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Remarks on Some Orientor Equations

1. We first recall fundamental definitions, taken partially from [2]. If (X, ϱ) is a metric space then by $\mathfrak{T}(X)$ we denote the family of all subsets of X and by $\mathrm{comp}(X)$ the set of all compact subsets of X. The set $\mathrm{comp}(X)$ is provided with the classical Hausdorff metric ϱ defined as follows: if $A, B \in \mathrm{comp}(X)$, then $\varrho(A, B) \overset{\mathrm{df}}{=} \max(\max\{d(a, B)\colon a \in A\}, \max\{d(b, A)\colon b \in B\})$, where $d(c, C) \overset{\mathrm{df}}{=} \inf\{\varrho(x, c)\colon x \in C\}$ for $c \in X$, $C \in \mathrm{comp}(X)$.

A mapping $F\colon X \to \mathrm{comp}(X)$ is said to be upper semi-continuous if for all sequences $\{x^p\}, \{y^p\} \subset X$, the conditions: $x^p \to x^0$, $y^p \to y^0$, $x^p \in F(y^p)$ imply: $x^0 \in F(y^0)$; F is called compact if for every bounded subset A of X the closure of the set $\bigcup\{F(x)\colon x \in A\}$ is compact in X. The mapping F is said to be completely continuous if it is upper semi-continuous and compact.

If E is a Banach space, then we put

$$cf(E) \overset{\mathrm{df}}{=} \{A \in \mathfrak{T}(E)\colon A \neq \emptyset,\ A \text{ is closed},\ A \text{ is convex}\},$$

and for $A, B \in \mathfrak{T}(E)$, $t \in R$ (= the real line):

$$A + B \overset{\mathrm{df}}{=} \{x + y\colon x \in A,\ y \in B\},\quad tA \overset{\mathrm{df}}{=} \{tx\colon x \in A\}.$$

We shall write $A + x^0$ in the place of $A + \{x^0\}$.

If $G, F\colon E \to cf(E)$ then $G \subset F$ means $G(x) \subset F(x)$ for $x \in E$, $F + G$ is the mapping from E into $cf(E)$ defined by the formula $(F + G)(x) = F(x) + G(x)$. A mapping $F\colon E \to cf(E)$ is called homogeneous if for every $x \in E$ and every $t \in R$, $F(tx) = tF(x)$.

Theorem L-O (Theorem 1 in the paper of A. Lasota and Z. Opial [2]). Let F and G be two completely continuous mappings from E into $cf(E)$, such that F is homogeneous, $G \subset F + K$, where $K\colon E \to cf(E)$ is a constant map defined by $x \mapsto \{y\colon |y| \leqslant r\}$, r fixed ($|\cdot|$ the norm in E), and moreover if $x \in F(x)$ then $x = 0$. Under these assumptions there exists $x \in E$ such that $x \in G(x)$.

Using the methods of A. Lasota and Z. Opial presented in [1] and [2], we shall now give some results concerning a functional-differential equation of the orientor type. A similar result is given in [3].

2. If Δ is a closed interval in R, then by $C^n(\Delta)$ we denote the space of all continuous mappings $u\colon \Delta \to R^n$ provided with the norm of the uniform convergence: $\|u\| = \max\{|u(t)|\colon t \in \Delta\}$, where $|u|$ is the Euclidean norm of u in R^n.

Let a, b, c be fixed real numbers such that $c \leqslant a < b$. By C we denote the set of all continuous mappings from $[a, b]$ into $[c, b]$. For a set $A \subset R^n$ we put $|A| = \sup\{|x|\colon x \in A\}$. If $F\colon R^n \to cf(R^n)$, then we put

$$\int_a^x F(t)\,dt \overset{\text{df}}{=} \{\int_a^x w(t)\,dt\colon w(t) \in F(t),\ w \in L^1_{[a,b]}\}$$

(see for instance [2]), where $L^1_{[a,b]}$ denotes the family of real summable functions on $[a, b]$.

Proposition. Let F and L be mappings from R^n into $cf(R^n)$ and from $C^n([a, b])$ into R^n respectively, and let $u \in C^n([a, b])$ and $r \in R^n$ be fixed. Then the two following conditions are equivalent, under the assumption that u is absolutely continuous:

(a) $$u'(x) \in F(x) \text{ for almost every } x \in [a, b], \text{ and } Lu = r;$$

(b) $$u(x) \in \int_a^x F(t)\,dt + Lu - r + u(a)\,.$$

The proof will be omitted (compare [1, 2]).

