ON THE CHAPLYGHIN METHOD FOR GENERALIZED
SOLUTIONS OF PARTIAL DIFFERENTIAL FUNCTIONAL
EQUATIONS

by Wojciech Czernous

Abstract. Initial boundary value problems for semilinear first-order par-
tial differential functional equations are considered. It is shown that under
natural assumptions on given functions there exists a Chaplyghin sequence
and it is convergent to the Carathéodory solution of the original problem.
Error estimates for approximate solutions are given. It is proved that the
Chaplyghin method for initial boundary value problems is equivalent to the
Newton method for a suitable integral functional equation.

1. Introduction. For any metric spaces $X$ and $Y$, by $C(X,Y)$, we denote
the class of all continuous functions from $X$ into $Y$. We will use vectorial
inequalities with the understanding that the same inequalities hold between
their corresponding components.

Let $a > 0$, $h_0 \in \mathbb{R}_+$, $\mathbb{R}_+ = [0, +\infty)$, $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$ and $h = (h_1, \ldots, h_n) \in \mathbb{R}_+^n$ be given, where $b_i > 0$ for $1 \leq i \leq n$. We define the sets

$$E = [0,a] \times [-b,b], \quad D = [-h_0,0] \times [-h,h].$$

Let $\bar{c} = (c_1, \ldots, c_n) = b + h$ and

$$E_0 = [-h_0,0] \times [-\bar{c},\bar{c}],$$

$$\partial_0 E = [0,a] \times ([-\bar{c},\bar{c}] \backslash (-b,b)), \quad \Omega = E_0 \cup E \cup \partial_0 E.$$ 

Suppose that $z : \Omega \to \mathbb{R}$ and $(t,x) \in E$ are fixed. We define the function

$$z_{(t,x)} : D \to \mathbb{R}$$

as follows

$$z_{(t,x)}(\tau,\xi) = z(t + \tau, x + \xi), \quad (\tau,\xi) \in D.$$
The function $z(t,x)$ is the restriction of $z$ to the set $[t-h_0,t] \times [x-h,x+h]$ and this restriction is shifted to the set $D$. Elements of the space $C(D,\mathbb{R})$ will be denoted by $w, \bar{w}$ and so on. We denote by $\| \cdot \|_0$ the supremum norm in the space $C(D,\mathbb{R})$. Let 

$$f = (f_1, \ldots, f_n) : E \to \mathbb{R}^n, \quad G : E \times C(D,\mathbb{R}) \to \mathbb{R}, \quad \varphi : E_0 \cup \partial_0 E \to \mathbb{R},$$

$$\alpha_0 : E \to \mathbb{R}, \quad \alpha' : E \to \mathbb{R}^n, \quad \alpha' = (\alpha_1, \ldots, \alpha_n)$$

be given functions. Write $\alpha(t,x) = (\alpha_0(t,x), \alpha'(t,x))$, $(t,x) \in E$. We require that $\alpha(t,x) \in E$ and $\alpha_0(t,x) \leq t$ for $(t,x) \in E$. We consider the problem consisting of the functional differential equation

$$(1) \quad \partial_t z(t,x) + \sum_{i=1}^n f_i(t,x) \partial_{x_i} z(t,x) = G(t,x,z(\alpha(t,x))),$$

and the initial-boundary condition

$$(2) \quad z(t,x) = \varphi(t,x) \text{ on } E_0 \cup \partial_0 E.$$ 

A function $\tilde{z} : [-h_0,\xi] \times [-\bar{c},\bar{c}] \to \mathbb{R}$, where $0 < \xi \leq a$, is a Carathéodory solution of problem (1), (2) if it is continuous and

(i) the derivatives $\partial_t \tilde{z}, \partial_x \tilde{z} = (\partial_{x_1} \tilde{z}, \ldots, \partial_{x_n} \tilde{z})$ exist almost everywhere on $[0,\xi] \times [-b,b]$,

(ii) $\tilde{z}$ satisfies (1) almost everywhere on $[0,\xi] \times [-b,b]$ and condition (2) holds.

Note that our hereditary setting contains well known delay structures as particular cases.

In the paper, we give sufficient conditions for the existence of a sequence $\{u^{(m)}\}$ and a function $u^*$ such that:

(i) $u^*$ is a Carathéodory solution of (1), (2) and $\{u^{(m)}\}$ is a sequence of solutions of some linear functional differential equations obtained by linearization of (1),

(ii) $\{u^{(m)}\}$ uniformly converges to $u^*$ and the convergence is of the Newton type.

This method of approximating solutions of differential equations was introduced by Chaplygin in [5]. The Chaplygin method was applied to systems of first-order partial differential equations in [9], [11] and it was extended in [10] onto the case of infinite systems. The method for partial differential-functional equations with another model of functional dependence was considered in [7], but linearization with respect to the classical argument only was allowed here. The Chaplygin method for classical solutions of semilinear functional-differential equations with initial boundary conditions was investigated in [6]. The results presented in [6], [7] have the following properties: it is
assumed that the Chaplygin sequences exist and the proofs of the convergence
are based on theorems on differential inequalities.

It is clear that the results and methods presented in [6] do not apply to
Carathéodory solutions of [1], [2].

The theory of the Chaplygin method for parabolic functional-differential
problems was developed in [1]–[3].

The aim of this paper is to give a further contribution to the Chaplygin
method for functional-differential problems. We prove that under suitable
assumptions on given functions there exists a sequence of approximate solutions
and it uniformly converges to a Carathéodory solution of [1], [2]. We establish
estimates for the difference between exact and approximate solutions of [1],
[2].

Note that our results are new also in the case when [1] reduces to the
equation without the functional dependence.

Throughout the paper, we use these general ideas for the Chaplygin
method which were introduced in [12].

