A NOTE ON PLURIPOLAR EXTENSIONS OF UNIVALENT FUNCTIONS

by Józef Siaciak

Abstract. In this note we present a detailed proof of a recent result due to Edlund and Jöricke (see Corollary 2 in [1]) saying that there exists a univalent function \( f \) in the unit disc \( D := \{ |z| < 1 \} \) smooth up to the boundary such that \( f \) does not have analytic continuation across any point of the unit circle while the pluripolar hull of its graph over \( D \) contains the graph of the function \( f_e(z) := 1/f(1/\bar{z}) \) univalent in \( D_e := \{ |z| > 1 \} \).

1. Introduction. Given a pluripolar subset of \( C^N \), its (global) pluripolar hull \( E^* \) is defined by the formula

\[
E^* := \bigcap_{U \in \mathcal{F}_E} \{ U(z) = -\infty \},
\]

where \( \mathcal{F}_E := \{ U \in PSH(C^N); U(z) = -\infty \text{ on } E \} \). A pluripolar set \( E \) is called complete pluripolar if there exists \( U \in PSH(C^N) \) such that \( E = \{ U(z) = -\infty \} \).

We say that a function \( f_2 \in \mathcal{O}(D_2) \) holomorphic in a domain \( D_2 \subset C^N \) is a pluripolar continuation of a function \( f_1 \in \mathcal{O}(D_1) \) holomorphic on a domain \( D_1 \subset C^N \), if \( \Gamma_{f_1}(D_1) \supset \Gamma_{f_2}(D_2) \), i.e. if for for every function \( U \in PSH(C^{N+1}) \) such that \( U(z, f_1(z)) = -\infty \) on \( D_1 \) we have \( U(z, f_2(z)) = -\infty \) on \( D_2 \).

If \( f \in \mathcal{O}(D) \) is a holomorphic function in a domain \( D \) in \( C^N \) then its graph \( \Gamma_f(D) \) is a pluripolar subset of \( C^{N+1} \). Given \( f \in \mathcal{O}(D) \), let \( f \) be the complete multivalued analytic function defined on a domain \( \tilde{D} \supset D \) such that \( f \) is its

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holomorphic branch on $D$. One can easily check that the pluripolar hull of \( \Gamma_f(D) \) contains

\[
\Gamma_f(\tilde{D}) := \{(z, w) \in C^N \times C; z \in D, w \in \tilde{f}(z)\},
\]

the graph of $\tilde{f}$ over $\tilde{D}$, i.e. $\Gamma_f(\tilde{D}) \supset \Gamma_{\tilde{f}}(\tilde{D})$.

The aim of this note is to prove the following slight improvement of Corollary 2 in [1].

**Theorem 1.1.** Let $E$ be a non-empty nowhere dense compact subset of the unit circle. There exists a conformal $C^\infty$-diffeomorphism

\[
f : D \mapsto G, \quad f(0) = 0,
\]

of the closure of the unit disk $D$ onto the closure of a domain $G \subset D$, strictly starlike with respect to 0, such that the following conditions are satisfied:

(a) $f$ does not have analytic continuation across any point of the unit circle;

(b) the set $E_1 := \tilde{G} \cap \partial D$ has positive Lebesgue measure, $E \subset E_1$ and the function $f_e(z) := 1/\tilde{f}(1/z)$, $z \in D_e := \{1/z; |z| < 1\}$, is a pseudo-continuation of $f$ across the set $f^{-1}(E_1)$;

(c) $\Gamma_f(D) = \Gamma_{f_e}(D_e \setminus \{\infty\}) \supset \Gamma_f(f^{-1}(E))$, i.e. the functions $f$ and $f_e$ are pluripolar continuations of each other across the graph of $f$ over the set $f^{-1}(E)$. In other words: if $P \in PSH(C^2)$ and $P(z, f(z)) = -\infty$ on $D$ (resp., $P(z, f_e(z)) = -\infty$ on $D_e \setminus \{\infty\}$) then $P(z, f_e(z)) = -\infty$ on $(D_e \setminus \{\infty\}) \cup f^{-1}(E)$ (resp., $P(z, f(z)) = -\infty$ on $D \cup f^{-1}(E)$).

