Abstract. It is proved that if $F$ is the Cartesian product of $n$ closed subsets $F_1, F_2, \ldots, F_n$ of $\mathbb{C}$ ($n \geq 2$) with $F_1 \neq \mathbb{C}$ and $F_2 \neq \mathbb{C}$, then for any two different points $a, b \in D := \mathbb{C}^n \setminus F$ there is a holomorphic mapping $f : \mathbb{C} \to D$ such that $f(0) = a$ and $f(1) = b$.

The purpose of this note is to prove the following

**Proposition 1.** Let $F$ be the Cartesian product of $n$ closed subsets $F_1, F_2, \ldots, F_n$ of $\mathbb{C}$ ($n \geq 2$) with $F_1 \neq \mathbb{C}$ and $F_2 \neq \mathbb{C}$. Then for any two different points $a, b \in D := \mathbb{C}^n \setminus F$ there is a holomorphic mapping $f : \mathbb{C} \to D$ such that $f(0) = a$ and $f(1) = b$.

In the particular case when $a \in (\mathbb{C} \setminus F_1) \times \mathbb{C}^{n-1}$ and $b \in \mathbb{C} \times (\mathbb{C} \setminus F_2) \times \mathbb{C}^{n-2}$, this proposition has been proved in [1] and the authors raised the question if it still holds for any two different points $a, b \in D := \mathbb{C}^n \setminus F$.

Proof. It suffices to prove the proposition for points $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ such that $a_2, b_2 \in D_2 := \mathbb{C} \setminus F_2$. It is trivial if $a_2 = b_2$. Let $a_2 \neq b_2$. Since $D_1 := \mathbb{C} \setminus F_1$ and $D_2$ are nonempty open sets, after linear changes in the first two complex planes, we may assume that $D_1$ contains the unit disc $\Delta \subset \mathbb{C}$, $a_1, b_1 \not\in \Delta$, $a_2 = 1, b_2 = -1$, and $D_2 \supset G := \{ z : |z - 1| < \varepsilon \text{ or } |z + 1| < \varepsilon \}$ for some $\varepsilon > 0$. Let

$$g_1(z) := \frac{1 - \exp(-z^2)}{z^2}, \quad g_2(z) := \frac{(1 - \exp(-z^2))^2}{z^3},$$

$$h_j(z) := z \int_0^\lambda g_j(zt)dt, \quad j = 1, 2, \quad \hat{f}_1(z) := \exp(2z^2 - 1)$$

(we shall choose the number $\lambda > 0$ later on). Note that the set $A := \{ z \in \mathbb{C} : |\hat{f}_1(z)| \geq 1 \}$ is the union of the sets $A_1 := \{ z \in \mathbb{C} : \text{Re}(z) > 0, 2\text{Re}(z^2) \geq 1 \}$.
and \( A_2 := \{ z \in \mathbb{C} : \text{Re}(z) < 0, 2 \text{Re}(z^2) \geq 1 \} \). Then there exist numbers \( \alpha_1 \in A_1 \) and \( \alpha_2 \in A_2 \) such that \( h_1(\alpha_1) = a_1 \) and \( h_1(\alpha_2) = b_1 \). Let \( z \in A_1 \). Since \( g_1(u) \) and \( g_2(u) \) are entire functions, we have

\[
|h_1(z) - h_1(1)| = \left| \int_{\lambda}^{1} g_1(u) du \right| < 2 \frac{|z - 1|}{\lambda} \int_{0}^{1} \frac{dt}{|1 + (z - 1)t|^2}
\]

(1)

\[
\leq 2 \frac{|z - 1|}{\lambda} \int_{0}^{1} \frac{dt}{(1 + |z - 1|t)^2} = 2 \frac{|z - 1|}{\lambda \text{Re}(z)} < \frac{2 \sqrt{2}}{\lambda},
\]

and

\[
|h_2(z) - h_2(1)| = \left| \int_{\lambda}^{1} g_2(u) du \right| < 4 \frac{|z - 1|}{\lambda^2} \int_{0}^{1} \frac{dt}{|1 + (z - 1)t|^3}
\]

(2)

\[
\leq 4 \frac{|z - 1|}{\lambda^2} \int_{0}^{1} \frac{dt}{(1 + |z - 1|t)^3} = 2 \frac{|z - 1| \text{Re}(z + 1)}{(\lambda \text{Re}(z))^2} < \frac{4}{\lambda^2}.
\]

Analogously, if \( z \in A_2 \), then

\[
|h_1(z) - h_1(-1)| < \frac{2 \sqrt{2}}{\lambda} \quad \text{and} \quad |h_2(z) - h_2(-1)| < \frac{4}{\lambda^2}.
\]

Note that

\[
h_1(1) = -h_1(-1) \xrightarrow{\lambda \to \infty} d_1 := \int_{0}^{\infty} g_1(t) dt > 0
\]

and

\[
h_2(1) = h_2(-1) \xrightarrow{\lambda \to \infty} d_2 := \int_{0}^{\infty} g_2(t) dt > 0.
\]

Now, it follows from (1), (2), (3), and the triangle inequality that for any \( \lambda \gg 1 \), we may find constants \( c_1 \) and \( c_2 \) \((c_1, c_2) \) tends to the solution of the system \( d_1 x_1 + d_2 x_2 = 1, -d_1 x_1 + d_2 x_2 = -1 \), when \( \lambda \to \infty \) such that if \( \hat{f}_2 = c_1 h_1 + c_2 h_2 \), then \( \hat{f}_2(\alpha_1) = 1 \), \( \hat{f}_2(\alpha_2) = -1 \), \( |\hat{f}_2(z) - 1| < \varepsilon \) for \( z \in A_1 \), and \( |\hat{f}_2(z) + 1| < \varepsilon \) for \( z \in A_2 \). Set \( l(z) = (\alpha_2 - \alpha_1)z + \alpha_1 \), \( f_j(z) = f_j(l(z)) \) for \( j = 1, 2 \), and \( f_j(z) = (b_j - a_j)z + a_j \) for \( 3 \leq j \leq n \). Then the mapping \( f := (f_1, f_2, \ldots, f_n) \) has the required properties.

\[\square\]

**Acknowledgments.** The author would like thank M. Jarnicki for the useful remarks on an earlier version of this note.

**References**


*Received  April 15, 2002*