2. Characteristics. Write

\[ \Delta^+_i = \{ x \in [-b, b] : x_i = b_i \}, \quad \Delta^-_i = \{ x \in [-b, b] : x_i = -b_i \}, \]

where \( 1 \leq i \leq n \), and

\[ \Delta = \bigcup_{i=1}^{n} (\Delta^+_i \cup \Delta^-_i). \]

We will need the following function spaces in our considerations. Write \( \Omega_t = \Omega \cap \left( [-h_0, t] \times \mathbb{R}^n \right) \) and \( E_t = [0, t] \times [-b, b] \), where \( 0 \leq t \leq a \). We will denote by \( \| \cdot \|_i \) the supremum norm in the spaces \( C(\Omega_t, \mathbb{R}) \). Analogously, we will use the
symbol \( \| \cdot \|_{(t)} \) to denote the supremum norm in \( C(E_t, \mathbb{R}) \). For \( x \in \mathbb{R}^n \), where \( x = (x_1, \ldots, x_n) \), we put

\[ \| x \| = \sum_{j=1}^{n} |x_j|. \]

Let \( p = (p_0, p_1) \in \mathbb{R}_+^2 \). Denote by \( J[p] \) the set of all functions \( \varphi \in C(E_0 \cup \partial_0 E, \mathbb{R}) \) such that \( |\varphi(t, x)| \leq p_0 \) on \( E_0 \cup \partial_0 E \) and

\[ |\varphi(t, x) - \varphi(t, \bar{x})| \leq p_1 |t - \bar{t}| + \| x - \bar{x} \| \] on \( E_0 \cup \partial_0 E \).

**Assumption H**: Suppose that

1) the function \( f : E \to \mathbb{R}^n \) is such that

\( f(\cdot, x) : [0, a] \to \mathbb{R}^n \) is measurable for each \( x \in [-b, b] \) and \( f(t, \cdot) : [-b, b] \to \mathbb{R}^n \) is continuous for almost all \( t \in [0, a] \),
(ii) there are $A_1, B_1 \in \mathbb{R}_+$ such that
\[ \|f(t, x)\| \leq A_1 \text{ on } E, \]
and
\[ \|f(t, x) - f(t, \bar{x})\| \leq B_1 \|x - \bar{x}\| \]
for $(t, x), (\bar{t}, \bar{x}) \in E$.

2) there is $\kappa > 0$ such that for $1 \leq i \leq n$:
\[ f_i(t, x) \leq -\kappa \text{ for } x \in \Delta_i^+, \]
\[ f_i(t, x) \geq \kappa \text{ for } x \in \Delta_i^- \]
for $t \in [0, a]$.

Suppose that $(t, x) \in E_c$. Consider the Cauchy problem
\[ \eta'(\tau) = f(\tau, \eta(\tau)), \quad \eta(t) = x. \]
Denote by $g(\cdot, t, x) = (g_1(\cdot, t, x), \ldots, g_n(\cdot, t, x))$ a Carathéodory solution of (3).

The function $g(\cdot, t, x)$ is the characteristic of equation (1).

Suppose that Assumption $H[f]$ is satisfied. Then $g(\cdot, t, x)$ exists on the interval $[0, c]$ and is unique. The existence and uniqueness of the above solution follow from classical theorems. Let $I(\cdot, t, x)$ be the domain of $g(\cdot, t, x)$ and $\delta(t, x)$ be the left endpoint of the maximal interval on which the characteristic $g(\cdot, t, x)$ is defined.

We prove a lemma on bicharacteristics.

**Lemma 2.1.** Suppose that Assumption $H[f]$ is satisfied and let $\varphi \in J[p]$, $c \in (0, a]$ be given. Then the solution $g(\cdot, t, x)$ exists on the interval $I(\cdot, t, x)$ such that for $\zeta = \delta(t, x)$ there is $\zeta = 0$ or $g(\zeta, t, x) \in \Delta$. The characteristics are unique on $I(\cdot, t, x)$. Moreover, the following estimate holds
\[ \|g(\tau, t, x) - g(\tau, \bar{t}, \bar{x})\| \leq \bar{C} \left[ |t - \bar{t}| + \|x - \bar{x}\| \right] \]
for $\tau \in I(\cdot, t, x) \cap I(\cdot, \bar{t}, \bar{x})$, $(t, x), (\bar{t}, \bar{x}) \in E_c$ and
\[ \bar{C} = \max\{1, A_1\} \exp\{cB_1\}. \]

**Proof.** The existence and uniqueness of solutions of (3) follow from the classical theorem on Carathéodory solutions of initial problems. The function $g(\cdot, t, x)$ satisfies the integral equation
\[ g(\tau, t, x) = x + \int_{t}^{\tau} f(s, g(s, t, x)) ds. \]
It follows from Assumption $H[f]$ that the function $g(\cdot, t, x) - g(\cdot, \bar{t}, \bar{x})$ satisfies the integral inequality

$$
\| g(\tau, t, x) - g(\tau, \bar{t}, \bar{x}) \| \leq \| x - \bar{x} \| + A_1|t - \bar{t}| + \left| \int_{\tau}^{\bar{t}} B_1 \| g(s, t, x) - g(s, \bar{t}, \bar{x}) \| ds \right|, \quad \tau \in I(t,x) \cap I(\bar{t},\bar{x}).
$$

Then we obtain (4) by the Gronwall inequality. This proves Lemma 2.1.

Now we give a lemma on a regularity of the function $\delta$.

**Lemma 2.2.** Suppose that Assumption $H[f]$ is satisfied and $\varphi \in J[p]$, $c \in (0, a]$. Then the function $\delta$ is continuous on $E_c$. Moreover, the following estimate holds

$$
|\delta(t, x) - \delta(\bar{t}, \bar{x})| \leq \frac{2\bar{C}}{\kappa} [t - \bar{t} + \|x - \bar{x}\|]
$$

for $(t, x), (\bar{t}, \bar{x}) \in E_c$.