2. Proof of Theorem 1.1  

First we shall prove the following

**Lemma 2.1.** Given a non-empty compact nowhere dense subset $E$ of the unit circle, one can find a domain $G \subset D$, strictly starlike with respect to 0, such that the following conditions are satisfied:

(a) $\partial G$ is a $C^\infty$-smooth Jordan curve which is real analytic at no of its points;

(b) $E \subset E_1 := \tilde{G} \cap \partial D, \lambda(E_1) > 0$ ($\lambda$ - the Lebesgue measure on $\partial D$);

(c) There exists a positive constant $m_1$ such that

\[
V_U(z) \equiv V_{\tilde{U}}(z) \geq m_1, \quad z \in E.
\]

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1It is clear that $f_e$ maps conformally the closure of $D_e$ onto the closure of $G_e := \{1/z; w \in G\}$, and $f(z) = f_e(z)$ for all $z \in f^{-1}(E_1)$, which implies that $f$ and $f_e$ are pseudo-continuations of each other across $f^{-1}(E_1)$. More information on pseudo-continuation may be found in [3].
where \( U := C \setminus (G \cup \overline{G_e}) \), \( G_e := \{ 1/z; z \in G \} \), \( V_U \) is the global extremal function of \( U \) (for the definition see [2] or [4]), and \( U := \bigcup_{j=1}^{\infty} U_j \), where the union is taken over all connected components of the open set \( U \).

**Proof of Lemma 2.1.** First we shall prove

**Claim 1.** Let \( E \) be a non-empty nowhere dense closed subset of the unit circle. There exists a sequence of open arcs \( \{ I_j \} \) of the unit circle with the following properties:

1. \( I_j \cap I_k = \emptyset \) (\( j \neq k \));
2. the set \( S := \bigcup_{j=1}^{\infty} I_j \) is dense on the unit circle;
3. the set \( \tilde{S} := \bigcup_{j=1}^{\infty} I_j \) does not intersect \( E \), and there exists \( m_1 > 0 \) such that \( V_S(z) = V_{\tilde{S}}(z) \geq m_1 \), \( z \in E \). In particular, the set \( \tilde{S} \) is thin at each point of \( E \);
4. \( \lambda(E_1) > 0 \), where \( E_1 := \partial D \setminus S \).

**Proof of Claim 1.** Let \( W = \{ w_n \} \) be a countable dense subset of \( \partial D \setminus E \). We shall choose arcs of the sequence \( \{ I_j \} \) inductively.

Let \( I_1 \) be an open arc with center \( w_1 \) such that no of its endpoints belongs to \( W \), and \( I_1 \cap E = \emptyset \). The number \( 2m_1 := \min \{ V_{I_1}(z); z \in E \cap \{ 0 \} \} \) is positive.

Fix \( k \geq 1 \). Suppose arcs \( I_1, \ldots, I_k \) with centers \( w_{n_1}, \ldots, w_{n_k} \) \( (n_1 = 1 < n_2 < \cdots < n_k) \) are already chosen in such a way that the following conditions are satisfied: \( I_j \cap I_l = \emptyset \) (\( j \neq l, j, l \leq k \)), no endpoint of \( I_j \) lies in \( W \), \( w_{n_{j+1}} \) is the element of \( W \setminus (I_1 \cup \cdots \cup I_j) \) with the smallest index, and

\[
V_{I_1 \cup \cdots \cup I_j}(z) \geq m_1 \left( 2 - \frac{1}{2} - \cdots - \frac{1}{2^j} \right), \quad z \in E \cap \{ 0 \}, \quad j = 1, \ldots, k.
\]

Let \( w_{n_{k+1}} \) be the element of \( W \setminus (I_1 \cup \cdots \cup I_k) \) with the smallest index. Let \( I_{k+1} \) be an open arc with center \( w_{n_{k+1}} \) whose endpoints do not belong to \( W \) and which is so short that

\[
V_{I_1 \cup \cdots \cup I_{k+1}}(z) \geq m_1 \left( 2 - \frac{1}{2} - \cdots - \frac{1}{2^{k+1}} \right), \quad z \in E \cap \{ 0 \}.
\]

It is clear that the sequence \( \{ I_k \} \) satisfies (1) and (2).

