THE WU METRIC IN ELEMENTARY REINHARDT DOMAINS

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Abstract. In the paper we give the explicit formula for the Wu metric in elementary Reinhardt domains.

Introduction. A new holomorphically invariant Hermitian pseudometric was introduced by H. Wu in paper [4]. We will call it the Wu pseudometric. Theorem 0 gives a short description of the pseudometric.

First let us introduce some notions concerning invariant metrics. Let $B^n$ be the unit ball in $\mathbb{C}^n$ and $\Delta := B^1$. Denote by $\beta_0$ the normalized Poincaré–Bergman metric of $B^n$ at the point 0, i.e.

$$\beta_0 = \sum_{j,k=1}^{n} dz_j \otimes d\bar{z}_k.$$

Define the Kobayashi–Royden pseudometric on a domain $D \subset \mathbb{C}^n$ at a point $x \in D$, for a vector $v \in \mathbb{C}^n$,

$$\kappa_D(x; v) := \inf \{ t > 0 | \exists \phi : U \to M \text{ holomorphic}, U \supset \Delta, \phi(0) = x, t\phi'(0) = v \};$$

and the integrated form of $\kappa_D$, for $x, y \in D$,

$$(f \kappa_D)(x, y) := \inf \{ L_{\kappa_D}(\alpha) | \alpha \text{ is a piecewise } C^1 \text{-curve in } D \text{ from } x \text{ to } y \},$$

where $L_{\kappa_D}(\alpha) := \int_0^1 \kappa_D(\alpha(t); \alpha'(t)) \, dt$ is the $\kappa_D$-length of a curve $\alpha$.

A domain $D \subset \mathbb{C}^n$ is called hyperbolic, if for any $x_0 \in D$ there exist a neighborhood $U \subset D$ of $x_0$ and a positive real number $C$ such that $\kappa_D(x; v) \geq C \|v\|$ for all $x \in U$ and $v \in \mathbb{C}^n$. We say that $D$ is complete if any $(f \kappa_D)$-Cauchy sequence $(x_\nu)_{\nu=1}^\infty \subset D$ converges to a point $x_0 \in D$ in the natural topology.
Theorem 0. \(\{4\}\) The Wu pseudometric \(h\), which is an (upper semicontinuous) semi-definite Hermitian metric, can be defined on every domain \(D \subset \mathbb{C}^n\) and has the following properties:

(a) For \(0 \in \mathbb{B}^n\), \(h_{\mathbb{B}^n,0} = \beta_0\).

(b) If the Kobayashi–Royden pseudometric \(\kappa_D\) vanishes identically on \(D\), then so does \(h_D\).

(c) If \(D\) is hyperbolic, then \(h_D\) is a Hermitian metric.

(d) If \(D\) is complete hyperbolic, then \(h_D\) is a continuous Hermitian metric.

(e) If \(G \subset \mathbb{C}^m\) is a domain and \(f : D \to G\) is a holomorphic mapping, then

\[
h_{G,f(x)}(f'(x)(u), f'(x)(v)) \leq \sqrt{n} h_{D,x}(u, v), \quad \text{for} \quad x \in D, \ u, v \in \mathbb{C}^n.\]

Due to the relation between the Wu and Kobayashi–Royden pseudometrics it was possible to find the effective formulas for the Wu metric. It was studied by C. K. Cheung and Kang–Tae Kim in \(\{1\}\) and \(\{2\}\) on Thullen domains. W. Zwonek computed the formulas for the Kobayashi–Royden pseudometric on elementary Reinhardt domains (in \(\{5\}\)). Using them, we give here the formula for the Wu metric (Theorem 7).

1. The Wu pseudometric. The idea and the sketch of the construction of the Wu pseudometric on complex manifolds was first given in \(\{4\}\) (see also \(\{1\}\)). We recall here the details of the construction on domains in \(\mathbb{C}^n\) as well as simple properties for the sake of completeness.

For a domain \(D \subset \mathbb{C}^n\) let us introduce the family of sets \((\mathcal{F}_x)_{x \in D}\), where

\[
\mathcal{F}_x := \{f : \mathbb{B}^n \to D \text{ holomorphic} | \quad f(0) = x, \ f'(0) : \mathbb{C}^n \to \mathbb{C}^n \text{ is an isomorphism}\}.
\]

For any \(f \in \mathcal{F}_x\) we can consider the Hermitian inner product on \(\mathbb{C}^n\) induced by the normalized Bergman metric:

\[
f_0\beta_0(u, v) := \beta_0(0) (f'(0)^{-1}(u), f'(0)^{-1}(v)), \quad u, v \in \mathbb{C}^n.
\]

Let \(\Psi_x := \{f_0\beta_0 | f \in \mathcal{F}_x\}\). Then the family \(\{\Psi_x\}_{x \in D}\) is a biholomorphic invariant of \(D\).

