On a solvable of some systems of rational difference equations

M. M. El-Dessoky\textsuperscript{1,2}

\textsuperscript{1}King Abdulaziz University, Faculty of Science, Mathematics Department, P. O. Box 80203, Jeddah 21589, Saudi Arabia.
\textsuperscript{2}Mansoura University, Faculty of Science, Mathematics Department, Mansoura 35516, Egypt.
E-mail: dessokym@mane.edu.eg.

Abstract

In this paper, we study the existence of solutions for a class of rational systems of difference equations of order four in four-dimensional case

\[
\begin{align*}
    x_{n+1} & = \frac{x_{n-3}}{\pm 1 \pm l_n x_{n-1} + l_n y_{n-2} + 2 x_{n-3}}, &
    y_{n+1} & = \frac{y_{n-3}}{\pm 1 \pm l_n x_{n-1} + l_n y_{n-2} + 2 y_{n-3}}, \\
    z_{n+1} & = \frac{z_{n-3}}{\pm 1 \pm l_n x_{n-1} + l_n y_{n-2} + 2 z_{n-3}}, &
    t_{n+1} & = \frac{t_{n-3}}{\pm 1 \pm l_n x_{n-1} + l_n y_{n-2} + 2 t_{n-3}},
\end{align*}
\]

with the initial conditions are real numbers. Also, we study some behavior such as the periodicity and boundedness of solutions for such systems. Finally, some numerical examples are given to verify our theoretical results and graphed by Matlab.

\textbf{Keywords:} difference equations system, recursive sequences, periodic solutions.
\textbf{Mathematics Subject Classification:} 39A10.

1 Introduction

The theory of discrete dynamical systems and difference equations developed greatly during the last twenty-five years of the twentieth century. Applications of discrete dynamical systems and difference equations have experienced enormous growth in many areas. Many applications of discrete dynamical systems and difference equations have appeared recently in the areas of biology, economics, physics, resource management,
and others. The theory of difference equations occupies a central position in applicable analysis. There is no doubt that the theory of difference equations will continue to play an important role in mathematics as a whole. Nonlinear difference equations of order greater than one are of paramount importance in applications. Such equations also appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations which model various diverse phenomena in biology, ecology, psychology, engineering, physics, probability theory, economics, genetics, physiology and resource management. It is very interesting to investigate the behavior of solutions of a system of higher-order rational difference equations and to discuss the local asymptotic stability of their equilibrium points [1 - 34]. There are many papers deal with the difference equations system. For example, the dynamical behavior of positive solution for the system

\[ x_{n+1} = \frac{x_{n-m+1}}{A + y_n y_{n-1} \cdots y_{n-m+1}}, \quad y_{n+1} = \frac{y_{n-m+1}}{A + x_n x_{n-1} \cdots x_{n-m+1}}, \]

has been studied by Sroysang in [1].

In [2] El-Metwally, presented solutions of the following sixteen systems of difference equations

\[ x_n = \frac{x_{n-2} y_{n-1}}{\pm x_{n-2} \pm y_{n-3}}, \quad y_n = \frac{y_{n-2} x_{n-1}}{\pm y_{n-2} \pm x_{n-3}}. \]

In [3] Stević et al. obtained the solutions of the following systems of rational difference equations

\[ x_n = \frac{x_{n-k} y_{n-t}}{b_n x_{n-k} + a_n y_{n-t-k}}, \quad y_n = \frac{y_{n-k} x_{n-t}}{d_n y_{n-k} + c_n x_{n-t-k}}. \]

In [4] Din has investigated the dynamics of a system of fourth-order rational difference equations

\[ x_{n+1} = \frac{\alpha_1 x_{n-3}}{\beta_1 + \gamma_1 y_n y_{n-1} x_{n-2} x_{n-3}}, \quad y_{n+1} = \frac{\alpha_2 y_{n-3}}{\beta_2 + \gamma_2 x_n x_{n-1} y_{n-2} y_{n-3}}. \]

Cinar [5] got the periodicity of the positive solutions of the nonlinear difference equations system

\[ x_{n+1} = \frac{1}{z_n}, \quad y_{n+1} = \frac{y_n}{x_{n-1} y_{n-1}}, \quad z_{n+1} = \frac{1}{x_{n-1}}. \]