To show (3) it is sufficient to observe that

\[
V_S(z) = V_{\tilde{S}}(z) = \lim_{n \to \infty} V_{I_1 \cup \cdots \cup I_n}(z), \quad z \in C,
\]

is a subharmonic function with logarithmic pole at \( \infty \), harmonic on \( C \setminus \tilde{S} \), continuous on \( D \cup \tilde{S} \), \( V_S(z) = 0 \) on \( \tilde{S} \), and \( V_S(z) \geq m_1 \) for all \( z \in E \).

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\(^2\)It is clear that for every \( j \geq 1 \) the component \( U_j \) is a simple connected Jordan domain symmetric with respect to the unit circle. One may assume that \( I_j \subset U_j \).

\(^3\)Recall that \( V_{I_1} \) is identical with the Green function of \( C \setminus I_1 \) with pole at \( \infty \).
To show (4) observe that
\[ V_S(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} V_S(e^{it}) dt = \frac{1}{2\pi} \int_{E_1} \frac{1 - |z|^2}{|e^{it} - z|^2} V_S(e^{it}) dt, \quad z \in D, \]
which implies \( \lambda(E_1) > 0 \).

The proof of our claim is completed. \( \square \)

Now we pass to the proof of Lemma 2.1.

Let \( \{I_j\} \) be a sequence of arcs satisfying the conditions of Claim 1. Let \( p \in C^\infty(R) \) be a positive real-valued function of class \( C^\infty \) on the real line such that \( 0 < p(t) \leq 1 \) on \( R \), and \( p \) is nowhere \( R \)-analytic, e.g. we can take \( p(t) = \frac{1}{1 + |h(e^{it})|^2}, \quad t \in R, \) where
\[ h(z) = \sum_{n=1}^\infty 2^{-2^n} z^{2^n}, \quad |z| \leq 1. \]

Without loss of generality we may assume \( 1 \in E \). Let \( e^{\alpha_j}, e^{\beta_j} \) be endpoints of \( I_j \), where \( 0 < \alpha_j < \beta_j < 2\pi \). Put
\[ r_j(t) := p(t) \exp \left[ -\frac{1}{1 - \left( \frac{2(t - \alpha_j)}{\beta_j - \alpha_j} - 1 \right)^2} \right], \quad \alpha_j \leq t \leq \beta_j, \]
\[ r_j(t) := 0, \quad t \in [0, 2\pi] \setminus (\alpha_j, \beta_j). \]

One can check that \( r_j \in C^\infty([0, 2\pi]) \) and \( r_j^{(k)}(t) = 0 \) for all \( k \geq 1 \) and for all \( t \in [0, 2\pi] \setminus (\alpha_j, \beta_j) \), i.e. the function \( r_j \) is flat at every point of the last set. Moreover, \( r_j \) is positive at every point of the open interval \( (\alpha_j, \beta_j) \) and not \( R \)-analytic at any point of the closed interval \( [\alpha_j, \beta_j] \). It is clear that \( r_j \) can be extended to \( R \) as a \( C^\infty \) periodic function with period \( 2\pi \).

Put
\[ r(t) := \sum_{j=1}^\infty \epsilon_j r_j(t), \quad t \in R, \]
where \( \epsilon_j > 0 \) is chosen so small that
\[ \epsilon_j |r_j^{(k)}(t)| < \frac{1}{2^j}, \quad k = 0, \cdots, j, \quad j \geq 1, \quad t \in R. \]

It is clear that \( 0 \leq r(t) < 1, \quad t \in R, \) \( r \in C^\infty(R) \), \( r \) is periodic with period \( 2\pi \), and nowhere \( R \)-analytic. Observe that if \( s \) is a boundary point of \( I_k \) then \( r(s) = cr_k(s) = 0 \). Each point \( t \in E \) is a limit point of such points \( s \). Hence \( \{r(t) = 0\} = \partial D \setminus S =: E_1. \)
The domain $G$ containing 0 in its interior and bounded by the curve $\gamma$ with the parametric representation
\[ z = \gamma(t) \equiv (1 - r(t)) e^{it}, \quad 0 \leq t \leq 2\pi, \]
is strictly starlike with respect to 0. Moreover, $E \subset E_1 := \bar{G} \cap \partial D \equiv \partial D \setminus S$, and $\partial G$ is a $C^\infty$-smooth Jordan curve nowhere $R$-analytic.