**Proof.** The continuity of $\delta$ follows from classical theorems on continuous dependence on initial conditions for Carathéodory solutions of initial problems. Now we prove (6). This estimate is obvious in the case $\delta(t, x) = \delta(\bar{t}, \bar{x}) = 0$ (i.e., in the case where solutions of problem (3) are defined on $[0, t]$ and $[0, \bar{t}]$). Suppose now that $0 \leq \delta(t, x) < \delta(\bar{t}, \bar{x})$. Then for $\zeta = \delta(\bar{t}, \bar{x})$ there is $g(\zeta, \bar{t}, \bar{x}) \in \Delta$ and there exists $i$, $1 \leq i \leq n$, such that $|g_i(\zeta, \bar{t}, \bar{x})| = b_i$. Two possibilities can occur: either (i) $g_i(\zeta, \bar{t}, \bar{x}) = b_i$ or (ii) $g_i(\zeta, \bar{t}, \bar{x}) = -b_i$. Consider case (i). Let $x = (x_1, \ldots, x_n)$, $\bar{x} = (x_1, \ldots, x_{i-1}, b_i, x_{i+1}, \ldots, x_n)$. There is

$$
|f_i(t, x) - f_i(t, \bar{x})| \leq B_1(b_i - x_i),
$$

for $(t, x) \in E_c$. Thus

$$
f_i(t, x) \leq -\frac{\kappa}{2}
$$

for $(t, x) \in E_c$ such that $b_i - x_i \leq \kappa(2B_1)^{-1}$. It follows from Lemma 2.1 that

$$
b_i - g_i(\zeta, t, x) = g_i(\zeta, \bar{t}, \bar{x}) - g_i(\zeta, t, x) \leq \frac{\kappa}{2B_1}
$$

for $(t, x), (\bar{t}, \bar{x}) \in E_c$ such that

$$
|t - \bar{t}| + \|x - \bar{x}\| \leq \frac{\kappa}{2B_1C}.
$$

Then we get

$$
f_i(\zeta, g(\zeta, t, x)) \leq -\frac{\kappa}{2} < 0
$$

and, consequently,

$$
\partial_t g_i(\delta(\bar{t}, \bar{x}), t, x) < 0
$$
for \((t,x),(\bar{t},\bar{x}) \in E_c\) satisfying \([8]\). From the above inequality and from the uniqueness of \(g(\cdot, t,x)\), it can easily be seen that \(g_i(\cdot, t,x)\) is decreasing on the interval \((\delta(t,x), \delta(\bar{t},\bar{x}))\). Therefore,
\[
b_i - g_i(\tau, t,x) \leq \frac{\kappa}{2B_1}
\]
and the estimate
\[
f_i(\tau, g(\tau, t,x)) \leq -\frac{\kappa}{2}
\]
holds for \(\tau \in (\delta(t,x), \delta(\bar{t},\bar{x}))\) and \((t,x),(\bar{t},\bar{x}) \in E_c\) such that \([8]\) is satisfied. Then
\[
-\frac{\kappa}{2} \delta(\bar{t},\bar{x}) - \delta(t,x)
\]
holds for \(\tau \in (\delta(t,x), \delta(\bar{t},\bar{x}))\) and \((t,x),(\bar{t},\bar{x}) \in E_c\) such that \([8]\) is satisfied. Thus the proof of \([6]\) for \((t,x),(\bar{t},\bar{x}) \in E_c\), such that \([8]\) holds, is complete in case (i). In a similar way we prove \([6]\) in case (ii). Let \((t,x),(\bar{t},\bar{x}) \in E_c\) be arbitrary. We put \(M = \|x - \bar{x}\| + |t - \bar{t}|\). There exists \(K \in \mathbb{N}\) such that
\[
(K - 1)\frac{\kappa}{2B_1C} < M \leq K\frac{\kappa}{2B_1C}.
\]
Let \(\varepsilon \in \mathbb{R}\), \(\varepsilon = \frac{1}{K}\). For \(j = 0, \ldots, K\), we put
\[
\bar{x}^{(j)} = j\varepsilon\bar{x} + (1 - j\varepsilon)x, \quad \bar{t}^{(j)} = j\varepsilon\bar{t} + (1 - j\varepsilon)t.
\]
Note that \((\bar{t}^{(0)}, \bar{x}^{(0)}) = (t,x), (\bar{t}^{(K)}, \bar{x}^{(K)}) = (\bar{t},\bar{x})\) and
\[
\|\bar{x}^{(j)} - \bar{x}^{(j+1)}\| + |\bar{t}^{(j)} - \bar{t}^{(j+1)}| = \frac{M}{K} \leq \frac{\kappa}{2B_1C}
\]
for \(j = 0, \ldots, K - 1\). It is easy to see that
\[
\|x - \bar{x}\| = \sum_{j=0}^{K-1} \|\bar{x}^{(j)} - \bar{x}^{(j+1)}\| \text{ and } |t - \bar{t}| = \sum_{j=0}^{K-1} |\bar{t}^{(j)} - \bar{t}^{(j+1)}|.
\]
Then there is
\[
|\delta(t,x) - \delta(\bar{t},\bar{x})| \leq \sum_{j=0}^{K-1} |\delta(\bar{t}^{(j)},\bar{x}^{(j)}) - \delta(\bar{t}^{(j+1)},\bar{x}^{(j+1)})|
\]
\[
\leq \sum_{j=0}^{K-1} \frac{2C}{\kappa} \left[|\bar{t}^{(j)} - \bar{t}^{(j+1)}| + \|\bar{x}^{(j)} - \bar{x}^{(j+1)}\|\right] = \frac{2C}{\kappa} \left[|t - \bar{t}| + \|x - \bar{x}\|\right].
\]
Thus we see that \([6]\) holds true for all \((t,x),(\bar{t},\bar{x}) \in E_c\). \(\square\)
3. Chaplygin sequences. We denote by $CL(D, \mathbb{R})$ the set of all continuous and real functions defined on $C(D, \mathbb{R})$ and by $\| \cdot \|_*$ the norm in $CL(D, \mathbb{R})$. We now formulate assumptions on $G$ and $\alpha$.