El-Dessoky et al. [6] studied the solutions of the difference equation systems

\[ x_{n+1} = \frac{x_{n-1}}{1 + y_n x_{n-1}}, \quad y_{n+1} = \frac{y_{n-1}}{1 + x_n y_{n-1}}, \quad z_{n+1} = \frac{z_{n-m}}{x_n y_n}. \]

Zkan et al. [7] investigated the periodical solutions of the following systems of third order rational difference equations

\[ x_{n+1} = \frac{y_{n-2}}{-1 \pm y_{n-2} x_{n-1} y_n}, \quad y_{n+1} = \frac{x_{n-2}}{-1 \pm x_{n-2} y_{n-1} x_n}, \quad z_{n+1} = \frac{x_{n-2} + y_{n-2}}{-1 \pm x_{n-2} y_{n-1} x_n}. \]
Yazlik et al. [8] studied the behaviour of solutions of the systems of difference equations

$$x_{n+1} = \frac{y_n-2x_n-3y_n-4}{y_nx_n-1(\pm 1 \pm y_n-2x_n-3y_n-4)}, \quad y_{n+1} = \frac{x_n-2y_n-3x_n-4}{x_ny_n-1(\pm 1 \pm x_n-2y_n-3x_n-4)}.$$ 

El-Dessoky et al. [9] investigated the form of the solution of the systems of difference equations

$$x_{n+1} = \frac{x_n-2}{\pm 1 + x_n-2z_n-1y_n}, \quad y_{n+1} = \frac{y_n-2}{\pm 1 + y_n-2x_n-1z_n}, \quad z_{n+1} = \frac{z_n-2}{\pm 1 + z_n-2y_n-1x_n},$$

Also, in [10], Kurbanli studied a three-dimensional system of rational difference equations

$$x_{n+1} = \frac{x_n-1}{x_n-1y_n-1}, \quad y_{n+1} = \frac{y_n-1}{y_n-1x_n-1}, \quad z_{n+1} = \frac{z_n}{z_n-1y_n-1}.$$ 

To be motivated by the above studies, our aim in this paper is to investigate the existence of solutions for the rational systems of difference equations of order four in four-dimensional case

$$x_{n+1} = \frac{x_n-3}{\pm 1 + t_nz_n-1y_n-2x_n-3}, \quad y_{n+1} = \frac{y_n-3}{\pm 1 + x_n-1z_n-2y_n-3},$$

$$z_{n+1} = \frac{z_n-3}{\pm 1 + y_nx_n-1t_n-2z_n-3}, \quad t_{n+1} = \frac{t_n-3}{\pm 1 + z_ny_n-1x_n-2t_n-3},$$

where $n \in \mathbb{N}_0$ and the initial conditions $x_i, y_i, z_i, t_i, i = -3, -2, -1, 0$ are arbitrary real numbers. Although we study the dynamics of these solutions such as the periodicity and boundedness and give some numerical examples for the systems.

## 2 Systems and The Expressions of Their Solutions

Here we interest to investigate the following system of some rational difference equations

$$x_{n+1} = \frac{x_n-3}{-1 - t_nz_n-1y_n-2x_n-3}, \quad y_{n+1} = \frac{y_n-3}{-1 - x_n-1z_n-2y_n-3},$$

$$z_{n+1} = \frac{z_n-3}{1 + y_nx_n-1t_n-2z_n-3}, \quad t_{n+1} = \frac{t_n-3}{1 + z_ny_n-1x_n-2t_n-3}.$$ 

(1)

where $n \in \mathbb{N}_0$ and the initial conditions $x_i, y_i, z_i, t_i, i = -3, -2, -1, 0$ are arbitrary real numbers.
Theorem 1  Assume that \( \{x_n, y_n, z_n, t_n\} \) are solutions of system (1). Then for \( n = 0, 1, 2, \ldots \), we obtain

