$. This yields (7) and completes the proof of Theorem 2.

4. Example. The conditions (i)—(xi) are satisfied for $T(x) = a\operatorname{tg}(bx+c)$ if $|ab|>1$. In fact, we have

$$|T'(x)| = \frac{|ab|}{\cos^2(bx+c)} \geqslant |ab| > 1,$$

$$\frac{|T''(x)|}{|T'(x)|^2} = \left|\frac{1}{a}\sin 2(bx+c)\right| \leqslant \frac{2}{|a|}, \text{ and}$$

$$\omega(x) = \sup \frac{|\psi_k'(x)|}{|I_k|} = \frac{|a|}{\pi(a^2+x^2)}.$$

Let $a_n = n^2$. We have

$$\int_{-\infty}^{\infty} \omega\, dm - \int_{-n^2}^{n^2} \omega\, dm = \left(\pi - 2\operatorname{arctg}\frac{n^2}{a}\right) b^{-1} = 0\left(\frac{1}{n^2}\right).$$

Therefore for $\delta>2$;

$$\sum_{n=1}^{\infty} \Big(\int_{-\infty}^{\infty} \omega\, dm - \int_{-a_n}^{a_n} \omega\, dm\Big)^{\frac{\delta}{2+\delta}} < \infty.$$

From the fact that

$$(P_T f)(x) = \sum_{k=-\infty}^{\infty} f\left(\frac{1}{b}\left(\operatorname{arctg}\frac{x}{a} - k\pi - c\right)\right) \frac{|a|}{\pi(a^2+x^2)|b|}$$

we obtain

$$g_\mu = P_T g_\mu = \sum_{k=-\infty} g_\mu\left(\frac{1}{b}\left(\operatorname{arctg}\frac{x}{a} - k\pi - c\right)\right) \frac{|a|}{\pi(a^2+x^2)|b|} \geqslant g_\mu\left(\frac{1}{b}\operatorname{arctg}\frac{x}{a} - \frac{c}{b}\right) \cdot \frac{|a|}{\pi(a^2+x^2)|b|}$$

$\geqslant \dfrac{\bar{c}}{a^2+x^2}$, where $\bar{c} = \dfrac{|a|}{\pi|b|} \inf\limits_{\left[-\frac{\pi}{2b}, \frac{\pi}{2b}\right]} g_\mu \geqslant \dfrac{|b|}{2\pi} \exp\left(-\varrho\dfrac{\pi}{|b|}\right)$ for some constant ϱ (see [3]).

Thus, since (x) is satisfied with $p = 2$ and $M_1 = \dfrac{2|b|}{\pi}$ therefore

$$\sum_{n=1}^{\infty} \Big(s^n n^p \big(\min_{[-a_n, a_n]} g_\mu\big)^{-1}\Big)^{\frac{\delta}{2+\delta}} \leqslant \sum_{n=1}^{\infty} \left(s^n n^2 \frac{(a^2+n^4)}{\bar{c}}\right)^{\frac{\delta}{2+\delta}} < \infty.$$

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Received March 16, 1981