Lag synchronization of complex dynamical networks with hybrid coupling via adaptive pinning control

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Abstract

This paper investigates the lag synchronization of two general complex dynamical networks with mixed coupling via pinning control. By applying the Lyaponov functional theory and mathematical analysis method, sufficient verifiable conditions of lag synchronization are obtained by adding controllers to a part of nodes. Here, the coupling configuration matrices are not required to be symmetric or irreducible and the inner connecting matrices are arbitrary real matrices. It is shown that the lag synchronization of the drive and response systems can be realized via the linear feedback pinning control and adaptive feedback pinning control. These results remove some restrictions on the node dynamics and the number of the pinned nodes. Numerical examples are presented to illustrate the effectiveness of the theoretical results.

Keywords:
Lag synchronization, Complex dynamical networks, Pinning control, Hybrid coupling

1. Introduction

As is known to all, a complex dynamical network is a set of coupled nodes interconnected by edges, in which each node represents a dynamical system. The structure of many real systems in nature can be described by the complex dynamical networks, such as social organizations, Internet, communication networks, food webs, disease transmission networks, the World Wide Web, power grids, and so on [1-3]. This has led to much interest to the studies of the complex networks. In particular, with the wide applications of the complex networks in fields of neural networks [4], biological systems [5], information science [6], secure communication [7-8], etc, synchronization of the complex network has been one of the main topics due to its realistic significance and study value.

In the existing literatures, synchronization and its control of complex dynamical networks have been widely and extensively studied, and a number of researchers have proposed many synchronization methods...
including linear state feedback control \[9\], pinning control \[10-14\], state observer based control \[15-16\], impulsive control \[17-18\], and adaptive control \[19\] and so forth. However, majority of the works in network synchronization focus on the inner synchronization \[20\], which means that all nodes in one network achieve a coherent behavior. Different from the inner synchronization, there is another kind of synchronization with the name of outer synchronization \[21-22\], which quickly became a focus point of researching since Li first proposed the concept of outer synchronization in 2007 \[23\]. In general, we can see several kinds of synchronization, such as, complete synchronization \[24\], phase synchronization \[25\], lag synchronization \[26\], generalized synchronization \[27\], projective synchronization \[28\] and so on. Among them, lag synchronization, which requires the states of response system to synchronize with the past states of the drive system, has been widely observed in many practical systems such as electronic circuits, lasers and neural systems \[29\]. It has been proved to be a reasonable scheme from the viewpoint of engineering applications and the characteristics of channel in secure communication, parallel image processing, and pattern storage \[30\]. Therefore, lag synchronization has become a hot topic and attracted much attention from authors in many fields \[26,31-33\]. For example, in Ref.\[34\], the author investigated the issue of the lag synchronization between two coupled networks by adding controllers to a part of nodes. Zhao et al.\[35\] considered the lag synchronization problem of two different complex networks based on the approach of state observer. Furthermore, in Ref.\[36\], the authors studied lag synchronization between two coupled networks via pinning control, including the linear and adaptive feedback pinning schemes.

Unfortunately, although the approach realized the lag synchronization for complex dynamical networks, there are still some problems which need to be studied. These include, (1) the coupling configuration matrices are always assumed to be irreducible and their off-diagonal entries nonnegative, and the inner connecting matrices are diagonal positive define. (2) It is very expensive and even impractical to apply controllers to all or many nodes, especially for the engineering applications. For this reason, as described in Ref.\[10\], in virtue of low-cost and easy implementation, it is significant to investigate that the drive and response networks are synchronized by pinning only a small portion of nodes of the network. (3) In a realistic network, since the speed of signal travel between nodes is limited and the network nodes may be required to have non-local interconnections such as telecommunications \[37,38\], the discrete delay coupling and distributed time coupling are inevitable factors in the network. Thus, the synchronization of complex networks with delayed coupling, which includes discrete delay coupling and distributed delay coupling, should be considered. Sufficient conditions for adaptive lag synchronization of complex dynamical network with discrete delayed coupling have been provided in Ref.\[33\]. To the best of our knowledge, up to now, there has been no literature concerning the problems of lag synchronization for complex dynamical networks with mixed coupling.

Inspired by the above mentioned discussions, in this paper, a lag synchronization method between two general complex dynamical networks with hybrid coupling by pinning control a small portion of nodes of the network...
network has been proposed. The main contributions of this paper are listed as follows: Firstly, the hybrid coupling, which is made up of non-delay coupling, discrete delay coupling and distributed delay coupling; Then, by applying the Lyaponov functional theory and mathematical analysis method, sufficient verifiable conditions are constructed for the lag synchronization of the drive and response networks. These results are less conservative and easy to verify through numerical simulation. Moreover, the coupling matrices are not necessary to be symmetric and irreducible, and without assuming diagonal or positive define of the inner linking matrices. In numerical simulation section, we verify that pinning only one node can realize lag synchronization adequately, and the node can be chosen according to the high-degree of vertex or the maximum norm of synchronization error.

The rest of this paper is organized as follows. In section 2, the complex dynamical network is introduced and some necessary definitions and lemmas are given. In section 3, the linear feedback pinning control and the adaptive feedback pinning control are designed, and the corresponding lag synchronization theorems are derived respectively. In section 4, two illustrative examples are provided to illustrate the effectiveness of the theoretical results. Finally, section 5 concludes this paper.

**Notation:** Throughout this paper, $I_n$ denotes an $n$-dimensional identity matrix, $\mathbb{R}^n$ denotes $n$-dimensional Euclidean space and $\mathbb{R}^{n \times n}$ is the set of all $n \times n$ real matrices. For symmetric matrices $X$ and $Y$, the notation $X > Y (X \geq Y)$ means that the matrix $X - Y$ is positive definite (nonnegative). The $\text{diag}\{\ldots\}$ denotes the block diagonal matrix. For a real symmetric matrix $P$, $\lambda_{\text{min}}(P)$ and $\lambda_{\text{max}}(P)$ denote the minimum and maximum eigenvalues of $P$. The superscript $T$ denotes matrix or vector transposition. The symmetric terms in a symmetric matrix are denoted by $\ast$. Matrices, if not explicitly stated, are assumed to have compatible dimensions.