\[
x_{4n-3} = (-1)^n x_{-3} \prod_{i=0}^{n-1} \frac{1 + (2i+1)x_{-3}y_{-2}z_{-1}t_{0}}{1 + (2i+1)x_{-3}y_{-2}z_{-1}t_{0}},
\]
\[
x_{4n-1} = (-1)^n x_{-1} \prod_{i=0}^{n-1} \frac{1 + (2i+2)x_{-3}y_{-2}z_{-1}t_{0}}{1 + (2i+2)x_{-3}y_{-2}z_{-1}t_{0}},
\]
\[
y_{4n-3} = (-1)^n y_{-3} \prod_{i=0}^{n-1} \frac{1 - (2i+1)x_{-3}y_{-2}z_{-1}t_{0}}{1 - (2i+1)x_{-3}y_{-2}z_{-1}t_{0}},
\]
\[
y_{4n-1} = (-1)^n y_{-1} \prod_{i=0}^{n-1} \frac{1 - (2i+2)x_{-3}y_{-2}z_{-1}t_{0}}{1 - (2i+2)x_{-3}y_{-2}z_{-1}t_{0}},
\]
\[
z_{4n-3} = z_{-3} \prod_{i=0}^{n-1} \frac{1 + (2i+1)x_{-3}y_{-2}z_{-1}t_{0}}{1 + (2i+1)x_{-3}y_{-2}z_{-1}t_{0}},
\]
\[
z_{4n-1} = z_{-1} \prod_{i=0}^{n-1} \frac{1 + (2i+1)x_{-3}y_{-2}z_{-1}t_{0}}{1 + (2i+1)x_{-3}y_{-2}z_{-1}t_{0}},
\]

and

\[
t_{4n-3} = t_{-3} \prod_{i=0}^{n-1} \frac{1 + (2i+1)x_{-3}y_{-2}z_{-1}t_{0}}{1 + (2i+1)x_{-3}y_{-2}z_{-1}t_{0}},
\]
\[
t_{4n-1} = t_{-1} \prod_{i=0}^{n-1} \frac{1 + (2i+1)x_{-3}y_{-2}z_{-1}t_{0}}{1 + (2i+1)x_{-3}y_{-2}z_{-1}t_{0}},
\]

where \( \prod_{i=0}^{n-1} A_i = 1 \).

Proof: For \( n = 0 \) the result holds. Now suppose that \( n > 1 \) and that our assumption holds for \( n - 1 \), that is,

\[
x_{4n-7} = (-1)^{n-1} x_{-3} \prod_{i=0}^{n-2} \frac{1 + (2i+2)x_{-3}y_{-2}z_{-1}t_{0}}{1 + (2i+2)x_{-3}y_{-2}z_{-1}t_{0}},
\]
\[
x_{4n-5} = (-1)^{n-1} x_{-1} \prod_{i=0}^{n-2} \frac{1 + (2i+3)x_{-3}y_{-2}z_{-1}t_{0}}{1 + (2i+3)x_{-3}y_{-2}z_{-1}t_{0}},
\]
\[
y_{4n-7} = (-1)^{n-1} y_{-3} \prod_{i=0}^{n-2} \frac{1 - (2i+2)x_{-3}y_{-2}z_{-1}t_{0}}{1 - (2i+2)x_{-3}y_{-2}z_{-1}t_{0}},
\]
\[
y_{4n-5} = (-1)^{n-1} y_{-1} \prod_{i=0}^{n-2} \frac{1 - (2i+3)x_{-3}y_{-2}z_{-1}t_{0}}{1 - (2i+3)x_{-3}y_{-2}z_{-1}t_{0}},
\]
\[
z_{4n-7} = z_{-3} \prod_{i=0}^{n-2} \frac{1 + (2i+2)x_{-3}y_{-2}z_{-1}t_{0}}{1 + (2i+2)x_{-3}y_{-2}z_{-1}t_{0}},
\]
\[
z_{4n-5} = z_{-1} \prod_{i=0}^{n-2} \frac{1 + (2i+3)x_{-3}y_{-2}z_{-1}t_{0}}{1 + (2i+3)x_{-3}y_{-2}z_{-1}t_{0}},
\]
\[ t_{4n-7} = t_{-3} \prod_{i=0}^{n-2} \frac{(1+2i)x_{-3}y_{-3}z_{-1}t_{0}}{(1+2i+1)x_{-3}y_{-3}z_{-2}t_{-1}z_{0}}, \quad t_{4n-6} = t_{-2} \prod_{i=0}^{n-2} \frac{(1+2i+1)x_{-3}y_{-3}z_{-2}t_{-1}x_{0}}{(1+2i+2)x_{-3}y_{-3}z_{-2}t_{-1}y_{0}}; \]