If (X, ϱ) and (Y, r) are two metric spaces, then in $X \times Y$ we shall consider the metric s as always defined by the formula:

$$s((x, y), (u, v)) = \varrho(x, u) + r(y, v)\,.$$

Hence, in particular, a mapping $H\colon R^n \times \mathcal{A} \to cf(R^n)$ where $\mathcal{A}$ is a family of compact subsets of R^n, is upper semi-continuous if and only if the following four conditions:

(1) $$x^k \in R^n,\ k = 0, 1, \ldots,\ x^k \to x^0 \text{ as } k \to \infty\,,$$

(2) $$y^k \in R^n,\ k = 0, 1, \ldots,\ y^k \to y^0 \text{ as } k \to \infty\,,$$

with respect to the Hausdorff metric,

(3) $$A^k \in \mathcal{A}^n,\ k = 0, 1, \ldots,\ A^k \to A^0 \text{ as } k \to \infty$$

(4) $$y^k \in H(x^k, A^k)\,, \quad k = 1, 2, \ldots\,,$$

imply the condition:

(5) $$y^0 \in H(x^0, A^0)\,.$$

Definition 1. (see [1]). We say that a function $h\colon [a, b] \times R^n \times \mathfrak{T}(R^n) \to cf(R^n)$ fulfils the condition of Carathéodory if and only if:

(i) for almost every $x \in [a, b]$, the mapping

(6) $$R^n \times \mathfrak{T}(R^n) \ni (u, A) \mapsto h(x, u, A) \in cf(R^n)$$

is upper semi-continuous,

(ii) for every $(u, A) \in R^n \times \mathfrak{I}(R^n)$, the mapping

$$[a, b] \ni x \mapsto h(x, u, A) \in cf(R) \tag{7}$$

is measurable.

(iii) there exist summable functions $\varrho_1, \varrho_2, \mu$: $[a, b] \to R$, such that

$$|h(x, u, A)| \leqslant \varrho_1(x)|u| + \varrho_2(x)|A| + \mu(x) \tag{8}$$

for $(x, u, A) \in [a, b] \times R^n \times \mathfrak{I}(R^n)$.

Definition 2. We say that a mapping Z: $C^n([a, b]) \to cf(C^n[a, b])$ fulfils the hypothesis (H) if the following conditions (9)-(11) hold:

$$v \in Z(u) \Rightarrow v(a) = u(a)\,, \tag{9}$$

$$Z \text{ is completely continuous}\,, \tag{10}$$

$$Z \text{ is homogeneous}\,. \tag{11}$$

3. Suppose now, that L: $C^n([a, b]) \to R^n$ is linear and continuous, f: $[a, b] \times R^n \times \mathfrak{I}(R^n) \to cf(R^n)$ fulfils the Carathéodory condition, Z is a mapping fulfilling (H) (see Def. 1 and 2), and furthermore, the mapping Φ: $[a, b] \to \text{comp}([c, b])$ = the set of all compact subsets of $[c, b]$, is such that for every $u \in C^n([c, b])$ the set

$$W = W(\Phi, f;\ u) = \{w \in L^1_{[a,b]}\colon\ w(x) \in f(x, u(x), u(\Phi(x)))\ \text{for}\ x \in [a, b]\} \tag{12}$$

is non-empty.

Remark 1. Putting

$$\Phi(x) = \{\varphi(x)\colon\ \varphi \in C, \varphi(t) \in [\varphi^0(t), \varphi^1(t)]\ \text{for}\ t \in [a, b]\}\,,$$

where $\varphi^0, \varphi^1 \in C$ are fixed and such that $\varphi^0(t) \leqslant \varphi^1(t)$ for $t \in [a, b]$, we obtain an example of Φ: $[a, b] \to \text{comp}([c, b])$ for which $W(\Phi, f;\ u) \neq \emptyset$ for any $u \in C^n([c, b])$ and any f fulfilling the Carathéodory condition.