### 2. Problem formulation and Preliminaries

Consider the following complex dynamical networks with hybrid time-varying delays coupling:

$$
\dot{x}_i(t) = f(x_i(t), x_i(t - \sigma(t))) + \sum_{j=1}^{N} c_{ij}^{(1)} \Gamma_1 x_j(t) + \sum_{j=1}^{N} c_{ij}^{(2)} \Gamma_2 x_j(t - \sigma(t)) + \sum_{j=1}^{N} c_{ij}^{(3)} \int_{t-d(t)}^{t} x_j(s)ds
$$

$$
i = 1, 2, \ldots, N,
$$

where $x_i(t) = [x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t)]^T \in \mathbb{R}^n$ stands for the drive state of the $i$th node. $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous nonlinear vector-valued function. $\sigma(t)$ is discrete time-varying delay, and $d(t)$ is distributed time-varying delay. $\Gamma_1, \Gamma_2$ and $\Gamma_3 \in \mathbb{R}^n \times \mathbb{R}^n$ represent the inner connecting matrix, the discrete-delay inner connecting matrix and the distributed-delay inner connecting matrix, respectively; $C^{(k)} = (\hat{c}_{ij}^{(k)}) \in \mathbb{R}^{N \times N}, k = 1, 2, 3$, represent the coupling configuration of the drive-response networks and satisfy the diffu-
sive coupling connections:

\[ \dot{c}_{ii}^{(k)} = - \sum_{j=1, j \neq i}^{\infty} c_{ij}^{(k)}, \quad i = 1, 2, \ldots, N, \quad k = 1, 2, 3, \]  

(2)

where \( c_{ij}^{(1)} \) are defined as follows: \( c_{ij}^{(1)} \geq 0 \) for \( j \neq i \), i.e. \( C^{(1)} \) is nonnegative diffusive.

**Remark 1.** In this paper, the coupling configuration matrices are not required to be identical, symmetric or irreducible. Moreover, different from Refs.[33-36], in our paper the non-delayed inner connecting matrix, the discrete-delay inner connecting matrix and the distributed-delay inner connecting matrix are arbitrary real matrices.

Throughout this paper, we make the following assumptions on time-varying delays and nonlinear function \( f \).

**Assumption 2.1.** \( 0 \leq \sigma(t) \leq \sigma, \quad 0 \leq d(t) \leq d, \) and \( \dot{\sigma}(t) \leq \dot{\sigma} < 1, \quad \dot{d}(t) \leq \mu < 1, \) where \( \sigma, d, \dot{\sigma} \) and \( \mu \) are constants.

**Assumption 2.2.** The nonlinear function \( f \) satisfies uniform semi-Lipschitz condition, i.e., there exists positive constants \( \alpha_1 \) and \( \alpha_2 \) such that

\[ (x - y)^T (f(x, \bar{x}) - f(y, \bar{y})) \leq \alpha_1 (x - y)^T (x - y) + \alpha_2 (\bar{x} - \bar{y})^T (\bar{x} - \bar{y}). \]  

(3)

for any \( x \in \mathbb{R}^n, \quad y \in \mathbb{R}^n, \quad \bar{x} \in \mathbb{R}^n, \quad \bar{y} \in \mathbb{R}^n. \)

It has been verified that many typical benchmark chaotic systems such as the Lorenz system, Chua’s system and the unified chaotic system satisfy Assumption 2.2.

Correspondingly, the response system is designed by

\[ \dot{y}_i(t) = f(y_i(t), y_i(t - \sigma(t))) + \sum_{j=1}^{N} c_{ij}^{(1)} \Gamma_1 y_j(t) + \sum_{j=1}^{N} c_{ij}^{(2)} \Gamma_2 y_j(t - \sigma(t)) + \sum_{j=1}^{N} c_{ij}^{(3)} \Gamma_3 \int_{t-d(t)}^{t} y_j(s) ds + u_i, \quad i = 1, 2, \ldots, N, \]  

(4)

where \( y_i(t) = [y_{i1}(t), y_{i2}(t), \ldots, y_{in}(t)]^T \in \mathbb{R}^n \) is the response state of the \( i \)th node, \( u_i(i = 1, 2, \ldots, N) \) are the controllers to be designed later, and other notations are the same as above.

The following definition and lemmas are useful in deriving our main results:

**Definition 2.1.** [34] The drive system (1) is said to lag synchronization with the response system (4) at time \( \tau \) if satisfies the following property:

\[ \lim_{t \to \infty} \| y_i(t) - x_i(t - \tau) \| = 0, \quad i = 1, 2, \ldots, N. \]  

(5)

where \( \tau \) is a given positive time delay.
Lemma 2.1. [39] For any constant matrix $W \in \mathbb{R}^{n \times n}$, $W^T = W > 0$, scalar $d > 0$, and vector function $\omega : [0, d] \to \mathbb{R}^n$ such that the integrations concerned are well defined, then
\[
d \int_0^d \omega^T(s)W\omega(s)ds \geq (\int_0^d \omega(s)ds)^T W (\int_0^d \omega(s)ds),\]

**Lemma 2.2.** [11] For a $n \times n$ matrix $A$, the following inequality holds:
\[
AA^T \leq \|A\|^2I.
\]

**Lemma 2.3.** [40] Assume that $A$, $B$ are $n$ by $n$ Hermitian matrices. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, and $\varepsilon_1 \geq \varepsilon_2 \geq \cdots \geq \varepsilon_n$ be eigenvalues of $A$, $B$ and $A + B$, respectively. Then one has
\[
\lambda_i + \mu_i \leq \varepsilon_i \leq \lambda_i + \mu_i, \quad i = 1, 2, \ldots, n.
\]

**Lemma 2.4.** [41] Assume that $Q = (q_{ij})_{n \times n}$ is symmetric. Let
\[
D = \text{diag}(d_1, d_2, \ldots, d_m, 0, 0, \ldots, 0), \quad Q - D = \begin{pmatrix} Q_{11} - D^* & Q_{12} \\ Q^T_{12} & Q_m \end{pmatrix}, \quad \text{and} \quad d = \min_{1 \leq i \leq m} d_i,
\]
where $1 \leq m \leq N$, $d_i > 0$, $i = 1, 2, \ldots, m$, $Q_m$ is the minor matrix of $Q$ by removing its first row-column pairs, $Q_{11}$ and $Q_{12}$ are matrices with appropriate dimensions, $D^* = \text{diag}(d_1, d_2, \ldots, d_m)$. When $d > \lambda_{\text{max}}(Q_{11} - Q_{12}Q_m^{-1}Q^T_{12})$, $Q - D < 0$ is equivalent to $Q_m < 0$.

