EXPONENTIAL STABILITY OF SOLUTIONS OF THE CAUCHY PROBLEM FOR A DIFFUSION EQUATION WITH ABSORPTION WITH A DISTRIBUTION INITIAL CONDITION

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Abstract. We establish the estimate of the $L^1$ norm of a solution of a diffusion equation with absorption with an initial condition given by a distribution with compact support.

Consider the Cauchy problem

$$\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u - V(x)u \\
u(t_0) &= \Lambda
\end{align*}$$

where $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$, $\Lambda \in \mathcal{D}'(\mathbb{R}^n)$, supp $\Lambda$ is compact. Denote $X = (0, \infty) \times \mathbb{R}^n$.

We call a function $u \in L^1(X)$ a solution of (1) iff (1) holds in the sense of distributions i.e. for all $\psi \in \mathcal{D}(X)$

$$\int_0^\infty \int_{\mathbb{R}^n} u(t,x) \left( \frac{\partial \psi}{\partial t}(t,x) + \Delta \psi(t,x) - V(x)\psi(t,x) \right) dx dt = 0.$$ 

We say that the solution of (1) satisfies the initial condition (2) if for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$

$$\lim_{t \to t_0} \int_{\mathbb{R}^n} u(t,x)\varphi(x)dx = \Lambda(\varphi).$$

When $\Lambda$ is a Dirac distribution, then a solution of \{(1), (2)\} is called a fundamental solution of (1).

By $(T(t))_{t \geq 0}$ we denote the Gaussian semigroup on $L^1(\mathbb{R}^n)$ given by

$$(T(t)f)(x) = (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(y)e^{-\frac{\|x-y\|^2}{4t}} dy.$$
Note that \((T(t))\) is a holomorphic contraction semigroup. \(\Delta\) is the generator of \((T(t))\) defined on its domain \(D(\Delta) = \{f \in L^1 : \Delta f \in L^1\}\) meant in the sense of distributions.

For \(0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)\) we define an operator \(\Delta - V\) as follows: let \(D(A_{\text{min}}) = D(\mathbb{R}^n)\) (the test functions on \(\mathbb{R}^n\)) and \(A_{\text{min}}f = \Delta f - Vf\). Then \(A_{\text{min}}\) is closable in \(L^1(\mathbb{R}^n)\) and we set \(\Delta - V = A_{\text{min}}\) in \(L^p(\mathbb{R}^n)\). Then \(\Delta - V\) generates a holomorphic semigroup \((S(t))_{t \geq 0}\) on \(L^1(\mathbb{R}^n)\).

We say that \(G \subset \mathbb{R}^n\) contains arbitrary large balls if for any \(r > 0\) there exists \(x \in \mathbb{R}^n\) such that the ball \(B(x,r) := \{y \in \mathbb{R}^n : |x - y| < r\}\) is included in \(G\). By \(G\) we denote the set of all open subsets of \(\mathbb{R}^n\) which contain arbitrary large balls. In \([1]\) W. Arendt and Ch. Batty proved the theorem on stability of a solution of equation \((1)\).

**Theorem 1.** Let \(0 \leq V \in (L^1 + L^\infty)(\mathbb{R}^n)\). If for each \(G \in \mathcal{G}\)
\[(3)\quad \int_G V(x)dx = \infty\]
then
\[
\inf\{\omega \in \mathbb{R} : \sup_{t \geq 0} e^{-\omega t}||S(t)|| < \infty\} < 0.
\]

So now, we can get an easy

**Corollary 2.** Let \(0 \leq V \in (L^1 + L^\infty)(\mathbb{R}^n)\). If for each \(G \in \mathcal{G}\) \(\Box\) holds, then there exist constants \(M, \omega > 0\) such that for all \(f \in L^1(\mathbb{R}^n)\) and for any initial time \(t_0 \in \mathbb{R}\) a distribution solution \(u(t,x)\) of the Cauchy problem \((1)\), \(u(t_0,x) = f(x)\) satisfies
\[
||u(t,\cdot)||_{L^1(\mathbb{R}^n)} \leq Me^{-\omega(t-t_0)}||f||_{L^1(\mathbb{R}^n)}.
\]