\[ t_{4n-5} = t_{-1} \prod_{i=0}^{n-2} \frac{(-1+2i)y_{-3}z_{-3}t_{-1}x_{0}}{(-1+2i+1)y_{-3}z_{-3}t_{-1}z_{0}}, \quad t_{4n-4} = t_{0} \prod_{i=0}^{n-2} \frac{(1+2i+1)x_{-3}y_{-3}z_{-2}t_{-1}x_{0}}{(1+2i+2)x_{-3}y_{-3}z_{-2}t_{-1}y_{0}}. \]

We deduce from System (1) that

\[ x_{4n-3} = \frac{x_{4n-7}}{-1-t_{4n-4}x_{4n-5}y_{4n-6}x_{4n-7}} = \frac{-1}{(-1)^{n-1}x_{-3} \prod_{i=0}^{n-2} \frac{(1+2i)x_{-3}y_{-3}z_{-1}t_{0}}{(1+2i+1)x_{-3}y_{-3}z_{-2}t_{-1}z_{0}}}\]

\[ = \frac{-1}{(-1)^{n-1}y_{-2} \prod_{i=0}^{n-2} \frac{(1+2i+1)x_{-3}y_{-3}z_{-2}t_{-1}x_{0}}{(1+2i+2)x_{-3}y_{-3}z_{-2}t_{-1}y_{0}} \left(1-t_{0} \prod_{i=0}^{n-2} \frac{(1+2i)x_{-3}y_{-3}z_{-2}t_{-1}x_{0}}{(1+2i+1)x_{-3}y_{-3}z_{-2}t_{-1}y_{0}}\right)}\]

\[ = \frac{(-1)^{n-1}x_{-3} \prod_{i=0}^{n-2} \frac{(1+2i)x_{-3}y_{-3}z_{-2}t_{-1}x_{0}}{(1+2i+1)x_{-3}y_{-3}z_{-2}t_{-1}y_{0}}}{(-1)^{n-1}x_{-3} \prod_{i=0}^{n-2} \frac{(1+2i)x_{-3}y_{-3}z_{-2}t_{-1}x_{0}}{(1+2i+1)x_{-3}y_{-3}z_{-2}t_{-1}y_{0}} - \frac{(1+2i+1)x_{-3}y_{-3}z_{-2}t_{-1}x_{0}}{(1+2i+2)x_{-3}y_{-3}z_{-2}t_{-1}y_{0}}}\]

\[ = (-1)^{n-1}x_{-3} \prod_{i=0}^{n-2} \frac{(1+2i)x_{-3}y_{-3}z_{-2}t_{-1}x_{0}}{(1+2i+1)x_{-3}y_{-3}z_{-2}t_{-1}y_{0}}; \]

\[ y_{4n-3} = \frac{y_{4n-7}}{-1-t_{4n-4}x_{4n-5}y_{4n-6}y_{4n-7}} = \frac{(-1)^{n-1}y_{-3} \prod_{i=0}^{n-2} \frac{(-1+2i)y_{3}z_{3}t_{1}x_{0}}{(-1+2i+1)y_{3}z_{3}t_{1}y_{0}}}{(-1)^{n-1}y_{-3} \prod_{i=0}^{n-2} \frac{(-1+2i)y_{3}z_{3}t_{1}x_{0}}{(-1+2i+1)y_{3}z_{3}t_{1}y_{0}} - \frac{(-1+2i)y_{3}z_{3}t_{1}x_{0}}{(-1+2i+1)y_{3}z_{3}t_{1}y_{0}}}\]

\[ = (-1)^{n-1}y_{-3} \prod_{i=0}^{n-2} \frac{(-1+2i)y_{3}z_{3}t_{1}x_{0}}{(-1+2i+1)y_{3}z_{3}t_{1}y_{0}}; \]
Also, we see from Eq.(1) that

\[ z_{4n-3} = \frac{z_{4n-7}}{1 + y_{4n-4}x_{4n-5} + \frac{z_{4n-6}z_{4n-7}}{1 + y_{4n-6}}}. \]

Finally from Eq.(1), we see that

\[ t_{4n-3} = 1 + \frac{z_{4n-4}x_{4n-5}y_{4n-6} + \frac{z_{4n-6}x_{4n-7}}{1 + y_{4n-6}}}{1 + z_{4n-3}t_{4n-7}}. \]

Similarly we can prove the other relations. This completes the proof.