Let \(Q\) be the space of all positive semi-definite Hermitian inner products on \(\mathbb{C}^n\) with the natural topology of the vector space of all sesquilinear forms on \(\mathbb{C}^n\). Define the partial ordering \(\ll\) in \(Q\):

\[
\alpha \ll \beta \quad \text{if} \quad \forall v \in \mathbb{C}^n : \alpha(v, v) \leq \beta(v, v),
\]

and also the set of lower bounds of \(\Psi_x\):

\[
l(\Psi_x) := \{\alpha \in Q | \forall \beta \in \Psi_x : \alpha \ll \beta\}.
\]
We are going to pick up an element of this set which will be the Wu pseudometric of the domain $D$ at the point $x$.

First, let us make some remarks:

**Lemma 1.** The set $l(\Psi_x)$ is a compact subset of $\mathbb{Q}$.

**Proof.** Fix any $\beta \in \Psi_x$. Let us define the set $l(\beta)$ of lower bounds of $\beta$:

$$l(\beta) := \{ \alpha \in \mathbb{Q} \mid \alpha < \beta \}.$$  

It suffices to prove that the set $l(\beta)$ is compact since $l(\Psi_x) = \bigcap_{\beta \in \Psi_x} l(\beta)$. Choose a basis $\{e_1, \ldots, e_n\}$ of $\mathbb{C}^n$ such that the matrix $[\beta(e_j, e_k)]_{j,k=1,\ldots,n}$ is diagonal.

Then, for any $\alpha \in l(\beta)$, we have $\alpha(e_j, e_j) \leq \beta(e_j, e_j)$ and also $|\alpha(e_j, e_k)|^2 \leq \alpha(e_j, e_j)\alpha(e_k, e_k) \leq \beta(e_j, e_j)\beta(e_k, e_k)$. This implies the boundedness of $l(\beta)$. The closedness is obvious.  

Let $K_x$ be the Kobayashi indicatrix at a point $x$:

$$K_x := \{ v \in \mathbb{C}^n \mid \kappa_D(x; v) < 1 \}$$

and for any $\alpha \in \mathbb{Q}$

$$B_\alpha := \{ v \in \mathbb{C}^n \mid |\alpha(v, v) < 1 \}$$

be the “$\alpha$-ball”. Notice that $B_\alpha \supset B_\beta$ if $\alpha \ll \beta$. Moreover, the following holds:

**Lemma 2.**

$$K_x = \bigcup_{\alpha \in \Psi_x} B_\alpha$$

**Proof.** Notice that, for a holomorphic mapping $f \in F_x$ such that $f'(0)(w) = v$, we have:

$$\kappa_D(x; v)^2 = \kappa_D(f(0); f'(0)(w))^2 \leq \kappa_{\mathbb{B}^n}(0; w)^2 = \beta_0(w, w) = f_1\beta_0(v, v).$$

In consequence, for any $\alpha \in \Psi_x$ we have $K_x \supset B_\alpha$.

To prove the other inclusion, fix $v \in \mathbb{C}^n \setminus \{0\}$ such that $\kappa_D(x; v) = \delta < 1$. Choose $\phi : U \to D$ from the definition of $\kappa_D(x; v)$ and $t \in (0, 1)$ such that $t\phi'(0) = v$. We may assume that $\phi'(0) \neq 0$. Define the mapping $f : \mathbb{B}^n \to \mathbb{C}^n$, $f(z_1, \ldots, z_n) := \phi(z_1) + c(0, z_2, \ldots, z_n)$. For $\epsilon > 0$ small enough we obtain $f \in F_x$. Moreover, we can check that $f'(0)^{-1}(v) = (t, 0, \ldots, 0)$. Thus $f_1\beta_0(v, v) = t^2 < 1$, which finishes the proof.  

Let

$$V_x := \bigcap_{\gamma \in (\Psi_x)} \{ v \in \mathbb{C}^n \mid \gamma(v, v) = 0 \}$$

and take the orthogonal supplement $I_x$ of the space $V_x$ in $\mathbb{C}^n$:

$$I_x \oplus V_x = \mathbb{C}^n.$$
By the Cauchy–Schwarz inequality, any \( \gamma \in l(\Psi_x) \) is of the form
\[
\gamma = \gamma_1 + \gamma_2, \quad \text{where} \quad \gamma_1 = \gamma|_{I_x \times l_x}, \quad \gamma_2 = \gamma|_{V_x \times V_x} \equiv 0,
\]
i.e. \( \gamma(u_1 + v_1, u_2 + v_2) = \gamma(u_1, u_2) \) for \( u_1, u_2 \in I_x, v_1, v_2 \in V_x \).