We shall show that, given $0 < m < m_1$, the coefficients $\epsilon_j$ in the formula (2) can be chosen so small that
\[ V_U(z) \equiv V_{C \setminus (\bar{G} \cup \bar{G} e)}(z) \geq m, \quad z \in E. \]
The function $V_S$, given by Claim 1, is non-negative in $C$, continuous at each point of $\tilde{S}$, and $V_S(z) = 0$ on $\tilde{S}$. It follows that, given $0 < \delta < m_1$, the set $U_\delta := \{ z ; V_S(z) < \delta \}$ is an open neighborhood of $\tilde{S}$. In particular, $I_j \subset U_\delta$ for every $j \geq 1$.

Hence one can choose coefficients $\epsilon_j$ so small that both (3) and the following condition (4) are satisfied
\[ \{(1 - \epsilon_j r_j(t)) e^{it}, \frac{e^{it}}{1 - \epsilon_j r_j(t)} \} \subset U_\delta, \quad \alpha_j \leq t \leq \beta_j, \quad j \geq 1. \]

It is clear that $\tilde{U} = \bigcup_1^\infty \tilde{U}_j \subset U_\delta$, where $U_j$ is the connected component of $U$ such that $I_j \subset U_j$. Hence $V_U(z) \geq V_{U_\delta} \equiv V_S - \delta \geq m := m_1 - \delta > 0, \quad z \in E$. This ends the proof of Lemma 2.1.

We shall need the following

**Lemma 2.2.** Given $0 < \rho < 1 < R$ and a closed subset $E$ of the unit circle, assume that $U$ is an open subset of $\{ \rho < |z| < R \}$ such that $V_U(z) \geq m = \text{const} > 0$ on $E$. Then for every $0 < \theta < 1$ there exists $0 < r_0 < \rho$ such that
\[ V_{D(0,r_0) \cup U}(z) \geq \theta m, \quad z \in E. \]

**Proof.** Put $M := \sup\{V_U(z) ; \quad |z| \leq R\}$. Given $0 < \epsilon < 1$,
\[ \varphi_\epsilon(z) := (1 - \epsilon) \log \frac{|z|}{R} + \epsilon V_U(z) \]
is a subharmonic function of the class $\mathcal{L}$ such that
\[ \varphi_\epsilon(z) \leq \begin{cases} 0, & z \in U, \\ (1 - \epsilon) \log \frac{r}{R} + \epsilon M, & |z| \leq r, \end{cases} \]
where $0 < r < \rho$. Hence, if $(1 - \epsilon) \log \frac{r}{R} + \epsilon M \leq 0$ (i.e. if $0 < \epsilon \leq \frac{\log \frac{R}{M + \log \frac{R}{r}}}{1 - \epsilon}$) then $\varphi_\epsilon(z) \leq V_{D(0,r) \cup U}(z)$ on $C$. Fix $0 < \theta < 1$. Then $\varphi_\epsilon(z) \geq \theta m$ on $E$, if
\[
\epsilon \geq \frac{\theta m + \log R}{m + \log R}.
\]
Choose \( r_0 = r \) with \( 0 < r < \rho \) so small that
\[
\frac{\theta m + \log R}{m + \log R} < \frac{\log \frac{R}{r}}{M + \log \frac{R}{r}}.
\]
Then \( V_{D(0,r_0)\cup \bar{U}}(z) \geq \phi_e(z) \geq \theta m \) on \( E \) for \( \epsilon \in \left( \frac{\theta m + \log R}{m + \log R}, \frac{\log \frac{R}{r}}{M + \log \frac{R}{r}} \right) \) which ends the proof of Lemma 2.2.