3. Main results

3.1. Lag synchronization via the linear feedback pinning control

In this subsection, we apply the linear feedback control to pin the lag synchronization. Without loss of generality, assume that the first $m(1 \leq m \leq N)$ nodes are selected and pinned with the linear controllers, which are described as
\[
\begin{cases}
    u_i = -\gamma_1 k_i e_i(t), & 1 \leq i \leq m, \\
    u_i = 0, & 1 + m \leq i \leq N,
\end{cases}
\]
where $\gamma_1 = ||\Gamma_1||$, $e_i(t) = y_i(t) - x_i(t - \tau)$, $k_i (i = 1, 2, \ldots, m) > 0$ are feedback gains.

According to Eq. (6), we obtain the following lag synchronization error system,
\[
\begin{cases}
    \dot{e}_i(t) = f(y_i(t), y_i(t - \sigma(t))) - f(x_i(t - \tau), x_i(t - \sigma(t) - \tau)) + \sum_{j=1}^{N} c_{ij}^{(1)} \Gamma_1 e_j(t) + \sum_{j=1}^{N} c_{ij}^{(2)} \Gamma_2 e_j(t - \sigma(t)) \\
    + \sum_{j=1}^{N} c_{ij}^{(3)} \Gamma_3 \int_{t-d(t)}^{t} e_j(s)ds - \gamma_1 k_i e_i(t), & 1 \leq i \leq m, \\
    \dot{e}_i(t) = f(y_i(t), y_i(t - \sigma(t))) - f(x_i(t - \tau), x_i(t - \sigma(t) - \tau)) + \sum_{j=1}^{N} c_{ij}^{(1)} \Gamma_1 e_j(t) + \sum_{j=1}^{N} c_{ij}^{(2)} \Gamma_2 e_j(t - \sigma(t)) \\
    + \sum_{j=1}^{N} c_{ij}^{(3)} \Gamma_3 \int_{t-d(t)}^{t} e_j(s)ds, & m + 1 \leq i \leq N.
\end{cases}
\]
Let $||\Gamma_2|| = \gamma_2$, $||\Gamma_3|| = \gamma_3$, $|\bar{C}(t)| = (|\bar{c}_{ij}(k)|)_{N \times N}$, ($k = 1, 2, 3$), $\rho_{min} = \lambda_{min}(\Gamma_1 + \Gamma_1^T)/2$, $\bar{C}(t) = diag(\bar{c}_{11}, \bar{c}_{22}, \ldots, \bar{c}_{NN})$, $K = diag(k_1, \ldots, k_m, 0, \ldots, 0)$, where $k_i$ ($1 \leq i \leq m$) are positive constants to be determined later. Then we have the following result.

**Theorem 3.1.** Suppose that the Assumption 2.1 and 2.2 hold, the drive system (1) and the response system (4) with linear controllers (6) can realize the lag synchronization, if there exist matrices $H_i = \text{diag}(h_i(1), h_i(2), \ldots, h_i(N)) \geq 0$, ($i = 1, 2$) such that the following LMI holds:

$$\Omega = \begin{pmatrix} \Omega_{11} & \frac{1}{2}\gamma_2|\bar{C}(t)| & \frac{1}{2}\gamma_3|\bar{C}(t)| \\ * & \alpha_2 I_N - (1 - \bar{\sigma})H_1 & 0 \\ * & * & -\frac{1}{d}H_2 \end{pmatrix} < 0,$$

(8)

where $\Omega_{11} = \alpha_1 I_N + (\rho_{min} - \gamma_1)\bar{C}(t) + \gamma_1\frac{C(t) + (C(t))^T}{2} - \gamma_1 K + H_1 + dH_2$.

**Proof.** Choose the following Lyapunov-Krasovskii functional candidate:

$$V(t) = V_1(t) + V_2(t)$$

(9)

where

$$V_1(t) = \frac{1}{2} \sum_{i=1}^{N} e_i^T(t)e_i(t)$$

$$V_2(t) = \sum_{i=1}^{N} h_i^{(1)} \int_{t-\sigma(t)}^{t} e_i^T(s)e_i(s)ds + \sum_{i=1}^{N} h_i^{(2)} \int_{t-d(t)}^{t} \int_{t+\theta}^{t} e_i^T(s)e_i(s)ds$$

Differentiating $V_1(t)$ along the trajectory of the error system (7), we have

$$\dot{V}_1(t) = \sum_{i=1}^{N} e_i^T(t)[f(y_i(t), y_i(t-\sigma(t))) - f(x_i(t-\tau), x_i(t-\sigma(t) - \tau))] + \sum_{j=1}^{N} \bar{c}_{ij}(k)\Gamma_1 e_j(t)$$

$$+ \sum_{j=1}^{N} \bar{c}_{ij}(2)\Gamma_2 e_j(t - \sigma(t)) + \sum_{j=1}^{N} \bar{c}_{ij}(3)\Gamma_3 \int_{t-d(t)}^{t} e_j(s)ds - \sum_{i=1}^{m} \gamma_1 k_i e_i^T(t)e_i(t)$$