**Lemma 3.**
(a) The set \( \bigcup_{\alpha \in \Psi_x} B_\alpha \cap I_x \) is bounded in \( I_x \).
(b) \( \text{span}\{v \in \mathbb{C}^n|\kappa_D(x; v) = 0\} \subset V_x \).

**Proof.** Suppose that the set \( \bigcup_{\alpha \in \Psi_x} B_\alpha \cap I_x \) is unbounded. Then there exist sequences \( (x_\nu)_{\nu=1}^\infty \subset \partial \mathbb{R}^n \cap I_x \) and \( (\alpha_\nu)_{\nu=1}^\infty \subset \Psi_x \) such that \( \lim_{\nu \to \infty} x_\nu = x_0 \in \partial \mathbb{R}^n \cap I_x \) and \( \alpha_\nu(x_\nu, x_\nu) < \frac{1}{\nu} \).

Hence we have \( 0 \leq \gamma(x_0, x_0) = \lim_{\nu \to \infty} \gamma(x_\nu, x_\nu) \leq \lim_{\nu \to \infty} \alpha_\nu(x_\nu, x_\nu) = 0 \),
for any \( \gamma \in l(\Psi_x) \). But this implies that \( x_0 \in V_x \) which cannot hold.

To prove the other part, it suffices to show that \( \{v|\kappa_D(x; v) = 0\} \subset \{v|\gamma(v,v) = 0\} \), for any \( \gamma \in l(\Psi_x) \). Fix \( v \neq 0 \) satisfying \( \kappa_D(x; v) = 0 \). Reasoning in the same way as in the proof of the previous lemma, for arbitrarily small \( t > 0 \), we can find an element \( \alpha \in \Psi_x \) such that \( \alpha(v,v) < t \). But \( \gamma \ll \alpha \)
for any \( \alpha \in \Psi_x \), hence \( \gamma(v,v) = 0 \).

**Lemma 4.** There exists the unique element \( h_x \) of the set \( l(\Psi_x) \) such that for any choice of a basis \( \{e_1, \ldots, e_m\} \) of the space \( I_x \)
\[
det[\gamma(e_j, e_k)]_{j,k=1,\ldots,m} \leq \det[h_x(e_j, e_k)]_{j,k=1,\ldots,m}, \quad \text{for any} \ \gamma \in l(\Psi_x).
\]

**Proof.** It is easy to see the existence of such an element (by Lemma 3) and independence of the choice of the basis. Suppose that \( h_1 \neq h_2 \) are two elements satisfying the above condition. Let \( h_0 := \frac{1}{2}h_1 + \frac{1}{2}h_2 \). We have \( h_0 \in l(\Psi_x) \).

Let us fix a basis \( \{e_1, \ldots, e_m\} \) of the space \( I_x \). Notice that there must be \( \det[h_l(e_j, e_k)]_{j,k=1,\ldots,m} \neq 0 \) for \( l = 1, 2 \). Indeed, define the Hermitian metric \( h_r(e_j, e_k) := 0 \) for \( j \neq k \), \( h_r(e_j, e_j) := r > 0 \) and put \( h_r|_{V_x \times V_x} \equiv 0 \). By Lemma 3(a), we can find \( r > 0 \) so small that \( \bigcup_{\beta \in \Psi_x} B_{\beta} \cap I_x \subset B_r \cap I_x \). Of course, \( h_r \in l(\Psi_x) \) and \( \det[h_r(e_j, e_k)]_{j,k=1,\ldots,m} = r^m > 0 \).

Let \( A \) and \( B \) be the matrices of \( h_1 \) and \( h_2 \), respectively, in this basis. However, since \( BA^{-1} \) is Hermitian, we could choose the basis \( \{e_1, \ldots, e_m\} \) in such a way that the matrix \( BA^{-1} \) is diagonal with strictly positive coefficients \( \lambda_1, \ldots, \lambda_m \) on the diagonal.