We shall also need the following Theorem due to Vitushkin \cite{Vitushkin}.

Let \( K \) be a compact subset of \( C \). Then \( C \setminus K \) is a (at most) countable union of open sets \( \{U_j\} \). The set \( \partial K := \bigcup_j \partial U_j \) is called exterior boundary of \( K \). Remaining part of the boundary \( \partial K \) is denoted by \( \partial_0 K \) and called interior boundary of \( K \).

**THEOREM 2.1.** (Vitushkin \cite{Vitushkin}). If the interior boundary of a compact set \( K \) is located on a countable union of Lyapunov’s arcs then \( \mathcal{A}(K) = \mathcal{R}(K) \), where \( \mathcal{A}(K) := \mathcal{C}(K) \cap \mathcal{O}(\text{int}K) \) and \( \mathcal{R}(K) := \{ f \in \mathcal{C}(K); f \text{ is a uniform limit of a sequence of rational functions} \} \).

Now we pass to the proof of Theorem 1.1. Let \( g : \bar{G} \mapsto \bar{D}, \ g(0) = 0 \), be the \( C^\infty \)-smooth conformal mapping of the closure of the domain \( G \) given by Lemma 2.1 onto the closure of the unit disk. The function \( g_e(z) = 1/g(1/z) \), \( z \in D_e \), is \( C^\infty \)-smooth and maps \( G_e \) conformally onto \( D_e \), \( g_e(\infty) = \infty \). Moreover, \( g(z) = g_e(z) \) on \( E_1 \).

The function \( F := g \cup g_e \) is continuous on \( G \cup G_e \) and holomorphic in \( G \cup G_e \).

Fix \( R > 1 \) so large that \( \{|z| = R\} \subset G_e \), and put \( U := C \setminus (\bar{G} \cup G_e) \). By Lemma 2.1, given \( m_1 \) with \( 0 < m_1 < m \), there exists \( r_0 > 0 \) such that \( \frac{1}{r_0} > R \), \( D(0,r_0) \subset G \) and \( V_{U \cup D(0,r_0)}(z) \geq m_1 \) on \( E \). It is clear that \( V_{U \cup D(0,r_0)}(z) \leq \log^+ \frac{|z|}{r_0} \) on \( C \). Since \( U = \{ \frac{1}{z} \; | \; z \in U \} \), the function \( v(z) := V_{U \cup D(0,r_0)}(\frac{1}{z})/\log \frac{R}{r_0} \) is subharmonic on \( C \setminus \{ 0 \} \), \( v(z) = 0 \) on \( U \cup D(0,1/r_0) \), \( v(z) \leq 1 \) for \( |z| \geq 1/R \), \( v(z) \geq \frac{m_1}{\log \frac{R}{r_0}} > 0 \) on \( E \), and \( v(z) > 0 \) for all \( z \in G_e \cup E \) with \( |z| < 1/r_0 \). Hence
\[
v(z) \leq h(z) \equiv h(z, U \cup D(\infty, \frac{1}{r_0}), D(\infty, \frac{1}{R})) \text{, } |z| \geq \frac{1}{R},
\]
where \( h \) denotes the (0-1)-extremal function for the domain \( D(\infty,1/R) \) and its subset \( U \cup D(\infty,1/r_0) \).
\footnote{Recall that if \( E \) is a subset of a domain \( D \), we put \( h(z, E, D) := \sup \{ u(z) \; | \; u \in SH(D), u \leq 0 \text{ on } E, u \leq 1 \text{ on } D \} , z \in D \).}
Put $K := (\bar{G} \cup \bar{G}_e) \cap \{|z| \leq \frac{1}{r_0}\}$. By the Vitushkin Theorem there exists a sequence of rational functions $\{F_n\}$ with poles in $U \cup D(\infty, \frac{1}{r_0})$ uniformly convergent to $F$ on $K$.

