A GENERAL ITERATIVE PROCESS FOR COMMON SOLUTIONS OF QUASI VARIATIONAL INCLUSION AND FIXED POINT PROBLEMS

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Abstract. In this paper, quasi-variational inclusion and fixed point problems are investigated based on a general iterative process. Strong convergence theorems are established in the framework of Hilbert spaces.

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1. Introduction

Fixed point problems have been found with an explosive growth in theoretical advances, algorithmic development and applications across all the discipline of pure and applied sciences, see [1-7] and the references therein. Analysis of these problems requires a blend of techniques from nonsmooth analysis, convex analysis, functional analysis and numerical analysis. As a result of the interaction between different branches of mathematical and engineering sciences, we now have a variety of techniques to suggest and analyze various iterative algorithms for solving these problems and related convex optimization problems. Variational inclusions involving two operators are useful and important extension and generalizations of the variational inequalities with a wide range of applications in industry, economics, decision sciences, ecology, mathematical and engineering sciences, see [8-13] and the references therein. It is well known that the projection method and its variant forms including the Wiener-Hopf equations cannot be extended and modified for solving the variational inclusions, which motivate us to use

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new techniques and methods. Resolvent techniques recently have been investigated by many authors in the framework of Hilbert spaces, see [14-23] and the references therein. The given operator is decomposed into the sum of two monotone operators whose resolvent is easy to evaluate than the resolvent of the original sum operator. Such type of methods are called the operator splitting methods and have proved to be very effective for solving inclusion problems involving two operators.

In this paper, we study a general iterative process for common solutions of quasi-variational inclusion and fixed point problems. Strong convergence theorems are established in the framework of Hilbert spaces. The organization of this paper is as follows. In Section 2, we provide some necessary preliminaries. In Section 3, the strong convergence theorem is established in the framework of Hilbert spaces. Some sub-results and applications are provided are provided to support our main results.

2. Preliminaries

Throughout this paper, we always assume that $H$ is a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let $B$ be a mapping on $H$. Recall that the following definitions. $B$ is said to be monotone, if for each $x, y \in H$, we have $\langle Bx - By, x - y \rangle \geq 0$. $B$ is said to be $\mu$-strongly monotone, if each $x, y \in H$, we have

$$\langle Bx - By, x - y \rangle \geq \mu \| x - y \|^2,$$

$B$ is said to be $\mu$-inverse-strongly monotone, if there exists a constant $\mu > 0$ such that

$$\langle Bx - By, x - y \rangle \geq \mu \| Bx - By \|^2.$$

$B$ is said to be relaxed $\delta$-cocoercive, if there exists a constant $\delta > 0$ such that

$$\langle Bx - By, x - y \rangle \geq (-\delta) \| Bx - By \|^2.$$

$B$ is said to be relaxed $(\delta, r)$-cocoercive, if there exist two constants $\delta, r > 0$ such that

$$\langle Bx - By, x - y \rangle \geq (-\delta) \| Bx - By \|^2 + r \| x - y \|^2.$$
Recall that a set-valued mapping $M : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Mx$ and $g \in My$ imply $\langle x - y, f - g \rangle \geq 0$. The monotone mapping $M : H \to 2^H$ is maximal if the graph of $G(M)$ of $T$ is not properly contained in the graph of any other monotone mapping.

In this article, we consider the following so-called quasi-variational inclusion problem: find an $u \in H$ for a given element $f \in H$ such that

$$f \in Bu + Mu,$$

where $B : H \to H$ and $M : H \to 2^H$ are two nonlinear mappings, see, for example, [19] and the references therein. A special case of problem (2.1) is to find an element $u \in H$ such that

$$0 \in Bu + Mu.$$

In this paper, we use $VI(H, B, M)$ to denote the solution of the problem (2.2). A number of problems arising in structural analysis, mechanics and economic can be studied in the framework of this class of variational inclusions.

Next, we consider two special cases of the problem (2.2).

(1) If $M = \partial \phi : H \to 2^H$, where $\phi : H \to \mathbb{R} \cup \{+\infty\}$ is a proper convex lower semi-continuous function and $\partial \phi$ is the sub-differential of $\phi$, then problem (2.2) is equivalent to finding $u \in H$ such that

$$\langle Bu, v - u \rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in H,$$

which is said to be the mixed quasi-variational inequality.