Hence
\[
det[h_0(e_j, e_k)] = \frac{1}{2^m} \det(A + B) = \frac{1}{2^m} \det((I + BA^{-1})A)
= \frac{1}{2^m} (\lambda_1 + 1) \ldots (\lambda_m + 1) \det A
\geq \sqrt{\lambda_1 \ldots \lambda_m} \det A = \sqrt{\det(BA^{-1})} \det A = \det A.
\]
The equality holds only if $\lambda_j = 1$, $j = 1, \ldots, m$ but that is impossible when $A \neq B$ (in our case). Thus, we obtain $\det[h_1(e_j, e_k)] = \det[h_2(e_j, e_k)] < \det[h_0(e_j, e_k)]$ which cannot hold.

We call this unique element $h_x$ the Wu pseudometric of the domain $D$ at the point $x$. (We will also write $h_x = h_{D,x}$.)

**Lemma 5.** If the Kobayashi–Royden pseudometric $\kappa_D(x; \cdot)$ is Hermitian at the point $x \in D$, then $h_{D,x}(v, v) = \kappa_D(x; v)^2$. In particular, this is the case when $D \subset \mathbb{C}$.

**Proof.** We will show that $K_x = B_{h_x}$, because each Hermitian metric is determined by its unit ball.

Let $\tilde{\kappa}_x \in \mathbb{Q}$ be such that $\tilde{\kappa}_x(v, v) = \kappa_D(x; v)^2$. We get $\tilde{\kappa}_x(v, v) = h_x(v, v) = 0$ for $v \in V_x$ by the definition of $V_x$. So, we only need to show that $K_x = B_{h_x}$ in $I_x$.

Choose any basis $\{e_1, \ldots, e_m\}$ of $I_x$. By the change of coordinates, we obtain the general property:

$$
\det[\alpha(e_j, e_k)]_{j,k=1,\ldots,m} dL^{2m}(B_\alpha \cap I_x) = C dL^{2m}(B_\beta \cap I_x),
$$

where the constant $C > 0$ depends only on the choice of the basis and $dL^{2m}$ is 2m-dimensional Lebesgue measure.

Lemma 2 implies $K_x \cap I_x \subset B_{h_x} \cap I_x$. Suppose that these unit balls are not equal, i.e. $L^{2m}(K_x \cap I_x) < L^{2m}(B_{h_x} \cap I_x)$. Then, due to the above property, there must be $\det[h_x(e_j, e_k)]_{j,k=1,\ldots,m} < \det[\tilde{\kappa}_x(e_j, e_k)]_{j,k=1,\ldots,m}$ but that contradicts Lemma 4.

Geometrically speaking, “$h_x$-ball” is the smallest possible ellipsoid containing the Kobayashi indicatrix.

**2. Elementary Reinhardt domains.** Let us introduce some notations concerning elementary Reinhardt domains. For $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n \setminus \{0\}$ define

$$D_\mu := \{z \in \mathbb{C}^n : |z_1|^\mu_1 \cdots |z_n|^\mu_n < 1, \text{ if } \mu_j < 0 \text{ then } z_j \neq 0\}.$$

We say that $\mu$ is of rational type if there exist $t > 0$ and $\nu \in \mathbb{Z}_+^n$ such that $\mu = t\nu$; otherwise $\mu$ is said to be of irrational type.

Without loss of generality we may assume that $\mu_1, \ldots, \mu_l < 0$, $\mu_{l+1}, \ldots, \mu_n > 0$, where $l \in \{0, \ldots, n\}$.
We remind the formulas for the Kobayashi–Royden pseudometric \( \kappa_{D_\mu} \) in elementary Reinhardt domain \( D_\mu \) (see [5]). Below, \( \gamma \) denotes 
\[
\gamma(x; v) = \frac{|v|}{1 - |x|^2}, \quad \text{for} \ x \in \Delta, \ v \in \mathbb{C}
\]
and \( \kappa_\Delta \) (where \( \Delta_* := \Delta \setminus \{0\} \)) is given by the formula (see [3])
\[
\kappa_\Delta(x; v) = -\frac{|v|}{2|x| \log |x|}, \quad \text{for} \ x \in \Delta_*, \ v \in \mathbb{C}.
\]

**Theorem 6.** Let \( (x, v) \in D_\mu \times \mathbb{C}^n \) and \( J := \{ j \in \{1, \ldots, n\} : x_j = 0 \} = \{ j_1, \ldots, j_k \} \). Define \( \tilde{\mu}_{l+1} := \min\{\mu_{l+1}, \ldots, \mu_n\} \).