(10)
Then from Assumption 2.2, we have the following estimations:

\[
\dot{V}_1(t) \leq \sum_{i=1}^{N} (\alpha_1 \|e_i(t)\|^2 + \alpha_2 \|e_i(t) - \sigma(t)\|^2) + \sum_{i=1}^{N} \sum_{j=1}^{N} e_i^T(t)\bar{c}_i^{(2)}(t) \bar{c}_i^{(1)} \Gamma_1 e_j(t) \\
+ \sum_{i=1}^{N} \sum_{j=1}^{N} e_i^T(t)\bar{c}_i^{(2)}(t) \bar{c}_i^{(1)} \Gamma_1 e_j(t) + \sum_{i=1}^{N} e_i^T(t)\bar{c}_i^{(1)} \Gamma_1 e_j(t) + \sum_{i=1}^{N} e_i^T(t)\bar{c}_i^{(1)} \Gamma_1 e_i(t) \\
+ \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} ||e_i(t)|| ||\bar{c}_i^{(2)}|| ||e_j(t) - \sigma(t)|| + \sum_{i=1}^{N} \sum_{j=1}^{N} ||e_i(t)|| ||\bar{c}_i^{(1)}|| ||\Gamma_1|| \int_{t-d(t)}^{t} e_j(s)ds \\
- \sum_{i=1}^{m} \gamma_1 k_i \bar{c}_i^T(t)e_i(t) \\
\leq \sum_{i=1}^{N} (\alpha_1 \|e_i(t)\|^2 + \alpha_2 \|e_i(t) - \sigma(t)\|^2) + \gamma_1 \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} ||e_i(t)|| ||\bar{c}_i^{(1)}|| ||e_j(t) - \sigma(t)|| + \sum_{i=1}^{N} \rho_{\min} \bar{c}_i^{(1)} e_i^T(t) e_i(t) \\
+ \gamma_2 \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} ||e_i(t)|| ||\bar{c}_i^{(1)}|| ||e_j(t) - \sigma(t)|| + \gamma_2 \sum_{j=1}^{N} \sum_{i=1}^{N} \sum_{j=1}^{N} ||e_i(t)|| ||\bar{c}_i^{(1)}|| ||\Gamma_1|| \int_{t-d(t)}^{t} e_j(s)ds \\
- \sum_{i=1}^{m} \gamma_1 k_i \bar{c}_i^T(t)e_i(t) \\
= e^T(t)(\alpha_1 I_N + (\rho_{\min} - \gamma_1)\bar{C}^{(1)}) + \gamma_1 \frac{(\bar{C}^{(1)})^T + (\bar{C}^{(1)})}{2} - \gamma_1 K e(t) + e^T(t)(\gamma_2|\bar{C}^{(2)}|e(t - \sigma(t))) \\
+ e^T(t - \sigma(t))(\alpha_2 I_N)e(t - \sigma(t)) + e^T(t)(\gamma_3|\bar{C}^{(3)}|)\bar{e}(t),
\]

where \(e(t) = (||e_1(t)||, ||e_2(t)||, \ldots, ||e_N(t)||)^T, e(t - \sigma(t)) = (||e_1(t - \sigma(t)||, ||e_2(t - \sigma(t)||, \ldots, ||e_N(t - \sigma(t)||)^T, \bar{e}(t) = (||\int_{t-d(t)}^{T} e_1(s)ds||, ||\int_{t-d(t)}^{T} e_2(s)ds||, \ldots, ||\int_{t-d(t)}^{T} e_N(s)ds||)^T.\)

By assumption 2.1 and lemma 2.1, calculating the time derivative of \(V_2(t)\) along the trajectories of system (7), we get

\[
V_2(t) \leq \sum_{i=1}^{N} h_i^{(1)}[e_i^T(t)e_i(t) - (1 - \sigma) e_i^T(t) - \sigma(t)]e_i(t - \sigma(t))] \\
+ \sum_{i=1}^{N} h_i^{(2)}[d_i^T(t)e_i(t) - (1 - \mu)\int_{t-d(t)}^{T} e_i^T(s)e_i(s)ds] \\
\sum_{i=1}^{N} h_i^{(1)}[e_i^T(t)e_i(t) - (1 - \sigma) e_i^T(t) - \sigma(t)]e_i(t - \sigma(t))] \\
+ \sum_{i=1}^{N} h_i^{(2)}[d_i^T(t)e_i(t) - \frac{1-\mu}{d}\int_{t-d(t)}^{T} e_i(s)ds][\int_{t-d(t)}^{T} e_i(s)ds] \\
= e^T(t)(H_1 + dH_2)e(t) - e^T(t - \sigma(t))(1 - \sigma)H_1 e(t - \sigma(t)) - \bar{e}(t)\frac{1-\mu}{d} H_2 \bar{e}(t)
\]

Let \(\xi(t) = (e^T(t), e^T(t - \sigma(t)), \bar{e}^T(t))^T, \Xi = -\Omega.\)
According to (8) and (10)-(12), it follows that
\[
\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) \leq -\xi^T(t)\Xi(t) \leq 0.
\] (13)

From (13), we get
\[
0 \leq \lambda_{\min}(\Xi)||\xi(t)||^2 \leq \xi^T(t)\Xi(t) \leq -\dot{V}(t)
\] (14)

Integrating (14) from 0 to \(t\), in view of \(V(t) > 0\), we obtain
\[
\int_0^t \lambda_{\min}(\Xi)||\xi(s)||^2 ds \leq -\int_0^t \dot{V}(s) ds = V(0) - V(t) \leq V(0) < +\infty
\]

By Barbalat lemma [42], we have
\[
\lambda_{\min}(\Xi)||e||^2 \leq \lambda_{\min}(\Xi)||\xi(t)||^2 \to 0,
\]
which implies that \(\lim_{t \to \infty} ||e(t)|| = 0\), then we can get \(\lim_{t \to \infty} (y_i(t) - x_i(t - \tau)) = 0\), \((i = 1, 2, \ldots, N)\). That is to say the drive system (1) lag synchronization with the response system (4) at time \(\tau\). This completes the proof.

**Remark 2.** For any given dynamical network with node dynamics \(f(x)\), the coupling matrices \(\bar{C}^{(1)}, \bar{C}^{(2)}, \bar{C}^{(3)}\) and \(\Gamma_1, \Gamma_2, \Gamma_3\) are known, so the positive constants \(\alpha_1, \alpha_2\) in Assumption 2.2 and \(\gamma_1, \gamma_2, \gamma_3, \rho_{\min}\) can be estimated by simple calculations. Thus, from condition (8), if the matrices \(H_1, H_2\) and the pinned nodes \(m\) are fixed, the feedback gains \(k_i\) can be estimated. However, the node dynamics and the coupling matrices are usually nonidentical for different dynamical networks. Therefore, the proposed pinning controllers with fixed feedback gains are not universal. In the following, an adaptive pinning strategy will be adopted to design universal controllers.