**Lemma 1.** If \( x_i, y_i, z_i, t_i, i = -3, -2, -1, 0 \) arbitrary real numbers and let \( \{x_n, y_n, z_n, t_n\} \) are solutions of system (1) then the following statements are true:

(i) If \( x_{-3} = 0 \), then we have \( x_{4n-3} = 0 \) and \( y_{4n-2} = (-1)^n y_{-2}, z_{4n-1} = z_{-1}, t_{4n} = t_0 \).

(ii) If \( x_{-2} = 0 \), then we have \( x_{4n-2} = 0 \) and \( y_{4n-1} = (-1)^n y_{-1}, z_{4n} = z_0, t_{4n-3} = t_{-3} \).

(iii) If \( x_{-1} = 0 \), then we have \( x_{4n-1} = 0 \) and \( y_{4n} = (-1)^n y_0, z_{4n-3} = z_{-3}, t_{4n-2} = t_{-2} \).
(iv) If \( x_0 = 0 \), then we have \( x_{4n} = 0 \) and \( y_{4n-3} = (-1)^n y_{-3} \), \( z_{4n-2} = z_{-2} \), \( t_{4n-1} = t_{-1} \).

(v) If \( y_{-3} = 0 \), then we have \( y_{4n-3} = 0 \) and \( x_{4n} = (-1)^n x_0 \), \( z_{4n-2} = z_{-2} \), \( t_{4n-1} = t_{-1} \).

(vi) If \( y_{-2} = 0 \), then we have \( y_{4n-2} = 0 \) and \( x_{4n-3} = (-1)^n x_{-3} \), \( z_{4n-1} = z_{-1} \), \( t_{4n} = t_0 \).

(vii) If \( y_{-1} = 0 \), then we have \( y_{4n-1} = 0 \) and \( x_{4n-2} = (-1)^n x_{-2} \), \( z_{4n} = z_0 \), \( t_{4n-3} = t_{-3} \).

(viii) If \( y_0 = 0 \), then we have \( y_{4n} = 0 \) and \( x_{4n-1} = (-1)^n x_{-1} \), \( z_{4n-3} = z_{-3} \), \( t_{4n-2} = t_{-2} \).

(ix) If \( z_{-3} = 0 \), then we have \( z_{4n-3} = 0 \) and \( x_{4n-1} = (-1)^n x_{-1} \), \( y_{4n} = (-1)^n y_0 \), \( t_{4n-2} = t_{-2} \).

(x) If \( z_{-2} = 0 \), then we have \( z_{4n-2} = 0 \) and \( x_{4n} = (-1)^n x_0 \), \( y_{4n-3} = (-1)^n y_{-3} \), \( t_{4n-1} = t_{-1} \).

(xi) If \( z_{-1} = 0 \), then we have \( z_{4n-1} = 0 \) and \( x_{4n-3} = (-1)^n x_{-3} \), \( y_{4n-2} = (-1)^n y_{-2} \), \( t_{4n} = t_0 \).

(xii) If \( z_0 = 0 \), then we have \( z_{4n} = 0 \) and \( x_{4n-2} = (-1)^n x_{-2} \), \( y_{4n-1} = (-1)^n y_{-1} \), \( t_{4n-3} = t_{-3} \).

(xiii) If \( t_{-3} = 0 \), then we have \( t_{4n-3} = 0 \) and \( x_{4n-2} = (-1)^n x_{-2} \), \( y_{4n-1} = (-1)^n y_{-1} \), \( z_{4n} = z_0 \).

(ivx) If \( t_{-2} = 0 \), then we have \( t_{4n-2} = 0 \) and \( x_{4n-1} = (-1)^n x_{-1} \), \( y_{4n} = (-1)^n y_0 \), \( z_{4n-3} = z_{-3} \).