(2) If $\phi$ is the indicator function of $C$, then problem (2.2) is equivalent to the classical variational inequality problem, denoted by $VI(C, B)$, is to find $u \in C$ such that

$$\langle Bu, v - u \rangle \geq 0,$$

for all $v \in C$.

Let $S$ be a nonlinear mapping on $H$. Recall that $S$ is said to be contractive with the coefficient $\alpha \in (0, 1)$ if

$$\|Sx - Sy\| \leq \alpha \|x - y\|, \quad \forall x, y \in H.$$
$S$ is said to be nonexpansive if

$$
\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in H.
$$

$S$ is said to be $k$-strictly pseudocontractive if there exists a constant $k \in [0, 1)$ such that

$$
\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|x - y - Sx + Sy\|^2, \quad \forall x, y \in H.
$$

The class of $k$-strictly pseudocontractive mappings was introduced by Browder and Petryshn [2] in 1967.

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space $H$:

$$
\min_{x \in F(S)} \left( \frac{1}{2} \langle Ax, x \rangle - h(x) \right),
$$

where $A$ is a linear bounded and strongly positive operator, $F(S)$ is the fixed point set of the nonexpansive mapping $S$ and $h$ is a potential function for $\gamma f$, i.e., $h'(x) = \gamma f(x)$ for $x \in H$.

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems. Recently, Marino and Xu [24] studied the following iterative scheme

$$
x_0 \in H, \quad x_{n+1} = (I - \alpha_n A)Sx_n + \alpha_n \gamma f(x_n), \quad n \geq 0.
$$

They proved $\{x_n\}$ generated by the above iterative scheme converges strongly to the unique solution of the variational inequality

$$
\langle x - x^*, (A - \gamma f)x^* \rangle \geq 0, \quad x \in F(S),
$$

which is the optimality condition for the minimization problem (2.4).

Recently, Zhang, Lee and Chan [15] considered problem (2.2). To be more precise, they proved the following theorem.

**Theorem ZLC.** Let $H$ be a real Hilbert space, $B : H \to H$ an $\alpha$-inverse-strongly monotone mapping, $M : H \to 2^H$ a maximal monotone mapping and $S : H \to H$ a nonexpansive mapping. Suppose that the set $F(S) \cap VI(H, B, M) \neq \emptyset$, where $VI(H, B, M)$ is the set of solutions of...
variational inclusion (2.2). Suppose \( x_0 = x \in H \) and \( \{x_n\} \) is the sequence defined by

\[
x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) S y_n, \\
y_n = J_{M, \lambda}(x_n - \lambda B x_n), \quad n \geq 0,
\]

where \( \lambda \in (0, 2\alpha) \) and \( \{\alpha_n\} \) is a sequence in \([0, 1]\) satisfying the following conditions:

(a) \( \lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty; \)

(b) \( \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty. \)

Then \( \{x_n\} \) converges strongly to \( P_{F(S) \cap \text{VI}(H,B,M)} x_0. \)

To prove our main results, we also need the following lemmas.

**Lemma 2.1.** [15] Let \( M : H \to 2^H \) be a multi-valued maximal monotone mapping, then the single-valued mapping \( J_{M, \lambda} : H \to H \) defined by \( J_{M, \lambda}(u) = (I + \lambda M)^{-1}(u), \forall u \in H \) is called the resolvent operator associated with \( M \), where \( \lambda \) is any positive number and \( I \) is the identity mapping. The resolvent operator \( J_{M, \lambda} \) associated with \( M \) is single-valued and nonexpansive for all \( \lambda > 0. \) \( u \in H \) is a solution of variational inclusion (2.2) if and only if \( u = J_{M, \lambda}(u - \lambda B u, \) \( \forall \lambda > 0, \) i.e.,

\[
\text{VI}(H,B,M) = F(J_{M, \lambda}(I - \lambda B)), \quad \forall \lambda > 0.
\]

**Lemma 2.2.** [25] Let \( \{x_n\} \) and \( \{y_n\} \) be bounded sequences in a Banach space \( X \) and let \( \beta_n \) be a sequence in \([0, 1]\) with

\[
0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1.
\]

Suppose that \( x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n \) for all \( n \geq 0 \) and \( \limsup_{n \to \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0. \) Then \( \lim_{n \to \infty} (\|y_n - x_n\| = 0. \)