### 3.2. Synchronization via the Adaptive Feedback Pinning Control

In this subsection, we use the adaptive feedback control to pin the lag synchronization. Without loss of generality, assume that the first \(m(1 \leq m \leq N)\) nodes are selected and pinned with the adaptive controllers, which are described as
\[
\begin{align*}
\dot{u}_i(t) &= -\gamma_1 k_i(t)e_i(t), \quad 1 \leq i \leq m, \\
\dot{k}_i(t) &= \delta_i e_i^T(t)e_i(t), \quad k_i(0) = 0, \quad 1 \leq i \leq m, \\
u_i(t) &= 0, \quad m + 1 \leq i \leq N,
\end{align*}
\] (15)

where \(\gamma_1 = ||\Gamma_1||\), \(e_i(t) = y_i(t) - x_i(t - \tau)\), and \(\delta_i\) are positive constants.
According to Eq. (15), we obtain the following lag synchronization error system,

\[
\begin{aligned}
\dot{e}_i(t) &= f(y_i(t), y_i(t - \sigma(t))) - f(x_i(t - \tau), x_i(t - \sigma(t) - \tau)) + \sum_{j=1}^{N} c_{ij}^{(1)} \Gamma_1 e_j(t) + \sum_{j=1}^{N} c_{ij}^{(2)} \Gamma_2 e_j(t - \sigma(t)) \\
&\quad + \sum_{j=1}^{N} c_{ij}^{(3)} \int_{t - d(t)}^{t} e_j(s) ds - \gamma_1 k_i(t)e_i(t), \quad 1 \leq i \leq m,
\end{aligned}
\]

\[
\begin{aligned}
\dot{e}_i(t) &= f(y_i(t), y_i(t - \sigma(t))) - f(x_i(t - \tau), x_i(t - \sigma(t) - \tau)) + \sum_{j=1}^{N} c_{ij}^{(1)} \Gamma_1 e_j(t) + \sum_{j=1}^{N} c_{ij}^{(2)} \Gamma_2 e_j(t - \sigma(t)) \\
&\quad + \sum_{j=1}^{N} c_{ij}^{(3)} \int_{t - d(t)}^{t} e_j(s) ds, \quad m + 1 \leq i \leq N,
\end{aligned}
\]

\[
\dot{k}_i(t) = \delta_i e_i^T(t) e_i(t), \quad \delta_i > 0, \quad 1 \leq i \leq m.
\]

Let \(||\Gamma_2|| = \gamma_2, \ ||\Gamma_3|| = \gamma_3, |\bar{C}^{(k)}| = (|\bar{C}_{ij}^{(k)}|)_{N \times N}, (k = 1, 2, 3), \rho_{\text{min}} = \lambda_{\text{min}}((\Gamma_1 + \Gamma_1^T)/2), \bar{C}^{(1)} = \text{diag}(c_{11}^{(1)}, c_{22}^{(1)}, \ldots, c_{NN}^{(1)}), K^* = \text{diag}(k_1^*, \ldots, k_m^*, 0, \ldots, 0)\), where \(k_i^* (1 \leq i \leq m)\) are positive constants to be determined later. Then we have the following result.

**Theorem 3.2.** Supposed that the Assumption 2.1 and 2.2. hold, the drive system (1) and the response system (4) with adaptive controllers (15) can realize the lag synchronization, if there exist matrices \(R_i = \text{diag}(r_1^{(i)}, r_2^{(i)}, \ldots, r_N^{(i)}) \geq 0, (i = 1, 2)\) such that the following LMI holds:

\[
\Phi = \begin{pmatrix}
\Phi_{11} & \frac{1}{2} \gamma_2 |\bar{C}^{(2)}| & \frac{1}{2} \gamma_3 |\bar{C}^{(3)}| \\
* & \alpha_2 I_N - (1 - \sigma) R_1 & 0 \\
* & * & -\frac{1 - \rho}{\rho_2} R_2
\end{pmatrix} < 0,
\]

where \(\Phi_{11} = \alpha_1 I_N + (\rho_{\text{min}} - \gamma_1) \bar{C}^{(1)} + \gamma_1 \frac{\bar{C}^{(1)} + (\bar{C}^{(1)})^T}{2} - \gamma_1 K^* + R_1 + dR_2\).

**Proof.** Choose the following Lyapunov-Krasovskii functional candidate as follows:

\[
V(t) = V_1(t) + V_2(t)
\]

