A NOVEL APPROACH TO BANACH CONTRACTION PRINCIPLE IN EXTENDED QUASI-METRIC SPACES

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Abstract. The purpose of this note is to give a natural approach to the extensions of the Banach Contraction Principle in metric spaces endowed with a partial order, a directed graph or a binary relation in terms of extended quasi-metric. This novel approach is new and may open the door to other new fixed point theorems. The case of multivalued mappings is also discussed and an analogue result to Nadler’s fixed point theorem in extended quasi-metric spaces is given.

1. Introduction

After the publication of the extension of the Banach Contraction Principle [3] to partially ordered metric spaces by Ran and Reurings [16], many mathematicians got interested into this new area of metric fixed point theory. Though this extension was known to Turinici [19, 20], what made the fixed point theorem of [16] interesting is the fact that its motivations come from concrete applications. Right after the publication of [16], Nieto and Rodríguez-López [15] improved the assumptions of the main result of [16] and used their new extension to solve some differential equations. In [11], Jachymski proposed directed graphs as a natural extension to partial orders. It is also worth mentioning the work of Ben-El-Mechaiekh [5] who gave a formulation of Ran and Reurings’ fixed point theorem when the metric space is endowed with a binary relation. Before we close these historical facts, let us point out that in fact the first attempt to generalize the Banach Contraction Principle to partially ordered metric spaces was carried by Turinici in [19, 20]. In this paper, we propose a novel approach to Banach Contraction Principle by considering its extension to the so-called extended quasi-metric spaces which allows the quasi-distance to take the value infinity [6].

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In terms of content, this paper overlaps in places with the following popular books on fixed point theory by Goebel and Kirk [10], by Khamsi and Kirk [12].

2. Basic results

First, we define the concept of an extended quasi-metric space.

**Definition 2.1.** [6] Let $X$ be an abstract set. The function $\bar{d} : X \times X \to [0, \infty]$ is called an extended quasi-distance if the following conditions are satisfied:

(i) $\bar{d}(x, y) = 0 \iff x = y$;

(ii) $\bar{d}(x, y) \leq \bar{d}(x, z) + \bar{d}(z, y)$, for all $x, y, z \in X$ (oriented triangle inequality).

In this case, the pair $(X, \bar{d})$ is called an extended quasi-metric space.

The reader may find an extensive list of examples on quasi-metric spaces in [6]. One of the problems dealing with quasi-metric spaces is the issue of a topology and more specifically the concepts of convergence, Cauchy and completeness [7, 17, 18].

**Definition 2.2.** Let $(X, \bar{d})$ be an extended quasi-metric space. A sequence $\{x_n\}$ in $X$ is said to be convergent if there exists $x \in X$ such that $\lim_{n \to +\infty} \bar{d}(x_n, x) = 0$. $X$ is said to be a $T_2$-space if and only if the limit of a sequence is unique, i.e., whenever $\lim_{n \to +\infty} \bar{d}(x_n, x) = \lim_{n \to +\infty} \bar{d}(x_n, y) = 0$ implies $x = y$. We will say that a subset $Y$ of $X$ is closed in $(X, \bar{d})$ if $Y$ contains the limit of any convergent sequence from $Y$.

Throughout this paper, we will assume that any extended quasi-metric space $(X, \bar{d})$ is a $T_2$-space. The concept of Cauchy sequences is little more complicated. We refer the interested reader to the paper [7] for more details.

**Definition 2.3.** [7] Let $(X, \bar{d})$ be an extended quasi-metric space. A sequence $\{x_n\}$ in $X$ is said to be Cauchy if and only if there exists a sequence $\{y_m\}$ such that for any $\varepsilon > 0$, there exists $N \geq 1$ such that for any $n, m > N$ we have

$$\bar{d}(x_n, y_m) < \varepsilon.$$
i.e., \( \lim_{n,m \to +\infty} \bar{d}(x_n, y_m) = 0 \). The sequence \( \{y_m\} \) is called a cosequence to \( \{x_n\} \).

The extended quasi-metric space \((X, \bar{d})\) is said to be complete if and only if any Cauchy sequence in \((X, \bar{d})\) is convergent.

Although this definition is kind of complicated, it allows for the following to be true:

**Proposition 2.1.** [7]

(i) Every convergent sequence in \((X, \bar{d})\) is a Cauchy sequence.

(ii) Every subsequence of a Cauchy sequence in a Cauchy sequence.