**Lemma 2.3.** [26] Assume that \( \{\alpha_n\} \) is a sequence of nonnegative real numbers such that \( \alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n, \) where \( \gamma_n \) is a sequence in \((0, 1)\) and \( \{\delta_n\} \) is a sequence such that

(a) \( \sum_{n=1}^{\infty} \gamma_n = \infty; \)

(b) \( \limsup_{n \to \infty} \delta_n / \gamma_n \leq 0 \) or \( \sum_{n=1}^{\infty} |\delta_n| < \infty. \)

Then \( \lim_{n \to \infty} \alpha_n = 0. \)

**Lemma 2.4.** [2] Let \( H \) be a real Hilbert space, \( C \) be a nonempty closed convex subset of \( H \) and \( S : C \to C \) be a strictly pseudocontractive mapping. Then \( I - S \) is demi-closed at zero. Let \( T : C \to C \) be a \( k \)-strict pseudo-contraction. Define \( S : C \to H \) by \( Sx = \alpha x + (1 - \alpha)Tx \) for each \( x \in C. \) Then, as \( \alpha \in [k, 1), \) \( S \) is nonexpansive such that \( F(S) = F(T). \)
Lemma 2.5. [27] Let $C$ be a closed convex subset of a strictly convex Banach space $E$. Let \{$T_i$\}_{i=1}^r, where $r$ denotes some positive integer, be a sequence of nonexpansive mappings on $C$. Suppose $\bigcap_{i=1}^r F(T_i)$ is nonempty. Let \{$\mu_i$\} be a sequence of positive numbers with $\sum_{i=1}^r \mu_i = 1$. Then a mapping $S$ on $C$ defined by $Sx = \sum_{i=1}^r \mu_i T_i x$ for $x \in C$ is well defined, nonexpansive and $F(S) = \bigcap_{i=1}^r F(T_i)$ holds.

3. Main results

Theorem 3.1. Let $H$ be a real Hilbert space and $M : H \to 2^H$ a maximal monotone mapping. Let $B : H \to H$ be a relaxed $(\delta, r)$-cocoercive and $\nu$-Lipschitz continuous mapping, and $T$ a $k$-strict pseudo-contraction on $H$. Let $f$ be a contraction of $H$ into itself with the contractive coefficient $\alpha$ ($0 < \alpha < 1$), $A$ a strongly positive linear bounded self-joint operator with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma} / \alpha$ and $\Omega = F(S) \cap VI(H, B, M) \neq \emptyset$. Let $x_1 \in H$ and \{$x_n$\} be a sequence generated by

$$
\begin{aligned}
&y_n = kTJ_{M, \lambda}(x_n - \lambda Bx_n) + (1 - k)J_{M, \lambda}(x_n - \lambda Bx_n), \\
x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + [(1 - \beta_n)I - \alpha_n A]y_n, \quad \forall n \geq 1,
\end{aligned}
$$

where \{$\alpha_n$\} and \{$\beta_n$\} are sequences in $(0, 1)$. Assume that $\lambda \in (0, \frac{2(r - \delta \nu^2)}{\nu^2})$, $r > \delta \nu^2$. If the control consequences \{$\alpha_n$\}, \{$\beta_n$\} satisfy the following restrictions: $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$, $0 < a \leq \beta_n \leq b < 1$, where $a$ and $b$ are two real numbers, then \{$x_n$\} converges strongly to $z \in \Omega$, which uniquely solve the following variational inequality

$$
\langle Az - \gamma f(z), z - w \rangle \leq 0, \forall w \in \Omega. \quad (3.1)
$$

Proof. Putting $S = kS + (1 - k)I$, we see that $S$ is nonexpansive with $F(S) = F(T)$. From the strong monotonicity of $A - \gamma f$, we get the uniqueness of the solution of the variational inequality (3.1). Suppose $z_1 \in \Omega$ and $z_2 \in \Omega$ both are solutions to (3.1). It follows that

$$
\langle Az_2 - \gamma f(z_2), z_2 - z_1 \rangle \leq 0
$$

and

$$
\langle \gamma f(z_1) - Az_1, z_1 - z_2 \rangle \geq 0.
$$
Adding up the two inequalities, we see that
\[
\langle z_2 - z_1, (A - \gamma f)z_1 - (A - \gamma f)z_2 \rangle \geq 0.
\]
The strong monotonicity of \(A - \gamma f\) implies that \(z_1 = z_2\) and the uniqueness is proved. Below we use \(z\) to denote the unique solution of (3.1).