where

\[
\begin{aligned}
V_1(t) &= \frac{1}{2} \sum_{i=1}^{N} e_i^T(t) e_i(t) + \gamma_1 \sum_{i=1}^{m} \frac{1}{2\delta_i} (k_i(t) - k_i^*)^2 \\
V_2(t) &= \sum_{i=1}^{N} r_1^{(i)} \int_{t - \sigma(t)}^{t} e_i^T(s) e_i(s) ds + \sum_{i=1}^{N} r_2^{(i)} \int_{t - d(t)}^{t} \int_{t + \theta}^{t} e_i^T(s) e_i(s) ds
\end{aligned}
\]
Calculating $V_1(t)$ along the trajectory of the error system (16), we have
\[
\dot{V}_1(t) = \sum_{i=1}^{N} e_i^T(t) f(y_i(t), y_i(t - \sigma(t))) - f(x_i(t - \tau), x_i(t - \sigma(t))) + \sum_{i=1}^{m} e_i^T(t) e_i(t) + \sum_{j=1}^{N} e_j^T(t) e_j(t - \sigma(t)) + \sum_{j=1}^{N} e_j^T(t) \sum_{i=1}^{N} c_{ij}^{(3)} e_i(t) + \sum_{i=1}^{m} \gamma_i k_i(t) e_i(t) + 2\gamma_1 \sum_{i=1}^{m} \frac{1}{2\delta_i} (k_i(t) - k_i^*) \dot{k}_i(t)
\]
\[
\leq \sum_{i=1}^{N} (\alpha_1 ||e_i(t)||^2 + \alpha_2 ||e_i(t - \sigma(t))||^2) + \sum_{i=1}^{N} \sum_{j=1}^{N} e_i^T(t) e_j^T(t) e_i(t) - \sum_{i=1}^{m} \gamma_i k_i(t) e_i(t) + 2\gamma_1 \sum_{i=1}^{m} \frac{1}{2\delta_i} (k_i(t) - k_i^*) \dot{k}_i(t)
\]
\[
\leq \sum_{i=1}^{N} (\alpha_1 ||e_i(t)||^2 + \alpha_2 ||e_i(t - \sigma(t))||^2) + \sum_{i=1}^{N} \sum_{j=1}^{N} ||e_i(t)|| ||e_j(t)|| ||e_j(t - \sigma(t))|| + \sum_{i=1}^{N} \sum_{j=1}^{N} ||e_i(t)|| ||e_j(t)|| ||e_j(t - \sigma(t))|| \int_{t-d(t)}^{t} e_j(s) ds
\]
\[
- \sum_{i=1}^{m} \gamma_i k_i^* e_j^T(t) e_i(t)
\]
\[
= e^T(t) (\alpha_1 I_N + (\rho_{\min} - \gamma_1) \bar{C}(1) + \frac{\bar{C}(1)}{2} - \gamma_1 K^*) e(t) + e^T(t) (\gamma_2 |\bar{C}(2)|) e(t - \sigma(t)) + e^T(t) (\alpha_2 I_N) e(t - \sigma(t)) + e^T(t) (\gamma_3 |\bar{C}(3)|) \bar{e}(t),
\]
where $e(t) = ([|e_1(t)||, |e_2(t)||, \ldots, |e_N(t)||]^T$, $e(t - \sigma(t)) = ([|e_1(t - \sigma(t)||, |e_2(t - \sigma(t)||, \ldots, |e_N(t - \sigma(t)||]^T$, $\bar{e}(t) = ([\int_{t-d(t)}^{t} e_1(s) ds||, \|\int_{t-d(t)}^{t} e_2(s) ds||, \ldots, \|\int_{t-d(t)}^{t} e_N(s) ds||]^T$.

By assumption 2.1 and lemma 2.1, calculating the time derivative of $V_2(t)$ along the trajectories of system
According to lemma 2.2., we have equivalent to corollary 3.1 are more easy to verify. As avoiding to solve the LMI (17), we have the following corollary, and the conditions of Remark 4.

Remark 3. From (17) and (19)-(21), we can see that

\[
\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) \leq \eta^T(t) \Phi \eta(t) \leq 0
\]

Then, similar to the proof Theorem 3.1, we have \(\lim_{t \to \infty} \|e(t)\| = 0\). This completes the proof.

**Remark 3.** In general, the strength of linear feedback must be maximal, which is a kind of waste in practice to some extent. Compared with linear control [34], the control gains of adaptive control increase according to the adaptive laws. Hence, adaptive control is more flexible.

**Remark 4.** As avoiding to solve the LMI (17), we have the following corollary, and the conditions of corollary 3.1 are more easy to verify.

First, we let \(R = \alpha_2 I_N - (1 - \sigma) R_1\), then by applying Schur complement lemma, the condition (17) is equivalent to

\[
\Phi_{11} - \frac{1}{4} \gamma_2^2 |\bar{C}^{(2)}| R^{-1} |\bar{C}^{(2)}|^T + \frac{d}{4} (1 - \mu) \gamma_2^2 |\bar{C}^{(3)}| R_2^{-1} |\bar{C}^{(3)}|^T < 0.
\]

Moreover, when \(R_1 = \frac{1}{1 - \sigma} (\alpha_2 + \frac{1}{2} \gamma_2 c_2) I_N\), \(R_2 = \frac{1}{2} \gamma_3 c_3 I_N\), where \(c_2 = ||(\bar{C}^{(2)})||\), \(c_3 = ||(\bar{C}^{(3)})||\).

According to lemma 2.2., we have

\[
\Phi_{11} - \frac{1}{4} \gamma_2^2 |\bar{C}^{(2)}| R^{-1} |\bar{C}^{(2)}|^T + \frac{d}{4} (1 - \mu) \gamma_2^2 |\bar{C}^{(3)}| R_2^{-1} |\bar{C}^{(3)}|^T
\]

\[
\leq \alpha I_N + \rho_{\min} - \gamma_1 |\bar{C}^{(1)}| + \frac{1}{2} \gamma_1 (\bar{C}^{(1)})^T - \gamma_1 K^*
\]

\[
= Q - \gamma_1 K^*
\]

where

\[
Q = \alpha I_N + \rho_{\min} - \gamma_1 |\bar{C}^{(1)}| + \frac{1}{2} \gamma_1 (\bar{C}^{(1)})^T
\]

\[
\alpha = \alpha_1 + \frac{1}{2(1 - \sigma)} (2\alpha_2 + \gamma_2 c_2 + (1 - \sigma) \gamma_2 c_2) + \frac{d}{2} \gamma_3 c_3 (2 - \mu).
\]
Next let
\[ Q - \gamma_1 K^* = \begin{pmatrix} Q_{11} - \tilde{K}^* & Q_{12} \\ Q_{12}^T & Q_m \end{pmatrix}, \]

where \( Q_m \) is the minor matrix of \( Q \) by removing its first \( m (1 \leq m \leq N) \) row-column pairs, \( Q_{11} \) and \( Q_{12} \) are matrices with appropriate dimensions, \( \tilde{K}^* = \text{diag}(k_1^*, k_2^*, \ldots, k_m^*) \). Now we can obtain the following corollary 3.1.