**Assumption H[$G$].** Suppose that

1) the function $G$: $E \times C(D, \mathbb{R}) \to \mathbb{R}$ is such that
   (i) $G(\cdot, x, w): [0, a] \to \mathbb{R}$ is measurable for each $(x, w) \in [-b, b] \times C(D, \mathbb{R})$,
   (ii) $G(t, \cdot): [-b, b] \times C(D, \mathbb{R}) \to \mathbb{R}$ is continuous for almost all $t \in [0, a]$,  

2) there are $A_2, B_2 \in \mathbb{R}_+$ such that for $(t, x, w) \in E \times C(D, \mathbb{R})$

   there is
   
   $$|G(t, x, w)| \leq A_2$$
   $$|G(t, x, w) - G(t, \bar{x}, \bar{w})| \leq B_2||x - \bar{x}|| + \|w - \bar{w}\|_0,$$

3) the Fréchet derivative $\partial_wG(t, x, w) \in CL(D, \mathbb{R})$ exists for $(t, x, w) \in E \times C(D, \mathbb{R})$,

4) there exist $A_3, B_3 \in \mathbb{R}_+$ such that

   $$\|\partial_wG(t, x, w)\|_* \leq A_3,$$
   $$\|\partial_wG(t, x, w) - \partial_wG(t, \bar{x}, \bar{w})\|_* \leq B_3\|w - \bar{w}\|_0$$

for $(t, x, w) , (t, x, \bar{w}) \in E \times C(D, \mathbb{R})$.

**Assumption H[$\alpha$].** Suppose that

1) the function $\alpha$: $E \to E$ is continuous,

2) there are $r_0, r_1 \in \mathbb{R}_+$ such that for $(t, x), (\bar{t}, \bar{x}) \in E$ there holds

   $$|\alpha_0(t, x) - \alpha_0(\bar{t}, \bar{x})| \leq r_0|t - \bar{t}| + ||x - \bar{x}||,$$
   $$|\alpha'(t, x) - \alpha'(\bar{t}, \bar{x})| \leq r_1|t - \bar{t}| + ||x - \bar{x}||.$$

Suppose that Assumptions [H[$\mathcal{J}$], H[$G$], H[$\alpha$]] are satisfied and $\varphi \in J[p]$. We consider a sequence $\{z^{(m)}\}, \Omega_c \rightarrow \mathbb{R}$, defined in the following way:

(i) $z^{(0)} \in C(\Omega_c, \mathbb{R}), z(t, x) = \varphi(t, x)$ for $(t, x) \in (E_0 \cup \partial_0 E) \cap ([-h_0, c] \times \mathbb{R}^n)$ and $z^{(0)}$ satisfies the Lipschitz condition with respect to $(t, x)$ on $\Omega_c$,

(ii) if $z^{(m)}: \Omega_c \rightarrow \mathbb{R}$ is known then $z^{(m+1)}$ is a Carathéodory solution of the linear equation

\begin{equation}
\frac{\partial z(t, x)}{\partial t} + \sum_{i=1}^{n} f_i(t, x) \partial x_i z(t, x) = G(t, x, z^{(m)}_{\alpha(t, x)}) + \partial_wG(t, x, z^{(m)}_{\alpha(t, x)})(z - z^{(m)})_{\alpha(t, x)}
\end{equation}

with the initial-boundary condition

\begin{equation}
z(t, x) = \varphi(t, x) \text{ for } (t, x) \in (E_0 \cup \partial_0 E) \cap ([-h_0, c] \times \mathbb{R}^n).
\end{equation}
We prove that there is $c \in (0, a]$ such that the sequence $\{z^{(m)}\}$ is well defined on $\Omega_c$ and converges to a Carathéodory solution of (1), (2).

Our considerations are based on the following idea. We transform problem (1), (2) into an abstract equation $F[z] = 0$ where $F : X \to X$ and $X$ is a Banach space. Then we consider a Newton method for the above equation and we prove that $\{z^{(m)}\}$ is a Newton sequence for the abstract equation.

Suppose that $(X, \| \cdot \|_X)$ is a Banach space and

$S = \{ u \in X : \| u - u_0 \|_X \leq \delta \}$,

where $u_0 \in X$ is arbitrary and $\delta > 0$. Let $F : S \to X$ be a given operator such that $F'(u)$ exists for $u \in S$. We consider the equation

(11) $F(u) = 0$

and the Newton method

(12) $u_0, \quad u_{m+1} = u_m - [F'(u_m)]^{-1}F(u_m), \quad m \geq 0$.

We will need the following theorem in our considerations.

**Theorem 3.1.** Suppose that $F : S \to X$ and

1) the Fréchet derivative $F'(u)$ exists for $x \in S$,

2) there is $K \in \mathbb{R}_+$ such that

\[ \| F'(u) - F'(\bar{u}) \|_* \leq K \| u - \bar{u} \|_X \quad \text{for} \quad u, \bar{u} \in S, \]

3) the operator $F'(u_0)$ has the inverse $\Gamma_0 = [F'(u_0)]^{-1}$ and there is $B \in \mathbb{R}_+$ such that $\| \Gamma_0 \| \leq B$,

4) for the initial element $u_0$, the estimate $\| \Gamma_0 F(u_0) \|_X \leq \eta$ holds,

5) the constants $B, K, \eta$ fulfill the inequality $\tilde{h} = BK\eta \leq 0, 5$,

6) for $\delta$, the following inequality holds:

\[ \frac{1 - \sqrt{1 - 2\tilde{h}}}{\tilde{h}} \eta \leq \delta. \]