From the condition \(\lambda \in (0, \frac{2(r - \mu^2)}{\mu^2})\), one has
\[
\| (I - \lambda B)x - (I - \lambda B)y \|^2 \leq \| x - y \|^2 - 2\lambda \| Bx - By \|^2 + r\| x - y \|^2 + \lambda^2 \nu^2 \| x - y \|^2 
\leq \| x - y \|^2,
\]
which implies the mapping \(I - \lambda B\) is nonexpansive. Taking \(x^* \in \Omega\), we find from Lemma 2.1 that \(x^* = J_{M, \lambda} (x^* - \lambda Bx^*)\). It follows that
\[
\| y_n - x^* \| = \| J_{M, \lambda} (x_n - \lambda Bx_n) - J_{M, \lambda} (x^* - \lambda Bx^*) \| \leq \| x_n - x^* \|.
\]
Note that from the conditions, we may assume, without loss of generality, that \(\alpha_n \leq (1 - \beta_n) \| A \|^{-1} \) for all \(n \geq 1\). Since \(A\) is a strongly positive linear bounded self-adjoint operator, we have \(\| A \| = \sup \{ |\langle Ax, x \rangle| : x \in H, \| x \| = 1 \}\). Now for \(x \in C\) with \(\| x \| = 1\), we see that
\[
0 \leq 1 - \beta_n - \alpha_n \| A \| \\
\leq 1 - \beta_n - \alpha_n \langle Ax, x \rangle \\
= ((1 - \beta_n)I - \alpha_n A)x, x),
\]
that is, \((1 - \beta_n)I - \alpha_n A\) is positive. It follows that
\[
\| (1 - \beta_n)I - \alpha_n A \| = \sup \{ \langle ((1 - \beta_n)I - \alpha_n A)x, x \rangle : x \in C, \| x \| = 1 \}
\leq \sup \{ 1 - \beta_n - \alpha_n \langle Ax, x \rangle : x \in C, \| x \| = 1 \}
\leq 1 - \beta_n - \alpha_n \bar{\gamma}.
\]
Set \(t_n = \mu Sx_n + (1 - \mu) y_n\). It follows that
\[
\| t_n - x^* \| \leq \mu \| Sx_n - x^* \| + (1 - \mu) \| y_n - x^* \| \leq \| x_n - x^* \|.
\]
We have
\[
\|x_{n+1} - x^*\| \leq \alpha_n \|\gamma f(x_n) - Ax^*\| + \beta_n \|x_n - x^*\| + \| (1 - \beta_n) I - \alpha_n A \| \|t_n - x^*\|
\]
\[
\leq \alpha_n \|\gamma f(x_n) - Ax^*\| + \beta_n \|x_n - x^*\| + (1 - \beta_n - \alpha_n \bar{\gamma}) \|t_n - x^*\|
\]
\[
\leq \alpha_n \|\gamma f(x_n) - \gamma f(x^*)\| + \alpha_n \|\gamma f(x^*) - Ax^*\| + \beta_n \|x_n - x^*\|
\]
\[
+ (1 - \beta_n - \alpha_n \bar{\gamma}) \|x_n - x^*\|
\]
\[
\leq \alpha \alpha_n \gamma \|x_n - x^*\| + \alpha_n \|\gamma f(x^*) - Ax^*\| + \beta_n \|x_n - x^*\|
\]
\[
+ (1 - \beta_n - \alpha_n \bar{\gamma}) \|x_n - x^*\|
\]
\[= [1 - \alpha_n (\bar{\gamma} - \alpha \gamma)] \|x_n - x^*\| + \alpha_n \|\gamma f(x^*) - Ax^*\|.
\]
This implies that
\[
\|x_n - x^*\| \leq \max \{\|x_1 - x^*\|, \|\gamma f(x^*) - Ax^*\| \},
\]
which gives that sequence \( \{x_n\} \) is bounded, so are \( \{y_n\} \) and \( \{t_n\} \). So
\[
\|y_{n+1} - y_n\| = \|J_{M, \lambda} (x_{n+1} - \lambda B x_{n+1}) - J_{M, \lambda} (x_n - \lambda B x_n)\| \leq \|x_{n+1} - x_n\|.
\]
It follows that
\[
\|t_{n+1} - t_n\| = \|\mu S x_{n+1} + (1 - \mu) y_{n+1} - [\mu S_k x_n + (1 - \mu) y_n]\|
\]
\[
\leq \mu \|S x_{n+1} - S_k x_n\| + (1 - \mu) \|y_{n+1} - y_n\|
\]
\[
\leq \|x_{n+1} - x_n\|.
\]
Setting \( x_{n+1} = \beta_n x_n + (1 - \beta_n) d_n \), we have
\[
d_n - d_{n+1}
\]
\[
= \frac{\alpha_n \gamma f(x_n) + [(1 - \beta_n) I - \alpha_n A] t_n}{1 - \beta_n} - \frac{\alpha_{n+1} \gamma f(x_{n+1}) + [(1 - \beta_{n+1}) I - \alpha_{n+1} A] t_n}{1 - \beta_{n+1}}
\]
\[
= \frac{\alpha_n}{1 - \beta_n} [\gamma f(x_n) - A t_n] + t_n - \frac{\alpha_{n+1}}{1 - \beta_{n+1}} [\gamma f(x_{n+1}) - A t_n] - t_{n+1}.
\]
It follows that
\[
\|d_{n+1} - d_n\| \leq \frac{\alpha_n}{1 - \beta_n} \|\gamma f(x_n) - A t_n\| + \|t_{n+1} - t_n\| + \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \|\gamma f(x_{n+1}) - A t_n\|,
\]
which yields that
\[
\|d_{n+1} - d_n\| - \|x_{n+1} - x_n\| \leq \frac{\alpha_n}{1 - \beta_n} \|\gamma f(x_n) - A t_n\| + \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \|\gamma f(x_{n+1}) - A t_n\|.
\]
It follows from the conditions that