**Corollary 3.1.** Supposed that the Assumption 2.1 and 2.2 hold, the drive system (1) and the response system (4) with adaptive controllers (15) can realize the lag synchronization, if the following two conditions are satisfied:

\[ k_i^* > \frac{1}{\gamma_1} \lambda_{\text{max}}(Q_{11} - Q_{12}Q_m^{-1}Q_{12}^T), \quad 1 \leq i \leq m, \tag{25} \]

and

\[ \lambda_{\text{max}}((\tilde{C}(1) + (\tilde{C}(1))^T)_{m} < \frac{\delta}{\gamma_1} \tag{26} \]

where \( \delta = \alpha + \lambda_{\text{max}}((\rho_{\text{min}} - \gamma_1)\tilde{C}(1))_{m} \).

**Proof.** From the lemma 2.4 and the condition (25), we can see that \( Q - \gamma_1 K^* < 0 \) is equivalent to \( Q_m < 0 \). So, we only need to prove that \( Q_m < 0 \).

By applying the lemma 2.3, we get

\[ \lambda_{\text{max}}(Q_m) \leq \delta + \gamma_1 \lambda_{\text{max}}((\tilde{C}(1) + (\tilde{C}(1))^T)_{m}. \tag{27} \]

From the condition (26), it is not difficult to see that \( \delta + \gamma_1 (\tilde{C}(1) + (\tilde{C}(1))^T)_{m} < 0 \). Then, in view of (27), we have \( \lambda_{\text{max}}(Q_m) < 0 \). Therefore, along with (23), the condition (17) is satisfied. This completes the proof.

**Remark 5.** As similar to the proof in corollary 3.1, our lag synchronization criterion of Theorem 3.1 is also easily verified and does not solve any linear matrix inequality. And the corresponding results are verified through a simulation experiment. However, from the magnified inequalities (24) and (27), we can see that the results of corollary 3.1 are more conservative than Theorem 3.2.

**Remark 6.** Compared with Ref.[36], about the design of the pinning controller \( u_i (i = 1, 2, \ldots, N) \), considered topological structure of the outer coupling matrix and the cost of the dynamical systems, pinning only one node is the best choice. Without loss of generality, we can choose the first node as the pinning node, and the corresponding results can be verified in the numerical simulation.

**Remark 7.** Different from some existing papers [31-36], the proposed conditions in this paper depend on the time-varying delays. Moreover, in [11], the time-varying delay meets \( \sigma(t) = d(t) \), which is a strong condition, and most of the situations do not have this property. Thus the results in this paper have less conservativeness and expand the results in the existing literatures.
4. Illustrative example

In this section, two numerical examples are given to illustrate the effectiveness of our results. Firstly, we consider the following time-delayed Chua’s circuit [43], which is described by

\[
\begin{align*}
\dot{x}_1(t) &= -\beta(1 + b)x_1(t) + \beta x_2(t) + \varphi(x_1(t)) \\
\dot{x}_2(t) &= x_1(t) - x_2(t) + x_3(t) \\
\dot{x}_3(t) &= -\rho x_2(t) - \omega x_3(t) - \rho \omega_0 \sin(v x_1(t - \sigma(t)))
\end{align*}
\]

(28)

where \(\varphi(x_1(t)) = -\frac{\beta}{2}(a-b)(|x_1(t)+1|-|x_1(t)-1|)\). When the parameters \(\beta = 10, b = -0.7831, a = -1.4325, \rho = 19.53, \omega = 0.1636, \omega_0 = 0.2, v = 0.5,\) and \(\sigma(t) = 0.02 + 0.01 \sin(10t)\). One can verify that the nonlinear function \(f\) satisfies (Assumption 2.2) with \(\alpha_1 = 12.5008\) and \(\alpha_2 = 0.2441\), and Fig.1 shows the time-delayed chua circuit is chaotic.

\[\text{Fig.1 The state trajectories of the Chua’s system.}\]

**Example 1:** Based on the above Chua’s circuit (28), we consider the drive-response systems (1) and (4) consisting of \(N = 4\) identical time-delayed chua systems with time-varying delays coupling to verify the correctness of Theorem 3.1. Now we choose the following coupling matrices:

\[
\bar{C}^{(1)} = \begin{pmatrix}
-13 & 4 & 3 & 6 \\
6 & -14 & 5 & 3 \\
4 & 6 & -15 & 5 \\
8 & 2 & 6 & -16
\end{pmatrix},
\quad
\bar{C}^{(2)} = \begin{pmatrix}
0.2 & 0.1 & -0.3 & 0 \\
0.3 & -0.1 & 0.5 & -0.7 \\
-0.5 & 0.1 & 0.5 & 0.1 \\
0.3 & -0.3 & -0.1 & 0.1
\end{pmatrix},
\quad
\bar{C}^{(3)} = \begin{pmatrix}
0.1 & -0.2 & 0.6 & -0.5 \\
0.3 & -0.6 & 0.2 & 0.1 \\
0 & 0.5 & 0.2 & -0.7 \\
0.2 & 0.1 & 0 & -0.3
\end{pmatrix},
\quad
\Gamma_1 = \begin{pmatrix}
3.5 & 0 & -0.1 \\
0 & 3.5 & -0.5 \\
0.1 & 0.6 & 3.5
\end{pmatrix},
\]

13
\[
\Gamma_2 = \begin{pmatrix}
-0.3 & -0.1 & 0.1 \\
0.1 & 0.2 & 0 \\
0 & -0.1 & 0.4
\end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix}
0.1 & -0.3 & 0.1 \\
-0.1 & 0.1 & -0.2 \\
0 & -0.1 & 0.3
\end{pmatrix}.
\]

For \( \tau = 0.5 \) and \( d(t) = 0.3 + 0.5\cos(5t) \), choose the initial conditions \( x_i(0) = (-1.8 + 0.5i, -0.9 + 0.5i, -4.7 + 0.5i)^T \) and \( y_i(0) = (1.8 + 0.5i, 0.9 + 0.5i, 4.7 + 0.5i)^T \), \( 1 \leq i \leq 4 \). Fig.2 shows the curves of error dynamics between the drive-response networks without controllers. It is clear that the complex dynamical networks cannot reach to synchronization.