Then

(i) there exists a solution of equation (11),

(ii) the Newton sequence (12) exists and there is $u^*$ such that

\[ u^* = \lim_{m \to \infty} u_m, \]

(iii) $f(u^*) = 0$ and the following estimate holds

\[ \| u^* - u_m \|_X \leq \frac{1}{2^{m-1}} (\tilde{h})^{2^{m-1}} \eta, \quad m \geq 0. \]

The proof of the above theorem is given in [8].
4. Newton method for integral-functional equations. Suppose that Assumptions $\mathcal{H}[f]$, $\mathcal{H}[G]$, $\mathcal{H}[\alpha]$ are satisfied and $\varphi \in J[p]$. Write

$$S(t,x) = (\delta(t,x), g(\delta(t,x), t, x)).$$

Let us consider the operator $F : C(\Omega_c, \mathbb{R}) \to C(\Omega_c, \mathbb{R})$ defined by

$$F[z](t,x) = z(t,x) - \varphi(S(t,x)) - \int_{\delta(t,x)}^{t} G(s, g(s, t, x), z_{\alpha(s, g(s, t, x))}) ds,$$

$$F[z](t,x) = \varphi(t,x) \text{ for } (t,x) \in E_0 \cup \partial_0 E.$$

It is clear that the equation

$$F[z](t,x) = 0$$

is obtained from (1), (2) by integrating (1) along characteristics. The existence result for (13) is based on the following method of successive approximations.

We consider a sequence $\{z^{(m)}\}$, $z^{(m)} : \Omega_c \to \mathbb{R}$, defined in the following way:

(A) $z^{(0)} \in C(\Omega_c, \mathbb{R})$, $z(t,x) = \varphi(t,x)$ for $(t,x) \in (E_0 \cup \partial_0 E) \cap ([-h_0, c] \times \mathbb{R}^n)$,

(B) if $z^{(m)} : \Omega_c \to \mathbb{R}$ is known, then $z^{(m+1)}$ is the solution of the equation

$$z(t,x) = \varphi(S(t,x)) + \int_{\delta(t,x)}^{t} G(P^{(m)}(s,t,x)) ds$$

$$+ \int_{\delta(t,x)}^{t} \partial w G(P^{(m)}(s,t,x))(z - z^{(m)})_{\alpha(s, g(s, t, x))} ds$$

with the initial-boundary condition

$$z(t,x) = \varphi(t,x) \text{ for } (t,x) \in (E_0 \cup \partial_0 E) \cap ([-h_0, c] \times \mathbb{R}^n),$$

where

$$P^{(m)}(s,t,x) = P[z^{(m)}](s,t,x)$$

and

$$P[z](s,t,x) = (s, g(s, t, x), z_{\alpha(s, g(s, t, x))}).$$

We prove that the above sequence is well defined and that it converges to a solution of (13).

**Lemma 4.1.** Suppose that Assumptions $\mathcal{H}[f]$, $\mathcal{H}[G]$, $\mathcal{H}[\alpha]$ are satisfied and $\varphi \in J[p]$. Then for $0 < c \leq \tilde{c}$, where

$$\tilde{c} = (2A_3)^{-1} \arcsinh\{A_3(2B_3\tilde{\eta})^{-1}\}, \quad \tilde{\eta} = ||z^{(0)}||_a + p_0 + aA_2,$$

there is:

1) equation (13) has a solution $z^* : \Omega_c \to \mathbb{R},$
2) the sequence \( \{z^{(m)}\} \) is well defined by (A), (B) on \( \Omega_c \) and
\[
\lim_{m \to \infty} z^{(m)}(t, x) = z^*(t, x) \quad \text{on } E_c,
\]

3) the following estimates hold
\[
\|z^* - z^{(m)}\|_c \leq \frac{\tilde{C}}{2^{m-1}} (2\tilde{h})^{2m-1} \eta, \quad m \geq 0,
\]
where
\[
(15) \quad \tilde{C} = \exp\{2A_3c\}, \quad \eta = 2(\|z(0)\|_c + p_0 + cA_2), \quad \tilde{h} = B_3(A_3)^{-1} \sinh(2A_3c) \eta \leq \frac{1}{2}.
\]

PROOF. We consider the Banach space \((C(\Omega_c, \mathbb{R}), \| \cdot \|_B)\), where
\[
\|v\|_c e^{-2A_3t} \leq \|v\|_B
\]
and, consequently,
\[
(16) \quad \|v\|_c \leq \tilde{C}\|v\|_B
\]
for \(v \in C(\Omega_c, \mathbb{R})\).
We apply Theorem [3.1] to prove the above properties of equation (13). It follows from Assumption \([H[G]\) that the Fréchet derivative \(F'[z]\) exists and
\[
(F'[z]v)(t, x) = v(t, x) - \int_{\delta(t, x)} \partial_w G(P[z](s, t, x))v_{\alpha(s, g(s, t, x))} ds \quad \text{for } (t, x) \in E_c,
\]
\[
(F'[z]v)(t, x) = v(t, x) \quad \text{for } (t, x) \in (E_0 \cup \partial_0 E) \cap ([-h_0, c] \times \mathbb{R}^n).
\]
We consider the operator \(U : C(\Omega_c, \mathbb{R}) \to C(\Omega_c, \mathbb{R})\) defined by
\[
U[v](t, x) = \int_{\delta(t, x)} \partial_w G(P[z](s, t, x))v_{\alpha(s, g(s, t, x))} ds \quad \text{on } E,
\]
and
\[
U[v](t, x) = 0, \quad (t, x) \in E_0 \cup \partial_0 E.
\]
Then
\[
|U[v](t, x)| \leq \int_{\delta(t, x)} |\partial_w G(P[z](s, t, x))v_{\alpha(s, g(s, t, x))} e^{-2A_3s} e^{2A_1s}| ds
\]
\[
\leq \|v\|_B \int_0^t A_3 e^{2A_3s} ds \leq \frac{1}{2} \|v\|_B e^{2A_1t}
\]
and, consequently,
\[
|U[v](t, x)| e^{-2A_3t} \leq \frac{1}{2} \|v\|_B
\]
It easily follows that the above relations are equivalent to

\[ \|U[v]\|_B \leq \frac{1}{2} \|v\|_B. \]