(vx) If \( t_{-1} = 0 \), then we have \( t_{4n-1} = 0 \) and \( x_{4n} = (-1)^n x_0 \), \( y_{4n-3} = (-1)^n y_{-3} \), \( z_{4n-2} = z_{-2} \).

(vxi) If \( t_0 = 0 \), then we have \( t_{4n} = 0 \) and \( x_{4n-3} = (-1)^n x_{-3} \), \( y_{4n-2} = (-1)^n y_{-2} \), \( z_{4n-1} = z_{-1} \).

**Proof:** The proof follows directly from the expressions of the solutions of system (1).

**Theorem 2** Assume that \( \{x_n, y_n, z_n, t_n\} \) are solutions of the system

\[
\begin{align*}
x_{n+1} &= \frac{x_{n-3}}{1 + t_n x_{n-1} y_{n-2} x_{n-3}}, \quad y_{n+1} = \frac{y_{n-3}}{1 - x_n t_{n-1} z_{n-2} y_{n-3}}, \\
z_{n+1} &= \frac{z_{n-3}}{1 - y_n x_{n-2} z_{n-3}}, \quad t_{n+1} = \frac{t_{n-3}}{1 + z_n y_{n-2} x_{n-3}},
\end{align*}
\]

with \( x_{-3} y_{-2} z_{-1} t_0 \neq \pm 1, t_{-3} x_{-2} y_{-1} z_0 \neq -1, t_{-3} x_{-2} y_{-1} z_0 \neq -\frac{1}{2}, z_{-3} t_{-2} x_{-1} y_0 \neq \pm 1, y_{-3} z_{-2} t_{-1} x_0 \neq 1, y_{-3} z_{-2} t_{-1} x_0 \neq \frac{1}{2} \), takes the form

\[
\begin{align*}
x_{4n-3} &= \frac{x_{-3}}{(1 + x_{-3} y_{-2} z_{-1} t_0)^n}, & x_{4n-2} &= \frac{x_{-2}(1 + t_{-1} z_{-2} y_{-1} z_0)^n}{(1 + z_{-2} y_{-1} z_0)^n}, \\
x_{4n-1} &= \frac{x_{-1}}{(1 + t_{-2} x_{-1} y_{-1} z_0)^n}, & x_{4n} &= x_0 (1 - y_{-1} z_{-2} t_{-1} x_0)^n, \\
y_{4n-3} &= \frac{y_{-3}}{(1 + y_{-3} x_{-2} z_{-1} t_0)^n}, & y_{4n-2} &= y_{-2} (1 + x_{-2} y_{-1} z_{-1} t_0)^n, \\
y_{4n-1} &= \frac{y_{-1}(1 + 2t_{-3} x_{-2} y_{-1} z_0)^n}{(1 + t_{-3} x_{-2} y_{-1} z_0)^n}, & y_{4n} &= y_0 (1 + z_{-3} t_{-2} x_{-1} y_0)^n.
\end{align*}
\]
\[ z_{4n-3} = \frac{z^3}{(1 - z \cdot 3t - 2x - 1y)^n}, \quad z_{4n-2} = \frac{z^2(1 + t \cdot 3t^2 - 2y \cdot 1z)^n}{(1 + 2y \cdot 3z \cdot 2t^2 - 1x_0)^n}, \]

\[ t_{4n-1} = \frac{t^2}{(1 + y \cdot 3z \cdot 2t - 1x_0)^n}, \quad z_4 = z_0 \left( 1 + t \cdot 3t^2 - 2y \cdot 1z_0 \right)^n; \]

\[ t_{4n-3} = \frac{t^2}{(1 + x \cdot 3t^2 - 2y \cdot 1z_0)^n}, \quad t_{4n-2} = t_{\cdot 2} \left( 1 - z \cdot 3t^2 - 2x \cdot 1y_0 \right)^n, \]

\[ t_{4n-1} = \frac{t \cdot 3}{(1 + y \cdot 3z \cdot 2t^2 - 1x_0)^n}, \quad t_4 = t_0 \left( 1 - x \cdot 3y \cdot 2z \cdot 1t_0 \right)^n. \]