We define now $F = F_{f,r,Z,\Phi,L}$: $C^n([c, b]) \to cf(C^n([c, b]))$, as follows:

$$F(u) = \{w \in C^n([c, b])\colon \tag{13}$$

1^0 $w|_{[a,b]}(x) \in \int\limits_a^x f(t, u(t), u(\Phi(t)))\,dt + L(u|_{[a,b]}) - r + u(a)$ for $x \in [a, b]$, and

2^0 $w|_{[c,a]} \in Z(u|_{[c,a]} + L(u|_{[a,b]}) - r)\}$.

Here $w|_{[c,a]}$ (and similar symbols) denotes the restriction of w to the set $[c, a]$; $L(u|_{[a,b]}) - r$ is considered as a constant map: $[c, a] \ni x \mapsto L(u|_{[a,b]}) - r \in R^n$.

From the assumption of the convexity of Z and f, it follows, that $F(u)$ is really a convex subset of $C^n([c, b])$ for every u. The closedness of $F(u)$ is also obvious. Note that if $w \in F(u)$, then $w(a) = u(a) + L(u|_{[a,b]}) - r$.

Lemma. If f, Z, Φ, L are as above, then F defined by (13) is for every $r \in L(C^n([a, b]))$ completely continuous.

Proof. In order to prove that F is upper semi-continuous we apply the reasoning given in [1], without any essential changes. Let $\{u^p\}$ and $\{z^p\}(\subset C^n([c,b]))$ be convergent uniformly to u^0 and z^0 respectively, i.e.: $\|u^p-u^0\|, \|z^p-z^0\|\to 0$ as $p\to\infty$. Let $z^p\in F(u^p)$ for $p=1,2,\ldots$ From the assumptions it directly follows that $z^0|_{[a,c]}\in Z(u^0|_{[c,a]}+L(u^0|_{[a,b]})-r)$. Moreover, there exists a sequence $\{v^p\}_{p=1,2\ldots}\subset(L^1_{[a,b]})^n$, such that $z^p(x)=\int_a^x v^p(t)\,dt+Lu^p-r+u^p(a)$ and

$$v^p(x)\in f\big(x,u^p(x),u^p(\Phi(x))\big)\ \text{for}\ p=1,2,\ldots,\quad x\in[a,b].$$

Since $\{u^p\}$ converges uniformly to u^0 in $[a,b]$ and f fulfils the Carathéodory condition, there exists a function $\bar v\in L^1_{[a,b]}$, such that $\|v^p(x)\|\leqslant\bar v(x)$ almost everywhere in $[a,b]$. By Lemma 2 from the paper [1], there exists a double sequence $\{\lambda_{ij}\}$ $(i=1,2,\ldots,j=i,i+1,\ldots)$ of real non-negative numbers, such that $\sum_{j=1}^{\infty}\lambda_{ij}=1$, $\lambda_{ij}=0$ for sufficiently large j (depending on i), and the sequence $\tilde v^i=\sum_{j=1}^{\infty}\lambda_{ij}v^j$ $(i=1,2,\ldots)$ converges almost everywhere in $[a,b]$ to a function $v^0\in(L^1_{[a,b]})^n$. We have $\tilde v^i(x)=\sum_{j=1}^{\infty}\lambda_{ij}v^j(x)\to v^0(x)$ almost everywhere in $[a,b]$, and then

$$\int_a^x\tilde v^i(t)\,dt=\sum_{j=1}^{\infty}\lambda_{ij}\int_a^x v^j(t)\,dt\to\int_a^x v^0(t)\,dt\ \text{as}\ i\to\infty.$$

We have

$$\sum_{j=1}^{\infty}\lambda_{ij}z^j(x)=\sum_{j=1}^{\infty}\lambda_{ij}\cdot\int_a^x v^j(t)\,dt+\sum_{j=1}^{\infty}\lambda_{ij}L(u^j)-r+\sum_{j=1}^{\infty}\lambda_{ij}u^j(a),\ \text{and then}\ z^0(x)$$