**Proof.** For \(t \geq t_0\) define a holomorphic semigroup \(S_0(t) = S(t-t_0)\). Then for any \(f \in L^1(\mathbb{R}^n)\) the function \(u(t,\cdot) = S_0(t)f\) is a solution in the sense of distributions of the problem \((1)\), \(u(t_0,\cdot) = f\), so by Theorem 1 there exists \(\omega > 0\) such that
\[
M := \sup_{t \geq 0}e^{\omega t}||S(t)|| < \infty.
\]
Consequently,
\[
sup_{t \geq t_0}e^{\omega(t-t_0)}||S_0(t)|| = sup_{t \geq t_0}e^{\omega(t-t_0)}||S(t-t_0)|| = M,
\]
so
\[
||u(t,\cdot)||_{L^1(\mathbb{R}^n)} = ||S_0(t)f||_{L^1(\mathbb{R}^n)} \leq ||S_0(t)|| \cdot ||f||_{L^1(\mathbb{R}^n)}
\]
which completes the proof. \(\Box\)

Our main result is
Theorem 3. Let \( 0 \leq V \in (L^1 + L^\infty)(\mathbb{R}^n) \). Let \( \Lambda \in \mathcal{D}'(\mathbb{R}^n) \), \( \text{supp} \Lambda \) is compact. Let \( t_0 \in \mathbb{R} \). Let \( u \) be a solution of the Cauchy problem \{(1), (2)\}. If for each \( G \in \mathcal{G} \) holds, then there exist constants \( M, \omega > 0 \) such that
\[
||u(t, \cdot)||_{L^1(\mathbb{R}^n)} \leq Me^{-\omega(t-t_0)}|\Lambda(1)|
\]

Proof. Consider a function \( h \in \mathcal{D}(\mathbb{R}^n) \) such that \( h \geq 0, ||h||_{L^1(\mathbb{R}^n)} = 1 \) and define
\[
h_\nu(x) := \nu^n h(\nu x).
\]
Then \( h_\nu \ast \Lambda \in \mathcal{D}(\mathbb{R}^n) \) with \( \text{supp}(h_\nu \ast \Lambda) \subset \text{supp} h_\nu + \text{supp} \Lambda \). Moreover \( [h_\nu \ast \Lambda] \to \Lambda \), where by \([f]\) we denote a distribution generated by a function \( f \).

Consider a sequence of the Cauchy problems \{(1), \( u(t_0) = h_\nu \ast \Lambda \)\}. Denote by \( u_\nu \) solutions in the sense of distributions of these problems. Thanks to Corollary 2 we have
\[
||u_\nu(t, \cdot)||_{L^1(\mathbb{R}^n)} \leq Me^{-\omega(t-t_0)}||h_\nu \ast \Lambda||_{L^1(\mathbb{R}^n)},
\]
Since \( \Lambda \) has a compact support, it can be uniquely extended to a continuous linear functional on \( C^\infty(\mathbb{R}^n) \). Moreover, let \( \Lambda^+ = \sup \{\Lambda, 0\}, \Lambda^- = \sup \{-\Lambda, 0\} \), then
\[
||h_\nu \ast \Lambda^+||_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} 1 \cdot (h_\nu \ast \Lambda^+)(x)dx = \left(1 \ast (h_\nu \ast \Lambda^+)\right)(0) = (1 \ast (h_\nu \ast \Lambda^+))(0) = \Lambda^+(1 \ast \Lambda^-) = \Lambda^+(1)
\]
where \( \tilde{\nu}(x) = \nu(-x) \), and similarly
\[
||h_\nu \ast \Lambda^-||_{L^1(\mathbb{R}^n)} = \Lambda^- (1)
\]
so
\[
||h_\nu \ast \Lambda||_{L^1(\mathbb{R}^n)} = |\Lambda(1)|.
\]
Hence
\[
(4) \quad ||u_\nu(t, \cdot)||_{L^1(\mathbb{R}^n)} \leq Me^{-\omega(t-t_0)}|\Lambda(1)|.
\]
Moreover, we have
\[
||u_\nu||_{L^1(X)} \leq \frac{M}{\omega} |\Lambda(1)|,
\]
so the sequence \( u_\nu \) is bounded in \( X \), and so is \(-Vu_\nu\).