![Fig.2 The error-state trajectory without controllers.](image)

However, by applying the linear feedback pinning control, we assume \( m = 1 \), i.e., the number of nodes to be controlled is 1. By simple computation, we obtain that \( \delta = 16.2876 \), \( -\frac{\delta}{\gamma_1} = -4.4549 \), \( \lambda_{\max}(\frac{C^{(1)} + (C^{(1)})^T}{2})_m = -5.8223 \), and \( \lambda_{\max}(Q_{11} - Q_{12}Q_mQ_{12}^T)/\gamma_1 = 16.4058 \). Then choose the appropriate feedback gain \( k_1 = 20 \). The corresponding simulation can be seen in Fig.3, which shows the drive system (1) and response system (4) can reach to synchronization by using the above controllers. Moreover, Figs.4-6 show that response network (4) approximately follows drive network (1) with transmit delay \( \tau = 0.5 \).
Fig. 3 The error-state trajectory by the linear pinning control.

Fig. 4 The state trajectories of $x_{i1}(t)$ and $y_{i1}(t)$ ($i = 1, 2, \ldots, 4$) under the linear pinning control.

Fig. 5 The state trajectory of $x_{i2}(t)$ and $y_{i2}(t)$ ($i = 1, 2, \ldots, 4$) under the linear pinning control.
From the above example we can see that the strength of linear feedback may be maximal, which is a kind of waste in practice to some extent. Compared with linear control, the control gains of adaptive control increase according to the adaptive laws. In the following, a simulation example of adaptive control will be given.

**Example 2**: Based on the above Chua’s circuit (28), we consider the drive-response systems (1) and (4) consisting of \( N = 5 \) identical time-delayed chua systems with time-varying delays coupling to verify the correctness of Theorem 3.2. Choose the following coupling matrices:

\[
\bar{C}(1) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
5 & -14 & 4 & 2 & 3 \\
3 & 5 & -15 & 4 & 3 \\
6 & 2 & 3 & -16 & 5 \\
4 & 3 & 3 & 7 & -17
\end{pmatrix}, \quad \bar{C}(2) = \begin{pmatrix}
-0.2 & 0.5 & 0 & 0.1 & -0.3 \\
0.3 & -0.2 & 0.5 & -0.4 & -0.2 \\
-0.6 & 0.1 & 0.5 & 0 & 0 \\
0 & -0.4 & -0.1 & 0.3 & 0.2 \\
0.3 & 0 & -0.1 & 0.1 & -0.3
\end{pmatrix},
\]

\[
\bar{C}(3) = \begin{pmatrix}
0.1 & -0.2 & 0.6 & -0.5 & 0 \\
0.3 & -0.6 & 0.2 & 0.3 & -0.2 \\
0 & 0.5 & 0.1 & -0.7 & 0.1 \\
0.2 & 0.1 & -0.1 & -0.7 & 0.5 \\
-0.3 & 0.2 & 0 & 0.4 & -0.1
\end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix}
4.5 & 0 & 0 \\
0 & 4.5 & -0.5 \\
0 & 0.6 & 4.5
\end{pmatrix},
\]

\[
\Gamma_2 = \begin{pmatrix}
-0.2 & -0.1 & 0 \\
0.1 & 0.3 & 0 \\
0 & 0 & 0.4
\end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix}
0.2 & -0.4 & 0.1 \\
-0.1 & 0.1 & -0.2 \\
0 & 0.1 & 0.3
\end{pmatrix}.
\]

For \( \tau = 0.5 \) and \( d(t) = \frac{e^{t}}{e^{t-1}} \), choose the initial conditions \( x_i(0) = (-1.8 + 0.5i, -0.9 + 0.5i, -4.7 + 0.5i)^T \) and \( y_i(0) = (1.8 + 0.5i, 0.9 + 0.5i, 4.7 + 0.5i)^T, \ 1 \leq i \leq 5 \). Fig.7 shows the curves of error dynamics between
the drive-response networks without controllers. It is clear that the complex dynamical networks cannot reach to synchronization.

Next, by applying the adaptive feedback pinning control, we assume \( m = 1 \). By simple computation, we obtain that \( \delta = 14.5236 \), \( \frac{A}{\gamma_1} = -3.2393 \), \( \lambda_{max}(\frac{C^{(1)} + (C^{(1)})^T_m}{2}) = -4.4912 \). We can see that the condition (26) of Corollary 3.1 is tenable. Therefore, from Corollary 3.1, the lag synchronization between the drive system (1) and the response system (4) can be realized by the adaptive controllers (15).

In the numerical simulations, we apply the adaptive controllers (15) to pin the first node of the response system (4) and let \( \delta_1 = 0.1 \). The corresponding simulation can be seen in Fig.8, which shows the drive system (1) and response system (4) can reach to synchronization by using the adaptive controllers, and the state trajectories of drive system and response system are described in Figs.9-11. One can find response system approximately retards drive network by transmit delay \( \tau = 0.5 \).
Fig. 9 The state trajectory of $x_{i1}(t)$ and $y_{i2}(t)$ ($i = 1, 2, \ldots, 5$) under the adaptive pinning control.

Fig. 10 The state trajectory of $x_{i2}(t)$ and $y_{i2}(t)$ ($i = 1, 2, \ldots, 5$) under the adaptive pinning control.

Fig. 11 The state trajectory of $x_{i3}(t)$ and $y_{i3}(t)$ ($i = 1, 2, \ldots, 5$) under the adaptive pinning control.
5. Conclusion

In this paper, lag synchronization between drive and response systems with mixed coupling has been investigated. By applying the Lyaponov functional theory and mathematical analysis method, sufficient verifiable conditions of lag synchronization are obtained by adding controllers to a part of nodes. Here, the coupling configuration matrices are not required to be symmetric or irreducible and the inner connecting matrices are arbitrary real matrices. It is shown that the lag synchronization of the drive and response systems can be realized via the linear feedback pinning control and adaptive feedback pinning control. These results remove some restrictions on the node dynamics and the number of the pinned nodes. Numerical examples are presented to illustrate the effectiveness of the theoretical results.

In addition, parameter mismatch is inevitable in practical implementations in the case of the noises or other artificial factors, and it plays an important role in the quality of synchronization. In the future, we will study the lag synchronization of complex dynamical networks with parameter mismatch.

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7. References