\[
\limsup_{n \to \infty} (\|d_n - d_{n+1}\| - \|x_n - x_{n+1}\|) \leq 0.
\]

Hence, \(\lim_{n \to \infty} \|x_n - d_n\| = 0\). One has \((1 - \beta_n)\|d_n - x_n\| = \|x_{n+1} - x_n\|\). One has \(\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0\). Since

\[
\alpha_n (\gamma f(x_n) - A t_n) + (1 - \beta_n) (t_n - x_n) = x_{n+1} - x_n,
\]

we further have

\[
(1 - \beta_n) \|t_n - x_n\| \leq \alpha_n \|\gamma f(x_n) - A t_n\| + \|x_{n+1} - x_n\|.
\]

From the conditions, we see

\[
\lim_{n \to \infty} \|x_n - t_n\| = 0.
\]

Now, we are in a position to prove that

\[
\limsup_{n \to \infty} \langle x_n - z, (\gamma f - A)z \rangle \leq 0,
\]

where \(z = P_{\Omega} [I - (A - \gamma f)] z\). To see this, we choose a subsequence \(\{x_{n_i}\}\) of \(\{x_n\}\) such that

\[
\limsup_{n \to \infty} \langle x_n - z, (\gamma f - A)z \rangle = \lim_{i \to \infty} \langle x_{n_i} - z, (\gamma f - A)z \rangle.
\]

Since \(\{x_{n_i}\}\) is bounded, there exists a subsequence \(\{x_{n_{i_j}}\}\) of \(\{x_{n_i}\}\) which converges weakly to \(w\). Without loss of generality, we can assume that \(x_{n_i} \rightharpoonup w\). Next, we show that \(w \in F(S) \cap VI(H, M, B)\). Define a mapping \(D\) by

\[
Dx = \mu Sx + (1 - \mu) J_M \lambda (I - \lambda B), \quad \forall x \in H.
\]

We see that \(D\) is nonexpansive such that

\[
F(D) = F(S) \cap F(J_M \lambda (I - \lambda B)) = F(S) \cap VI(H, B, M).
\]

We obtain \(\lim_{n \to \infty} \|Dx_{n_i} - x_{n_i}\| = 0\). It follows that \(w \in F(D)\). That is,

\[
w \in F(S) \cap VI(H, M, B) = F(T) \cap VI(H, M, B).
\]