Let \( 0 < \tau < \infty \), denote \( Q_\tau := (0, \tau) \times \mathbb{R}^n \). Now, we need the following lemma which can be found in [2].

Lemma 4. Consider the mapping \( K \) defined by
\[
K : L^1(\mathbb{R}^n) \times L^1(Q_\tau) \ni (u_0, f) \mapsto u = T(t)u_0 + \int_0^t T(t - \tau)f(\tau)d\tau \in L^1(Q_\tau),
\]
i. e. \( u \) is the solution of the Cauchy problem
\[
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u &= f \\
u(0, x) &= u_0(x)
\end{align*}
\]
Then \( K \) is a compact operator.
Obviously, 

\[ u_\nu = K(i_\nu, -Vu_\nu), \]

so by Lemma 4 there exist a subsequence still denoted by \( u_\nu \) and a function \( u_\tau \in L^1(Q_\tau) \) such that \( u_\nu \to u_\tau \) in \( L^1(Q_\tau) \). Let \( u = \bigcup_{\tau > 0} u_\tau \). Since \( u_\nu \) are the solutions of (1), for any \( \psi \in D(X) \)

\[ \left| \int_0^\infty \int_{\mathbb{R}^n} u_\nu \left( \frac{\partial \psi}{\partial t} + \Delta \psi - V_\psi \right) dx dt - \int_0^\infty \int_{\mathbb{R}^n} u \left( \frac{\partial \psi}{\partial t} + \Delta \psi - V_\psi \right) dx dt \right| \leq \]

\[ \left| \int_0^\infty \int_{\mathbb{R}^n} |u_\nu - u| \left( \frac{\partial \psi}{\partial t} + \Delta \psi - V_\psi \right) dx dt \right| \leq C \int_0^\infty \int_{\mathbb{R}^n} |u_\nu - u| dx dt \to 0, \]

so \( u \) is a solution of (1) in the sense of distributions.

Moreover, by Riesz-Fischer theorem there exists a subsequence still denoted by \( u_\nu \) which converges pointwise almost everywhere to \( u \) so

\[ \|u_\nu(t, \cdot)\|_{L^1(\mathbb{R}^n)} \to \|u(t, \cdot)\|_{L^1(\mathbb{R}^n)}. \]

Let \( \varphi \in D(\mathbb{R}^n) \). Denote \( \Sigma = \sup \{|\varphi(x)| : x \in \mathbb{R}^n\} \). Let \( \varepsilon > 0 \). Then there exists \( N \) such that \( ||h_N \ast \Lambda(\varphi) - \varphi(0)|| \leq \frac{\varepsilon}{3} \) and

\[ \int_{\mathbb{R}^n} |u_N(t, x) - u(t, x)| dx \leq \frac{\varepsilon}{3 \Sigma}. \]

For \( N \) there exists \( \delta > 0 \) such that if \( 0 < t - t_0 < \delta \)

\[ \int_{\mathbb{R}^n} |u_N(t, x) - (h_N \ast \Lambda)(x)| dx \leq \frac{\varepsilon}{3 \Sigma}. \]

Then

\[ \left| \int_{\mathbb{R}^n} u(t, x) \varphi(x) dx - \Lambda(\varphi) \right| \leq \int_{\mathbb{R}^n} |u(t, x) - u_N(t, x)| \cdot |\varphi(x)| dx + \]

\[ \int_{\mathbb{R}^n} |u_N(t, x) - (h_N \ast \Lambda)(x)| \cdot |\varphi(x)| dx + ||h_N \ast \Lambda(\varphi) - \Lambda(\varphi)|| \leq \varepsilon, \]

so \( u \) is the solution of the Cauchy problem for \{1, 2\}, which completes the proof.

References


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