$$=\int_a^x v^0(t)\,dt+L(u^0)-r+u^0(a).$$

From the uniform convergence of $\{u^p\}$ to u^0, we obtain $u^p(\Phi(x))\to u^0(\Phi(x))$ in the sense of the Hausdorff metric. In virtue of the upper semi-continuity of f we finally obtain $v^0\in f\big(x,u^0(x),u^0(\Phi(x))\big)$, which means that $z^0\in F(u^0)$. Let now A be a bounded subset of $C^n([c,a])$. Consider the closure of $\bigcup\{F(u)\colon u\in A\}$ and denote it by $F[A]$. Let $\{z^p\}$ be the sequence of elements belonging to $F[A]$. Directly from the assumptions we have the compactness of the closure of the set: $\bigcup\{Z(u|_{[c,a]}+Lu-r)\colon u\in A\}$ (here $Lu=L(u|_{[a,b]})$). Denote this closure by $Z[A;L,r]$. Hence we can assume that $\{z^p|_{[c,a]}\}$ converges uniformly to a function $\tilde z\in Z[A;L,r]$.

There exist sequences $\{u^p\}\subset A$ and $\{v^p\}\subset(L^1_{[a,b]})^n$, such that $z^p(x)=\int_a^x v^p(t)\,dt+$ $+Lu^p-r+u^p(a)$, $v^p(x)\in f\big(x,u^p(x),u^p(\Phi(x))\big)$, for $x\in[a,b]$, $p=1,2,\ldots$.

From the Carathéodory condition (see (iii)) it follows that

$$\Big|\int_a^x v^p(t)\,dt\Big|\leqslant\int_a^x\big(\varrho_1(t)|u^p(t)|+\varrho_2(t)|u^p(\Phi(t))|+\mu(t)\big)\,dt$$

and then (since A is bounded) the family $\{\int_a^x v^p(t)dt\}$ is a family of equi-absolutely continuous functions. Hence, a subsequence of $\{z^p\}$ which converges uniformly may be chosen, because obviously, convergent sequences of $\{Lu^p\}$ and $\{u^p(a)\}$ may be chosen. Thus, $F[A]$ is compact, and the proof of Lemma is completed.

4. Let f, g: $[a, b]\times R^n\times \mathfrak{I}(R^n)\to cf(R^n)$ be two mappings fulfilling the Carathéodory condition, and let Z, Φ, L be as in the third section. Consider problems

$$u'(x)\in f(x, u(x), u(\Phi(x))) \text{ almost everywhere in } [a, b] \quad u|_{[c,a]}\in Z(u),\ L(u|_{[a,b]})=0 \tag{14}$$

and

$$u'(x)\in g(x, u(x), u(\Phi(x))) \text{ almost everywhere in } [a, b] \quad u|_{[c,a]}\in Z(u),\ L(u|_{[a,b]})=r \tag{15}$$

where $r\in L(C^n([a, b]))$ is arbitrarily fixed.
By a solution of (14) (resp. (15)) we mean any absolutely continuous function u: $[c, a]\to R^n$ fulfilling (14) (resp. (15)).

Theorem. Suppose the above assumptions on g, f, Z, Φ, L and suppose moreover that f is homogeneous with respect to $(u, A)\in R^n\times\mathfrak{I}(R^n)$, $g\subset f+K$ on $[a, b]\times R^n\times\mathfrak{I}(R^n)$, where K: $[a, b]\times R^n\times\mathfrak{I}(R^n)\to cf(R^n)$ is a map defined by $(x, u, A)\mapsto\{y\in R^n: |y|\leqslant\varrho(x)\}$, where ϱ: $[a, b]\to[0, \infty)$ is measurable (this means that putting $K(x)=K(x, u, A)$ we have a measurable function $x\mapsto|K(x)|$). Under the above assumptions, if the problem (14) has the unique solution $u=0$, then for every $r\in L(C^n([a, b]))$, the problem (15) has at least one solution.

Proof. In virtue of Proposition, the proof is reduced to a simple application of Theorem L-O. It is easy to see that:

1° $F_{g,r,Z,\Phi,L}\subset F_{f,0,Z,\Phi,L}+K$,

2° $F_{f,0,Z,\Phi,L}$ is homogeneous,

3° If $u\in F_{f,0,Z,\Phi,L}$, then $u=0$,

4° $F_{g,r,Z,\Phi,L}$ and $F_{f,0,Z,\Phi,L}$ are completely continuous (see Lemma).

Then all assumptions of Theorem L-O, for $F=F_{f,0,Z,\Phi,L}$ and $G=F_{g,r,Z,\Phi,L}$, are satisfied, and then the conclusion of this theorem holds, which means that the assertion of our theorem holds too. The proof is completed.

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