It follows that

\[
\limsup_{n \to \infty} \langle x_n - z, (\gamma f - A)z \rangle = \lim_{i \to \infty} \langle x_{n_i} - z, (\gamma f - A)z \rangle = \langle w - z, (\gamma f - A)z \rangle \leq 0.
\]
Finally, we show that \( x_n \to z \), as \( n \to \infty \). Indeed,

\[
\|x_{n+1} - z\|^2 = \langle x_{n+1} - z, \alpha_n \gamma f(x_n) + \beta_n x_n + [(1 - \beta_n)I - \alpha_n A]t_n - z \rangle \\
= \beta_n \langle x_{n+1} - z, x_n - z \rangle + \alpha_n \langle x_{n+1} - z, \gamma f(x_n) - Az \rangle \\
+ \langle x_{n+1} - z, [(1 - \beta_n)I - \alpha_n A](t_n - z) \rangle \\
\leq \alpha_n \langle x_{n+1} - z, \gamma f(z) - Az \rangle + \alpha_n \gamma \langle x_{n+1} - z, f(x_n) - f(z) \rangle \\
+ \beta_n \|x_n - z\| \|x_{n+1} - z\| + (1 - \beta_n - \alpha_n \gamma) \|x_n - z\| \|x_{n+1} - z\| \\
\leq \alpha_n \gamma \langle f(z) - Az, x_{n+1} - z \rangle + \gamma \alpha_n \|x_n - z\|^2 + \|x_{n+1} - z\|^2 \\
+ (1 - \alpha_n \gamma) \|x_n - z\| \|x_{n+1} - z\| \\
\leq \alpha_n \gamma \langle f(z) - Az, x_{n+1} - z \rangle + \gamma \alpha_n \|x_n - z\|^2 + \|x_{n+1} - z\|^2 \\
+ \left(1 - \frac{1 - \alpha_n \gamma}{2}\right) \|x_n - z\|^2 + \frac{1}{2} \|x_{n+1} - z\|^2 + \alpha_n \gamma \langle f(z) - Az, x_{n+1} - z \rangle,
\]

which implies that

\[
\|x_{n+1} - z\|^2 \leq 2 \alpha_n \langle f(z) - Az, x_{n+1} - z \rangle + [1 - \alpha_n(\gamma - \alpha \gamma)] \|x_n - z\|^2.
\]

Using Lemma 2.3, we find that \( \lim_{n \to \infty} \|x_n - z\| = 0 \). This completes the proof.

From Theorem 3.1, the following results are not hard to derive.

**Corollary 3.2.** Let \( H \) be a real Hilbert space and \( M : H \to 2^H \) a maximal monotone mapping. Let \( B : H \to H \) be a relaxed \( (\delta, r) \)-cocoercive and \( \nu \)-Lipschitz continuous mapping, and \( T \) a nonexpansive mapping on \( H \). Let \( f \) be a contraction of \( H \) into itself with the contractive coefficient \( \alpha \) (\( 0 < \alpha < 1 \)), \( A \) a strongly positive linear bounded self-joint operator with the coefficient \( \tilde{\gamma} > 0 \). Assume that \( 0 < \gamma < \tilde{\gamma}/\alpha \) and \( \Omega = F(S) \cap VI(H, B, M) \neq \emptyset \). Let \( x_1 \in H \) and \( \{x_n\} \) be a sequence generated by

\[
\begin{aligned}
\gamma_n &= TJ_{M, \lambda}(x_n - \lambda Bx_n), \\
x_{n+1} &= \alpha_n \gamma f(x_n) + \beta_n x_n + [(1 - \beta_n)I - \alpha_n A]y_n, \quad \forall n \geq 1,
\end{aligned}
\]
where \( \{\alpha_n\} \) and \( \{\beta_n\} \) are sequences in \((0,1)\). Assume that \( \lambda \in (0, \frac{2(r-\delta v^2)}{v^2}) \), \( r > \delta v^2 \). If the control consequences \( \{\alpha_n\}, \{\beta_n\} \) satisfy the following restrictions: \( \lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, 0 < a \leq \beta_n \leq b < 1 \), where \( a \) and \( b \) are two real numbers, then \( \{x_n\} \) converges strongly to \( z \in \Omega \), which uniquely solve the following variational inequality \( \langle Az - \gamma f(z), z - w \rangle \leq 0, \forall w \in \Omega \).