**Proof:** For \( n = 0 \) the result holds. Now suppose that \( n > 0 \) and that our assumption holds for \( n - 1 \). That is

\[ x_{4n-7} = \frac{x^3}{(1 + x \cdot 3y \cdot 2z \cdot 1t_0)^n - 1}, \quad x_{4n-6} = x_0 \left( 1 - y \cdot 3z \cdot 2t - 1x_0 \right)^n - 1; \]

\[ y_{4n-7} = \frac{y^2}{(1 + y \cdot 3z \cdot 2t \cdot 1x_0)^n - 1}, \quad y_{4n-6} = y_0 \left( 1 + z \cdot 3t^2 - 2x \cdot 1y_0 \right)^n - 1; \]

\[ z_{4n-7} = \frac{z^2}{(1 + z \cdot 3t \cdot 2x \cdot 1y_0)^n - 1}, \quad z_{4n-6} = z_0 \left( 1 + t \cdot 3x \cdot 2y \cdot 1z_0 \right)^n - 1; \]

\[ t_{4n-7} = \frac{t^2}{(1 + t \cdot 3x \cdot 2y \cdot 1z_0)^n - 1}, \quad t_{4n-6} = t_0 \left( 1 - z \cdot 3t \cdot 2x \cdot 1y_0 \right)^n. \]

It follows from system (2) that

\[ x_{4n-3} = \frac{x_{4n-7}}{1 + t_{4n-4} x_{4n-5} - 5y_{4n-4} - 6x_{4n-4} - 7} = \frac{x^3}{(1 + x \cdot 3y \cdot 2z \cdot 1t_0)^n - 1} \]

\[ = \frac{x^3}{(1 + x \cdot 3y \cdot 2z \cdot 1t_0)^n - 1}, \quad y_{4n-2} = \frac{x_{4n-7}}{1 - x_{4n-4} \cdot 3y_{4n-4} \cdot 3z_{4n-4} \cdot 6y_{4n-6}} = \frac{x_{4n-7}}{1 + y \cdot 3z \cdot 2t \cdot 1x_0)^n - 1} \]

\[ = \frac{y^2}{1 - x_{4n-4} \cdot 3z_{4n-4} \cdot 6y_{4n-6}} = \frac{y^2}{1 - x_{4n-4} \cdot 3y \cdot 2z \cdot 1t_0)^n - 1} \]

\[ = \frac{y^2}{1 - x_{4n-4} \cdot 3y \cdot 2z \cdot 1t_0)^n - 1} = y \cdot 2 \left( 1 + x \cdot 3y \cdot 2z \cdot 1t_0 \right)^n. \]

\[ z_{4n-1} = \frac{z_{4n-5}}{1 - y_{4n-4} \cdot 2x_{4n-4} \cdot 3y_{4n-4} \cdot 4z_{4n-5} - 7} = \frac{z \cdot 1}{1 - x \cdot 3y \cdot 2z \cdot 1t_0)^n - 1} \]

\[ = \frac{z \cdot 1}{1 - x \cdot 3y \cdot 2z \cdot 1t_0)^n - 1}, \quad t_{4n} = \frac{t_{4n-4}}{1 + t_{4n-4} \cdot 3y_{4n-4} \cdot 3z_{4n-4} \cdot 6y_{4n-6}} = \frac{t_{4n-4}}{1 + y \cdot 3z \cdot 2t \cdot 1x_0)^n - 1} \]

\[ = \frac{t \cdot 2}{1 + y \cdot 3z \cdot 2t \cdot 1x_0)^n - 1} = t_0 \left( 1 - z \cdot 3t \cdot 2x \cdot 1y_0 \right)^n. \]
Also, we can prove the other relations similarly. The proof is complete.