(iii) If \((X, \bar{d})\) is an extended metric space, then Definition 2.3 is equivalent to the usual definition of Cauchy sequence.

According to [7], the Sorgenfrey line is a complete quasi-metric space, the so called Kofner plane and Pixley-Roy space, considered as quasi-metric spaces, are also complete.

3. **Banach Contraction Principle in extended quasi-metric spaces**

In order to discuss the Banach Contraction Principle in extended quasi-metric spaces, we will need to introduce the concept of Lipschitzian mappings in these spaces.

**Definition 3.1.** Let \((X, \bar{d})\) be an extended quasi-metric space. A mapping \(T : X \to X\) is said to be Lipschitzian if there exists \(k > 0\) such that

\[
\bar{d}(T(x), T(y)) \leq k \bar{d}(x, y),
\]

for all \(x, y \in X\). If \(k < 1\), then \(T\) is said to be a contraction mapping. A point \(x \in X\) is called a fixed point of \(T\) whenever \(T(x) = x\).

**Remark 3.1.** Let \((X, \bar{d})\) be an extended quasi-metric space and \(T : X \to X\) a \(\bar{d}\)-contraction mapping. Then for any fixed points \(x\) and \(y\) of \(T\), we have \(x = y\) whenever \(\bar{d}(x, y) < \infty\).

The following technical lemma will be useful when studying the Banach contraction principle in extended metric spaces.
Theorem 3.1. Let \((X, \bar{d})\) be a complete extended quasi-metric space. Let \(T : X \to X\) be a contraction. Set \(X_T = \{ x \in X ; \, \bar{d}(x, T(x)) < \infty \}\). For any \(x_0 \in X_T\), the orbit \(\{T^n(x_0)\}\) is Cauchy. Moreover if \(\{T^n(x_0)\}\) converges to \(x \in X\), then \(T(x) = x\), i.e., \(x\) is a fixed point of \(T\).

Proof. Since \(T\) is a contraction mapping, there exists \(k \in (0, 1)\) such that
\[
\bar{d}(T(x), T(y)) \leq k \bar{d}(x, y) \quad \text{for any } x, y \in X.
\]
Hence for any \(x, y \in X\) and \(n \in \mathbb{N}\), we have
\[
\bar{d}(T^n(x), T^n(y)) \leq k^n \bar{d}(x, y).
\]
Let \(x_0 \in X_T\), i.e., \(\bar{d}(x_0, T(x_0)) < \infty\). Then
\[
\bar{d}(T^n(x_0), T^{n+1}(x_0)) \leq k^n \bar{d}(x_0, T(x_0)),
\]
for any \(n \in \mathbb{N}\). Hence for any \(n, h \in \mathbb{N}\), we have
\[
\bar{d}(T^n(x_0), T^{n+h+1}(x_0)) \leq k^n \bar{d}(x_0, T^{h+1}(x_0)) \leq k^n \sum_{i=0}^{i=h-1} \bar{d}(T^i(x_0), T^{i+1}(x_0)),
\]
which implies
\[
\bar{d}(T^n(x_0), T^{n+h+1}(x_0)) \leq \frac{k^n}{1-k} \bar{d}(x_0, T(x_0)).
\]
Hence \(\{T^n(x_0)\}\) is Cauchy, since \(k < 1\) and \(\bar{d}(x_0, T(x_0)) < \infty\), with \(\{T^n(x_0)\}\) as its cosequence. Since \((X, \bar{d})\) is a complete extended quasi-metric space, \(\{T^n(x_0)\}\) converges to \(x \in X\). We have
\[
\bar{d}(T^{n+1}(x_0), T(x)) \leq k \bar{d}(T^n(x_0), x), \quad \text{for any } n \in \mathbb{N}.
\]
Hence \(\{T^{n+1}(x_0)\}\) converges to \(T(x)\) and \(x\). Since \((X, \bar{d})\) is a \(T_2\)-space, we conclude that \(T(x) = x\). \qed

In the next example, we discuss how a partially ordered metric space may be endowed with an extended quasi-metric structure. Though this example is given for a metric space endowed with a partial order but the same ideas may be used for metric spaces endowed with a directed graph or a binary relation. In fact, this example will shed some light for a better understanding of the fixed point
Example 3.1. Let \((X, d, \preceq)\) be a partially ordered metric space. Define \(\bar{d} : X \times X \to [0, +\infty)\) by

\[
\bar{d}(x, y) = \begin{cases} 
  d(x, y), & \text{if } x \preceq y; \\
  +\infty, & \text{otherwise}.
\end{cases}
\]