It follows that \( \|U\|_* \leq \frac{1}{2} < 1 \) and \( F'[z] \) exists. Moreover, there is

\[ \| (F'[z])^{-1} \| \leq 2, \]

where \( z \in C(\Omega_c, \mathbb{R}) \). The Newton method for equation \([13]\) with the starting function \( z^{(0)} \) has the form

\[ z^{(m+1)}(t, x) = z^{(m)}(t, x) - \left( F'[z^{(m)}] \right)^{-1} F[z^{(m)}](t, x), \quad m \geq 0. \]

It easily follows that the above relations are equivalent to

\[ z^{(m+1)}(t, x) = \varphi(S(t, x)) + \int_t^T G(P^{(m)}(s, t, x)) \, ds \]

\[ + \int_{\delta(t,x)} \partial_w G(P^{(m)}(s, t, x))(z^{(m+1)} - z^{(m)})_{\alpha(s,g(s,t,x))} \, ds, \quad (t, x) \in E_c \]

and

\[ z^{(m+1)}(t, x) = \varphi(t, x) \quad \text{for} \quad (t, x) \in (E_0 \cup \partial E) \cap ([-h_0, c] \times \mathbb{R}^n). \]

Then the sequence \( \{z^{(m)}\} \) defined by relations (A), (B) is the Newton sequence for equation \([13]\).

We will write an estimate of the Lipschitz constant for the Fréchet derivative \( F' \). According to \([17]\), there is

\[ |(F'[z]v - F'[\bar{z}]v)(t, x)| \]

\[ = \left| \int_{\delta(t, x)} \left[ \partial_w G(P[z](s, t, x)) - \partial_w G(P[\bar{z}](s, t, x)) \right] v_{\alpha(s,g(s,t,x))} \, ds \right| \]

\[ \leq \int_{\delta(t, x)} \|\partial_w G(P[z](s, t, x)) - \partial_w G(P[\bar{z}](s, t, x))\|_* \|v_{\alpha(s,g(s,t,x))}\|_0 \, ds \]

\[ \leq \int_{\delta(t, x)} B_3 \|z - \bar{z}\|_{\alpha(s,g(s,t,x))} \|v_{\alpha(s,g(s,t,x))}\|_0 \, ds \]

\[ \leq \int_{\delta(t, x)} B_3 \|z - \bar{z}\|_{e^{-2A_3s}} \|v\|_{e^{-2A_3s}} e^{4A_3s} \, ds \]

\[ \leq \|z - \bar{z}\|_B \|v\|_B \int_0^t B_3 e^{4A_3s} \, ds = \|z - \bar{z}\|_B \|v\|_B B_3 (4A_3)^{-1} (e^{4A_3t} - 1). \]

Then

\[ \|F'[z]v - F'[\bar{z}]v)(t, x)\| e^{-2A_3t} \leq \|z - \bar{z}\|_B \|v\|_B B_3 (2A_3)^{-1} \sinh(2A_3t) \]
and, consequently,
\[ \|F'(\bar{z}) - F'(\bar{z})\|_B \leq \|\bar{z} - \bar{z}\|_B B_3(2A_3)^{-1} \sinh(2A_3t) \]
and
\[ \|F'(\bar{z}) - F'(\bar{z})\|_B \leq \|\bar{z} - \bar{z}\|_B B_3(2A_3)^{-1} \sinh(2A_3t). \]
Suppose that \( \eta > 0 \) is such a constant that
\[ \|z(1) - z(0)\|_B \leq \eta, \]
and that \( c \) is a constant small enough for the following inequality to hold:
\[ (18) \tilde{h} = B_3(2A_3)^{-1} \sinh(2A_3c) \eta \leq \frac{1}{2}. \]
We put
\[ \delta = 1 - \sqrt{1 - 2\tilde{h}} \eta. \]
Now by Theorem 3.1 and (16), all the assertions of the Lemma 4.1 follow.

Now we prove that differential-functional equations (9) with the initial-boundary condition
\[ (19) z(t, x) = \varphi(t, x) \text{ for } (t, x) \in (E_0 \cup \partial_0 E) \cap ([t_0, a] \times \mathbb{R}^n) \]
are equivalent to integral equations
\[ (20) z(t, x) = \varphi(S(t, x)) + \int_0^t \int_{\delta(t,x)} G(P(m)(s, t, x)) ds \]
with initial-boundary condition (19).
Suppose that \( z^{(m+1)}: \Omega \rightarrow \mathbb{R} \) is a Carathéodory solution of (9), (19) and there is \( b^{(m+1)} \) such that
\[ |z^{(m+1)}(t, x) - z^{(m+1)}(\bar{t}, \bar{x})| \leq b^{(m+1)}|t - \bar{t}| + |x - \bar{x}|. \]
It follows easily that \( z^{(m+1)} \) satisfies (20). We prove that classical solutions of (20) satisfy (9) almost everywhere on \( E_c \). We state a lemma on regularity of the Newton sequence \( \{z^{(m)}\} \).