(i) Assume that \( \mu \) is of rational type and \( l < n \), then
\[
\kappa_{D_\mu}(x; v) = \begin{cases} 
\gamma\left( \left( x^{\mu} \right)^{1/\tilde{\mu}_{l+1}}; \left( x^{\mu} \right)^{1/\tilde{\mu}_{l+1}} \sum_{j=1}^{n} \frac{\mu_j v_j}{x_j} \right) & \text{if} \ J = \emptyset \\
\left( |x_1|^{\mu_1} \cdots |v_{j_1}|^{\mu_{j_1}} \cdots |v_{j_k}|^{\mu_{j_k}} \cdots |x_n|^{\mu_n} \right)^{1/\tilde{\mu}_{j_1} + \cdots + \tilde{\mu}_{j_k}} & \text{if} \ J \neq \emptyset.
\end{cases}
\]

(ii) Assume that \( \mu \) is of irrational type and \( l < n \), then
\[
\kappa_{D_\mu}(x; v) = \begin{cases} 
\gamma\left( \left( \prod_{j=1}^{n} |x_j|^{\mu_j} \right)^{1/\tilde{\mu}_{l+1}}; \left( \prod_{j=1}^{n} |x_j|^{\mu_j} \right)^{1/\tilde{\mu}_{l+1}} \sum_{j=1}^{n} \frac{\mu_j v_j}{x_j} \right) & \text{if} \ J = \emptyset \\
\left( |x_1|^{\mu_1} \cdots |v_{j_1}|^{\mu_{j_1}} \cdots |v_{j_k}|^{\mu_{j_k}} \cdots |x_n|^{\mu_n} \right)^{1/\tilde{\mu}_{j_1} + \cdots + \tilde{\mu}_{j_k}} & \text{if} \ J \neq \emptyset.
\end{cases}
\]

(iii) Assume that \( \mu \) is of rational type and \( l = n \), then
\[
\kappa_{D_\mu}(x; v) = \kappa_\Delta\left( x^{\mu}; \sum_{j=1}^{n} \frac{\mu_j v_j}{x_j} \right).
\]

(iv) Assume that \( \mu \) is of irrational type and \( l = n \), then
\[
\kappa_{D_\mu}(x; v) = \kappa_\Delta\left( |x_1|^{\mu_1} \cdots |x_n|^{\mu_n}; |x_1|^{\mu_1} \cdots |x_n|^{\mu_n} \sum_{j=1}^{n} \frac{\mu_j v_j}{x_j} \right).
\]

Now, we state the main result:

**Theorem 7.** Let \( (x, v) \in D_\mu \times \mathbb{C}^n \) and \( J := \{ j \in \{1, \ldots, n\} : x_j = 0 \} \). Then
\[
h_{D_\mu,x}(v, v) = \begin{cases} 
\kappa_{D_\mu}(x; v)^2 & \text{if} \ \#J \leq 1 \\
0 & \text{if} \ \#J \geq 2,
\end{cases}
\]
where \( \#J \) denotes the cardinality of \( J \).
Proof. Notice that whenever \( \# J \leq 1 \), the Kobayashi–Royden pseudometric is Hermitian (both \( \gamma \) and \( \kappa_\Delta \) are Hermitian) and use Lemma 5. Otherwise, the linear span of the set of zeros of the Kobayashi–Royden pseudometric coincides with \( \mathbb{C}^n \). Thus, by Lemma 3(b), the Wu pseudometric is zero.

The Wu metric \( h_D \) is not necessarily continuous with respect to the variable \( x \in D \) which is illustrated by the following example.

Example. For the domain \( D_{(2,1)} = \{ (x_1, x_2) \in \mathbb{C}^2 \mid |x_1|^2 |x_2| < 1 \} \), we have

\[
h_{D_{(2,1)}, (x_1, x_2)}(1, 1) = \frac{|2x_1 x_2 + x_1^2|^2}{(1 - |x_1^2 x_2|^2)^2}, \quad \text{if } x_1, x_2 \neq 0,
\]

while \( h_{D_{(2,1)}, (0, x_2)}(1, 1) = |x_2|^2 \neq \lim_{x_1 \to 0} h_{D_{(2,1)}, (x_1, x_2)}(1, 1) = 0, \) if \( x_2 \neq 0 \).

However, the following is true:

Theorem 8. The Wu metric \( (x, u, v) \mapsto h_{D_{\mu}, x}(u, v) \) is real analytic in the domain \( \tilde{D}_{\mu} \times \mathbb{C}^2 \), where \( \tilde{D}_{\mu} := D_{\mu} \setminus \{ (x_1, \ldots, x_n) \in \mathbb{C}^n \mid x_1 \ldots x_n = 0 \} \).

Proof. The formulas in Theorem 6 and 7 imply the fact directly.

References


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