It is easy to check that \((X, \bar{d})\) is an extended quasi-metric space. Next, we discuss the convergence of sequences in \((X, \bar{d})\) and how it connects to \((X, d)\). First, it is clear that if \(\{x_n\}\) converges to \(x\) in \((X, \bar{d})\), then it also converges to \(x\) in \((X, d)\). This will imply that \((X, \bar{d})\) is a \(T_2\)-space. Next, we investigate Cauchy sequences in \((X, \bar{d})\). Let \(\{x_n\}\) be a Cauchy sequence in \((X, \bar{d})\). Then there exists a cosequence \(\{y_m\}\) in \(X\) such that \(\lim_{n, m \to \infty} \bar{d}(x_n, y_m) = 0\). There exists \(N_0 \geq 1\) such that for any \(n, m \geq N_0\) we have \(\bar{d}(x_n, y_m) < +\infty\). In particular, we have \(x_n \preceq y_m\), for any \(n, m \geq N_0\). Moreover, the condition \(\lim_{n, m \to \infty} \bar{d}(x_n, y_m) = 0\) implies that both sequences \(\{x_n\}\) and \(\{y_m\}\) are Cauchy in \((X, d)\). Therefore, if \((X, d)\) is complete, then \(\{x_n\}\) and \(\{y_m\}\) are convergent to the same limit \(x \in X\). In order to show that \(\{x_n\}\) converges to \(x\) in \((X, \bar{d})\), we will need to assume that order intervals are closed. Recall that order intervals are any subset \((\preceq, x] = \{y \in X; y \preceq x\}\) or \([x, \rightarrow) = \{y \in X; x \preceq y\}\), for any \(x \in X\). In this case, we will have

\(x_n \preceq x \preceq y_m\),

for any \(n, m \geq N_0\). Hence we have \(\bar{d}(x_n, x) = d(x_n, x)\), for any \(n \geq N_0\). Clearly, we will have \(\lim_{n \to +\infty} \bar{d}(x_n, x) = 0\), i.e., \(\{x_n\}\) converges to \(x\) in \((X, \bar{d})\). Therefore, if \((X, d)\) is complete and the order intervals are closed, then \((X, \bar{d})\) is a complete quasi-metric space. Next, we investigate Lipschitzian mappings in \((X, \bar{d})\). Let \(T : X \to X\) be a Lipschitzian mapping. Then there exists \(k > 0\) such that

\[d(T(x), T(y)) \leq k \bar{d}(x, y),\]
for any \( x, y \in X \). Fix \( x, y \in X \) such that \( \bar{d}(x, y) < +\infty \). Then \( \bar{d}(T(x), T(y)) < +\infty \) holds. In other words, if \( x \preceq y \), then \( T(x) \preceq T(y) \). This is the definition of a monotone increasing mapping. Moreover, we have

\[
\bar{d}(T(x), T(y)) = d(T(x), T(y)) \leq k d(x, y) = k \bar{d}(x, y).
\]

Therefore if \( T \) is a Lipschitzian mapping in \((X, \bar{d})\), then \( T \) is a Lipschitzian monotone mapping in \((X, \preceq, d)\) with the same Lipchitz constant. The converse is also true. In particular, \( T \) is a monotone contraction mapping in \((X, \preceq, d)\) if and only if \( T \) is a contraction mapping in \((X, \bar{d})\). Putting all these results together, we get an analogue result to Theorem 3.1 in partially ordered metric spaces.

**Theorem 3.2.** Let \((X, d, \preceq)\) be a complete partially ordered metric space. Assume that order intervals are closed. Let \( T : X \to X \) be a contraction mapping. Set \( X_T = \{ x \in X ; x \preceq T(x) \} \). Then \( T \) has a fixed point provided \( X_T \neq \emptyset \).

In the next section, we discuss Nadler’s fixed point theorem for multivalued contractions in extended quasi-metric spaces.

4. **NADLER’S FIXED POINT THEOREM IN EXTENDED METRIC SPACES**

The multivalued version of the Banach contraction principle was given by Nadler [14].