Suppose that \( z: \Omega \rightarrow \mathbb{R} \) is continuous. For \( t \in [0, a] \), we define
\[ [z]_{L_1 t} = \sup \left\{ \frac{|z(s, y) - z(\bar{s}, \bar{y})|}{|s - \bar{s}| + |y - \bar{y}|} : (s, y), (\bar{s}, \bar{y}) \in \Omega, (s, y) \neq (\bar{s}, \bar{y}) \right\}. \]

**Lemma 4.2.** Suppose that Assumptions \( H[f], H[G] \) and \( H[\alpha] \) are satisfied and that
1) \( \varphi \in J[p], c \in (0, a], \)
2) \( z(0) \in C(\Omega, \mathbb{R}) \) is given such that \( \|z(0)\|_{L_c} < \infty \),

3) the sequence \( z^{(m)} \) is well defined by (A), (B) on \( \Omega_c \) for \( m \geq 0 \).

Then there is

\[
\text{(21)} \quad \left[ |z^{(m)}| \right]_{L_c} < \infty \quad \text{for} \quad m \geq 0.
\]

**Proof.** We prove (21) by induction. It follows from assumption 2) that (21) is satisfied for \( m = 0 \). Supposing now that condition (21) holds for a given \( m \geq 0 \), we will prove that the function \( z^{(m+1)} \) given by (A), (B) satisfies (21). We put

\[
\text{(22)} \quad L^{(m)} = \left[ |z^{(m)}| \right]_{L_c}.
\]

From assumption 3) it follows that there exist constants \( d^{(m)}, d^{(m+1)} \in \mathbb{R}_+ \) such that

\[
\text{(23)} \quad |z^{(m)}(t, x)| \leq d^{(m)} \quad \text{and} \quad |z^{(m+1)}(t, x)| \leq d^{(m+1)} \quad \text{on} \quad \Omega_c.
\]

Put

\[
\psi(\tau) = \left[ |z^{(m+1)}| \right]_{L_c} \quad \text{for} \quad \tau \in [0, c].
\]

We will write an integral inequality for function \( \psi : [0, c] \to \mathbb{R}_+ \). Let \( \tau \in [0, c] \) be arbitrary and \( (t, x), (\bar{t}, \bar{x}) \in \Omega_c \), \( (t, x) \neq (\bar{t}, \bar{x}) \). According to (22), (23), Assumptions \( H[f], H[G] \) and \( H[\alpha] \) and Lemmas 2.1, 2.2, the following integral inequality holds:

\[
\text{(24)} \quad |z^{(m+1)}(t, x) - z^{(m+1)}(\bar{t}, \bar{x})| \leq \bar{A}[|t - \bar{t}| + ||x - \bar{x}||]
\]

\[
\quad + \int_{\delta(\bar{t}, \bar{x})}^\tau A_3||(z^{(m+1)})_{\alpha(s,g(s,t,x))} - (z^{(m+1)})_{\alpha(s,g(s,\bar{t},\bar{x}))}||_0 ds
\]

where

\[
\bar{A} = \bar{A}_1 + \bar{A}_2
\]

and

\[
\bar{A}_1 = p_1 \left[ (1 + A_1)\tilde{C} + C \right] + \left[ A_2 + A_3 d \right] \left( 1 + \tilde{C} \right),
\]

\[
\bar{A}_2 = c\tilde{C} \left[ B_2 + B_3 d \right] \left( 1 + \tilde{L} \right) + c\tilde{C}A_3 \tilde{L}
\]

and

\[
\tilde{C} = \frac{2\tilde{C}}{\kappa}, \quad \tilde{L} = L^{(m)}(r_0 + r_1), \quad \tilde{d} = d^{(m)} + d^{(m+1)}.
\]
Note that
\[ \| (z^{(m+1)}_{\alpha(s,g(s,t,x)}) - (z^{(m+1)}_{\alpha(s,g(s,\bar{t},\bar{x})})} \|_0 \]
\[ \leq \left[ |z^{(m+1)}| \right]_{L,s} (r_0 + r_1) \| g(s,t,x) - g(s,\bar{t},\bar{x}) \|
\]
\[ \leq \left[ |z^{(m+1)}| \right]_{L,s} (r_0 + r_1) \bar{C} \| t - \bar{t} \| + \| x - \bar{x} \|. \]

Then we get
\[ |z^{(m+1)}(t,x) - z^{(m+1)}(\bar{t},\bar{x})| \leq \bar{A} \| t - \bar{t} \| + \| x - \bar{x} \|
\]
\[ + A_3(r_0 + r_1) \bar{C} \| t - \bar{t} \| + \| x - \bar{x} \| \int_{\delta(\bar{t},\bar{x})}^t \left[ |z^{(m+1)}| \right]_{L,s} ds \]
and, consequently,
\[ \frac{|z^{(m+1)}(t,x) - z^{(m+1)}(\bar{t},\bar{x})|}{\| t - \bar{t} \| + \| x - \bar{x} \|} \leq \bar{A} + A_3(r_0 + r_1) \bar{C} \int_{\delta(\bar{t},\bar{x})}^t \left[ |z^{(m+1)}| \right]_{L,s} ds \]
for \((t,x),(\bar{t},\bar{x}) \in E_\tau, (t,x) \neq (\bar{t},\bar{x})\). According to the definition of \(|| \cdot ||_{L,\tau}\),
there is
\[ \left[ |z^{(m+1)}| \right]_{L,\tau} \leq \bar{A} + A_3(r_0 + r_1) \bar{C} \int_{\delta(\bar{t},\bar{x})}^t \left[ |z^{(m+1)}| \right]_{L,s} ds. \]
Then
\[ \psi(\tau) \leq \bar{A} + A_3(r_0 + r_1) \bar{C} \int_{0}^{\tau} \psi(s) ds. \]
Now from the Gronwall inequality it follows that
\[ \psi(\tau) \leq \bar{A} \exp \{ \tau A_3(r_0 + r_1) \bar{C} \} \]
and, consequently,
\[ \left[ |z^{(m+1)}| \right]_{L,c} < \infty, \]
which completes the proof of the Lemma 4.2. \qed

**Theorem 4.3.** Suppose that Assumptions [H[f], H[G], H[\alpha]] are satisfied and \( \varphi \in J[p] \).
Then there is \( c > 0 \) such that

1) the sequence \( \{z^{(m)}\} \) is well defined by (i), (ii) on \( \Omega_c \) and there exists \( z^* \in C(\Omega_c, \mathbb{R}) \) such that
\[ \lim_{m \to \infty} z^{(m)}(t,x) = z^*(t,x) \text{ uniformly on } E_c, \]
2) the following estimates hold:

\[ \| z^* - z^{(m)} \|_c \leq \tilde{C} \frac{2^{m-1}}{2^{2m-1}} \tilde{h}, \quad m \geq 0 \]

where \( \tilde{C}, \tilde{h}, \eta \) are given by (15).