Fix a function $P \in \text{PSH}(C^2)$ such that $P(z, g(z)) = -\infty$ on $G$. Let $a$ be a fixed point of $G_e \cup E$ with $|a| < 1/r_0$. It remains to show that $P(a, g_e(a)) = -\infty$.

Observe that $f_n(z) := F_n(z) + F(a) - F_n(a) \to g(z)$ uniformly on $\{|z| = \frac{1}{r_0}\}$. The sequence $\{f_n\}$ is uniformly bounded on the set $D(0, 1/r_0) \setminus U$. Therefore the sequence $v_n(z) := P(z, f_n(z))$ is uniformly upper bounded on this set.

Put $\Omega_n := \bigcup_{j=1}^{k_n} U_j$, where $k_n$ is so large that all poles of the function $f_n$, lying in $U$, are located in $\Omega_n$. By the maximum principle

$$\sup\{|f_n(z)|; z \in D(0, 1/r_0) \setminus \Omega_n\} = \sup\{|f_n(z)|; \zeta \in D(0, 1/r_0) \setminus U\}$$

for all $n \geq 1$. The function $v_n$ is subharmonic on an open neighborhood of the set $\bar{D}(0, 1/r_0) \setminus \Omega_n$. Put $C := \sup_{n \geq 1} \sup\{v_n(z); z \in D(0, 1/r_0) \setminus U\}$, and $M_n := \max\{v_n(z); |z| = \frac{1}{r_0}\}$. Then $C$ is finite and $M_n \to -\infty$ as $n \to \infty$.

The function $h(z, D(\infty, 1/r_0) \cup \Omega_n, D(\infty, 1/R))$ is harmonic in the domain $\{|z| < \frac{1}{r_0}\} \setminus \Omega_n$ and continuous in its closure, vanishes on $\{|z| = 1/r_0\} \cup \partial \Omega_n$, and is equal to 1 on $\{|z| = 1/R\}$. Hence, by two constant theorem

$$v_n(z) \leq C + (M_n - C)h(z, D(\infty, \frac{1}{r_0}) \cup \Omega_n, D(\infty, \frac{1}{R}))$$

for all $z$ in $\{\frac{1}{R} \leq |z| \leq \frac{1}{r_0}\} \setminus \Omega_n$.

One can check that $h(z, D(\infty, 1/r_0) \cup \Omega_n, D(\infty, 1/R)) \geq h(z, D(\infty, 1/r_0) \cup U, D(\infty, 1/R)) \geq v(z)$, $n \geq 1, |z| \geq 1/R$. Therefore

$$P(a, g_e(a)) = P(a, f_n(a)) \leq C + (M_n - C)v(a), \quad n \geq n_1(a),$$

where $n_1(a)$ is so large that $M_n - C < 0$ for $n \geq n_1(a)$. It follows that $P(a, g_e(a)) = -\infty$.

By the same method one can show that if $P(z, g_e(z)) = -\infty$ on $G_e$ then $P(z, g(z)) = -\infty$ on $G \cup E$. Namely, it is sufficient to observe that the function $v(z) = V_{U \cup D(0, r_0)}(z)/\log \frac{R}{r_0}$ is subharmonic in $C$, harmonic on $C \setminus \bar{D}(0, r_0) \cup U$, $v(z) = 0$ on $U \cup D(0, r_0)$, $v(z) \leq 1$ on $\{|z| \leq R\}$, $v(z) \geq 1/\log \frac{R}{r_0}$ on $E$, and $v(z) > 0$ for all $z \in G \cup E$ with $|z| > r_0$. Hence

$$v(z) \leq h(z, U \cap D(0, r_0), D(0, R)), \quad |z| \leq R.$$
Corollary. Put $f := g^{-1}$, $f_e := g_e^{-1}$. Then

$$f : \bar{D} \mapsto \bar{G}, \quad f_e : \bar{D} \mapsto \bar{G}_e$$

are conformal diffeomorphisms satisfying all the assertions of Theorem 1.1.

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Jagiellonian University
Institute of Mathematics
Reymonta 4
30-059 Kraków
Poland

e-mail: jozef.siciak@im.uj.edu.pl