**Theorem 4.1.** [14] Let \((M, d)\) be a complete metric space. Denote by \( CB(M) \) the set of all nonempty closed bounded subsets of \( M \). Let \( T : M \to CB(M) \) be a multivalued contraction mapping, i.e., there exists \( k \in [0, 1) \) such that

\[
H(T(x), T(y)) \leq k d(x, y),
\]

for all \( x, y \in M \), where \( H \) is the Pompeiu-Hausdorff metric on \( CB(M) \). Then \( T \) has a fixed point in \( M \).

Following the publication of Nadler’s fixed point theorem, many mathematicians gave a number of its extensions and generalizations; see for instance [9, 13] and references cited therein. In this section, we discuss an extension of Theorem 4.1 to extended metric spaces. In light of the Example 3.1, our extension gives a simpler and novel approach to the recent interest around monotone multivalued contraction mappings [1, 4].
Definition 4.1. Let \((X, \bar{d})\) be an extended quasi-metric space and \(C(X)\) be the class of all nonempty closed subsets of \((X, \bar{d})\). A multivalued map \(T : X \to C(X)\) is called a contraction mapping if there exists \(k \in (0, 1)\) such that for any \(x, y \in X\) and any \(u \in T(x)\), there exists \(v \in T(y)\) such that
\[
\bar{d}(u, v) \leq k \bar{d}(x, y).
\]
A point \(x \in X\) is called a fixed point of \(T\) if \(x \in T(x)\).

The following property will be needed to prove the existence of fixed points for multivalued contractive mappings in quasi-metric spaces.

Definition 4.2. Let \((X, \bar{d})\) be an extended quasi-metric space. We will say that \(X\) satisfies the property \((P_1)\) if for any \(\{x_n\}\) in \(X\) which converges to \(x\), and \(\{y_n\} \subseteq X\) such that \(\lim_{n \to \infty} \bar{d}(x_n, y_n) = 0\), we have \(\lim_{n \to \infty} \bar{d}(y_n, x) = 0\).

Now we are ready to give the multivalued version of Theorem 3.1.

Theorem 4.2. Let \((X, \bar{d})\) be a complete extended quasi-metric space. Assume that \(X\) satisfies the property \((P_1)\). Let \(T : X \to C(X)\) be a contraction mapping. Assume that \(X_T := \{x \in X; \bar{d}(x, u) < \infty \text{ for some } u \in T(x)\} \neq \emptyset\). Then \(T\) has a fixed point.

Proof. Let \(x_0 \in X_T\). Then there exists \(x_1 \in T(x_0)\) such that \(\bar{d}(x_0, x_1) < \infty\). Since \(T\) is a contraction mapping, there exists \(k \in (0, 1)\) such that for any \(x, y \in X\) and any \(u \in T(x)\), there exists \(v \in T(y)\) such that
\[
\bar{d}(u, v) \leq k \bar{d}(x, y).
\]
Hence there exists \(x_2 \in T(x_1)\) such that
\[
\bar{d}(x_1, x_2) \leq k \bar{d}(x_0, x_1).
\]
By induction, we construct a sequence \(\{x_n\}\) such that \(x_{n+1} \in T(x_n)\) and
\[
\bar{d}(x_n, x_{n+1}) \leq k \bar{d}(x_n, x_{n-1}) \leq k^n \bar{d}(x_0, x_1),
\]
for any \(n \in \mathbb{N}\). Hence for any \(n, h \in \mathbb{N}\), we have
\[
\bar{d}(x_n, x_{n+h}) \leq \frac{k^n}{1 - k} \bar{d}(x_0, x_1).
\]
Hence \( \{x_n\} \) is Cauchy, since \( k < 1 \) and \( \bar{d}(x_0, x_1) < \infty \), with \( \{x_m\} \) as its consequence. Since \((X, \bar{d})\) is a complete extended quasi-metric space, \( \{x_n\} \) converges to \( x \in X \). Since \( T \) is a contraction, there exists \( y_n \in T(x) \) such that

\[
\bar{d}(x_{n+1}, y_n) \leq k \bar{d}(x_n, x),
\]

for any \( n \in \mathbb{N} \). Hence \( \lim_{n \to \infty} \bar{d}(x_{n+1}, y_n) = 0 \) holds. Since \( X \) satisfies the property \((P_1)\), and \( \{x_{n+1}\} \) converges to \( x \), we conclude that \( \lim_{n \to \infty} \bar{d}(y_n, x) = 0 \). Since \( T(x) \) is closed, we conclude that \( x \in T(x) \), i.e., \( x \) is a fixed point of \( T \) as claimed. \( \square \)

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