**Corollary 3.3.** Let \( H \) be a real Hilbert space and \( M : H \to 2^H \) a maximal monotone mapping. Let \( B : H \to H \) be a relaxed \((\delta, r)\)-cocoercive and \( v \)-Lipschitz continuous mapping. Let \( f \) be a contraction of \( H \) into itself with the contractive coefficient \( \alpha \) \((0 < \alpha < 1)\), \( A \) a strongly positive linear bounded self-joint operator with the coefficient \( \tilde{\gamma} > 0 \). Assume that \( 0 < \gamma < \tilde{\gamma}/\alpha \) and \( VI(H, B, M) \neq \emptyset \). Let \( x_1 \in H \) and \( \{x_n\} \) be a sequence generated by

\[
x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + [(1 - \beta_n)I - \alpha_n A]J_{M, \lambda}(x_n - \lambda Bx_n), \quad \forall n \geq 1,
\]

where \( \{\alpha_n\} \) and \( \{\beta_n\} \) are sequences in \((0,1)\). Assume that \( \lambda \in (0, \frac{2(r-\delta v^2)}{v^2}) \), \( r > \delta v^2 \). If the control consequences \( \{\alpha_n\}, \{\beta_n\} \) satisfy the following restrictions: \( \lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, 0 < a \leq \beta_n \leq b < 1 \), where \( a \) and \( b \) are two real numbers, then \( \{x_n\} \) converges strongly to \( z \in VI(H, B, M) \), which uniquely solve the following variational inequality \( \langle Az - \gamma f(z), z - w \rangle \leq 0, \forall w \in VI(H, B, M) \).

Let \( C \) be a nonempty closed and convex subset of \( H \) and \( A : C \to H \) be a mapping. Recall that the classical variational inequality is to find an \( x \in C \) such that

\[
\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C.
\]

(3.2)

The solution set of (3.2) is denoted by \( VI(C,A) \). It is known that \( x \) is a solution to (3.2) iff \( x \) is a fixed point of the mapping \( P_C(I - rA) \), where \( I \) denotes the identity on \( H \). If \( A \) is inverse-strongly monotone, then \( P_C(I - rA) \) is nonexpansive. Moreover, if \( C \) is bounded, closed and convex, then the existence of solutions of the variational inequality is guaranteed by the nonexpansivity of the mapping \( P_C(I - rA) \). Let \( i_C \) be a function defined by \( i_C(x) = 0, x \in C, i_C(x) = \infty, x \notin C \). It is easy to see that \( i_C \) is a proper lower and semicontinuous convex function on \( H \), and the subdifferential \( \partial i_C \) of \( i_C \) is maximal monotone. Define the resolvent \( J^i_C := (I + r \partial i_C)^{-1} \) of the
subdifferential operator $\partial i_C$. Letting $x = J^i_C y$, we find that

$$y \in x + r \partial i_C x \iff y \in x + r N_C x \iff x = P_C y,$$

where $N_C x := \{e \in H : \langle e, v - x \rangle, \forall v \in C\}$. Putting $M = \partial i_C$ in Theorems 3.1, we find the following results immediately.

**Corollary 3.4.** Let $C$ be a nonempty closed and convex subset of a real Hilbert space $H$. Let $B : C \to H$ be a relaxed $(\delta, r)$-cocoercive and $\nu$-Lipschitz continuous mapping, and $T$ a $k$-strict pseudo-contraction on $C$. Let $f$ be a contraction of $C$ into itself with the contractive coefficient $\alpha$ ($0 < \alpha < 1$), $A$ a strongly positive linear bounded self-joint operator with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$ and $\Omega = F(S) \cap VI(C, B) \neq \emptyset$. Let $x_1 \in C$ and $\{x_n\}$ be a sequence generated by

$$\begin{align*}
y_n &= kTP_C(x_n - \lambda Bx_n) + (1 - k)P_C(x_n - \lambda Bx_n), \\
x_{n+1} &= \alpha_n \gamma f(x_n) + \beta_n x_n + [(1 - \beta_n)I - \alpha_n A]y_n, \quad \forall n \geq 1,
\end{align*}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. Assume that $\lambda \in (0, \frac{2(r-\delta \nu^2)}{\nu^2})$, $r > \delta \nu^2$. If the control consequences $\{\alpha_n\}$, $\{\beta_n\}$ satisfy the following restrictions; $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < a \leq \beta_n \leq b < 1$, where $a$ and $b$ are two real numbers, then $\{x_n\}$ converges strongly to $z \in \Omega$, which uniquely solve the following variational inequality $\langle Az - \gamma f(z), z - w \rangle \leq 0, \forall w \in \Omega$.

**References**