**Proof.** It follows from Lemma 4.1 that the sequence \( \{ z^{(m)} \} \) defined by (20) and (19) exists and converges on \( \Omega_c \) for some \( c > 0 \). Furthermore, from Lemma 4.2 and the chain rule differentiation Lemma (cf. [4]), there follows:

\[ \frac{d}{d\tau} z^{(m+1)}(\tau, g(\tau, t, x)) = \partial_t z^{(m+1)}(\tau, g(\tau, t, x)) + n \sum_{i=1}^{n} f_i(\tau, g(\tau, t, x)) \partial_{x_i} z^{(m+1)}(\tau, g(\tau, t, x)) \]

for almost all \( \tau \in I(t, x) \) and almost all \( (t, x) \in E_c \), where \( m \geq 0 \).

By integrating the above equation on \( [\delta(t, x), t] \) with respect to \( \tau \), we get

\[ z^{(m+1)}(t, x) = \phi(S(t, x)) + \int_{\delta(t, x)}^{t} \left[ \partial_t z^{(m+1)}(\tau, g(\tau, t, x)) + \sum_{i=1}^{n} f_i(\tau, g(\tau, t, x)) \partial_{x_i} z^{(m+1)}(\tau, g(\tau, t, x)) \right] d\tau \]

almost everywhere on \( E_c \).

Suppose now that \( z^{(m+1)} \) satisfies (20), almost everywhere on \( E_c \) and condition (19). Then from the above equation, we get

\[ \int_{\delta(t, x)}^{t} \left[ \partial_t z^{(m+1)}(\tau, g(\tau, t, x)) + \sum_{i=1}^{n} f_i(\tau, g(\tau, t, x)) \partial_{x_i} z^{(m+1)}(\tau, g(\tau, t, x)) \right] d\tau = G(P^{(m)}(\tau, t, x)) + \partial_w G(P^{(m)}(\tau, t, x))(z^{(m+1)} - z^{(m)})_{(\tau, g(\tau, t, x))} \]

almost everywhere on \( E_c \). We put \( \zeta = \delta(t, x) \). For a given \( x \in [-b, b] \), let us put \( y = g(\zeta, t, x) \). It follows from Lemma 2.7 that \( g(\tau, t, x) = g(\tau, \zeta, y) \) for almost all \( \tau \in I(t, x), t \in [0, c] \) and \( x = g(t, \zeta, y) \). Then

\[ \int_{\zeta}^{t} \left[ \partial_t z^{(m+1)}(\tau, g(\tau, \zeta, y)) + \sum_{i=1}^{n} f_i(\tau, g(\tau, \zeta, y)) \partial_{x_i} z^{(m+1)}(\tau, g(\tau, \zeta, y)) \right] d\tau = G(P^{(m)}(\zeta, y)) + \partial_w G(P^{(m)}(\zeta, y))(z^{(m+1)} - z^{(m)})_{(\tau, g(\tau, \zeta, y))} \]

almost everywhere on \( E_c \). Differentiating the above relation with respect to \( t \), we get

\[
\partial_t z^{(m+1)}(t,g(t,\zeta,y)) + \sum_{i=1}^{n} f_i(t,g(t,\zeta,y))\partial_{x_i} z^{(m+1)}(t,g(t,\zeta,y))
\]

\[
= G(P^{(m)}(t,\zeta,y)) + \partial_u G(P^{(m)}(t,\zeta,y))(z^{(m+1)} - z^{(m)})_{\alpha(t,g(t,\zeta,y))}
\]

almost everywhere on \( E_c \). Then we use the relation \( x = g(t,\zeta,y) \) and we infer that \( z^{(m+1)} \) satisfies (9) almost everywhere on \( E_c \).

Now we prove that relations (9) and (19) imply (20) and (19). Suppose that \( z^{(m+1)} \) is a Carathéodory solution of (9), (19) and that \( z^{(m+1)} \) satisfies the Lipschitz condition with respect to \( (t,x) \). Then (27) holds true almost everywhere on \( E_c \). Now, taking relation (9) along characteristics with \( z = z^{(m+1)} \) and applying it to the right hand side of (27), we get (20).

We proved that the sequences \( \{z^{(m)}\} \) defined on the one hand by (i), (ii) and on the other by (A), (B) are equivalent on \( \Omega_c \) for \( c \in (0,a] \). From Lemma 4.1 there follows that there exists \( c \in (0,a] \) such that the second sequence exists and estimate (14) holds. The assertion of Theorem 4.3 follows from the equivalence of the above sequences. This completes the proof.

**Remark 4.4.** The above result can be extended on initial boundary value problems for the following systems

\[
\partial_t z_i(t,x) = \sum_{j=1}^{n} f_{ij}(t,x)\partial_{x_j} z_i(t,x) = G_i(t,x,z_i(t,x)), \; i = 1, \ldots, k,
\]

where \( z = (z_1, \ldots, z_k) \).

It is important in our considerations that we do not assume monotonicity conditions for given functions with respect to the functional variable.

**References**


Received May 23, 2005

Gdańsk University of Technology
Department of Differential Equations
ul. Gabriela Narutowicza 11–12
80-952 Gdańsk
Poland