**Theorem 3** Let \( \{x_n, y_n, z_n, t_n\} \) are solutions of the system

\[
\begin{align*}
x_{n+1} &= \frac{x_{n-3}}{-1+t_n z_{n-1} y_n - 2x_{n-3}}; \quad y_{n+1} = \frac{y_{n-3}}{1-t_n x_n - 2y_{n-3}}; \\
z_{n+1} &= \frac{z_{n-3}}{1+y_n x_n - 1z_{n-2} z_{n-3}}; \quad t_{n+1} = \frac{t_{n-3}}{1+2y_n x_n - 2t_{n-3}}.
\end{align*}
\]

with the initial values are arbitrary real numbers satisfies \( x_{-3}y_{-2}z_{-1}t_0 \neq 1, \) \( x_{-3}y_{-2}z_{-1}t_0 \neq \frac{1}{2}, \) \( z_{-3}t_{-2}x_{-1}y_0 \neq -1, \) \( z_{-3}t_{-2}x_{-1}y_0 \neq -\frac{1}{2}, \) \( t_{-3}x_{-2}y_{-1}z_0 \neq \pm 1, \) \( y_{-3}z_{-2}t_{-1}x_0 \neq \pm 1. \)

Then the solution are given by the following formula for \( n = 0, 1, 2, \ldots, \)

\[
\begin{align*}
x_{4n-3} &= \frac{x_{-3}}{-1+x_{-3}y_{-2}z_{-1}t_0}^n, \quad x_{4n-2} = (-1)^n x_{-2} (1 + t_{-3}x_{-2}y_{-1}z_0)^n, \\
x_{4n-1} &= \frac{(-1)^n x_{-1}(1+2z_{-3}t_{-2}x_{-1}y_0)^n}{(1+z_{-3}t_{-2}x_{-1}y_0)^n}, \quad x_{4n} = x_0 (1 + y_{-3}z_{-2}t_{-1}x_0)^n, \\
y_{4n-3} &= \frac{(-1)^n y_{-3}}{(1+y_{-3}z_{-2}t_{-1}x_0)^n}, \quad y_{4n-2} = (-1)^n y_{-2} (1 + x_{-3}y_{-2}z_{-1}t_0)^n, \\
y_{4n-1} &= \frac{y_{-1}}{(-1+2x_{-3}y_{-2}z_{-1}t_0)^n}, \quad y_{4n} = (-1)^n y_0 (1 + z_{-3}t_{-2}x_{-1}y_0)^n, \\
z_{4n-3} &= \frac{z_{-3}}{(1+z_{-3}t_{-2}x_{-1}y_0)^n}, \quad z_{4n-2} = z_{-2} (1 + y_{-3}z_{-2}t_{-1}x_0)^n, \\
z_{4n-1} &= \frac{(-1)^n z_{-1}(1+2x_{-3}y_{-2}z_{-1}t_0)^n}{(1+x_{-3}y_{-2}z_{-1}t_0)^n}, \quad z_{4n} = (-1)^n z_0 (1 + t_{-3}x_{-2}y_{-1}z_0)^n, \\
t_{4n-3} &= \frac{t_{-3}}{(1+t_{-3}x_{-2}y_{-1}z_0)^n}, \quad t_{4n-2} = \frac{t_{-2}(1+z_{-3}t_{-2}x_{-1}y_0)^n}{(1+2z_{-3}t_{-2}x_{-1}y_0)^n}, \\
t_{4n-1} &= \frac{t_{-1}}{(-1+y_{-3}z_{-2}t_{-1}x_0)^n}, \quad t_{4n} = (-1)^n t_0 (1 + x_{-3}y_{-2}z_{-1}t_0)^n.
\end{align*}
\]

**Proof:** As the proof of Theorem 2.

Here for confirming the results of this section, we consider an interesting numerical examples of the systems (1) - (2).

**Example 1.** We consider the system (1) with the initial conditions \( x_{-3} = 0.6, \) \( x_{-2} = 3, \) \( x_{-1} = 0.9, \) \( x_0 = 1.3, \) \( y_{-3} = 2, \) \( y_{-2} = 1.3, \) \( y_{-1} = -0.5, \) \( y_0 = 0.1, \) \( z_{-3} = 1.1, \) \( z_{-2} = 0.6, \) \( z_{-1} = -0.7, \) \( z_0 = 1.5, \) \( t_{-3} = -2, \) \( t_{-2} = 0.9, \) \( t_{-1} = -3 \) and \( t_0 = 0.8. \) (See Fig. 1).

![Figure 1. Plot the behavior of the solution of system (1).](image-url)
Example 2. See Figure (2) for an example for the system (2) with the initial values
\( x_{-3} = 0.46, \ x_{-2} = 0.23, \ x_{-1} = 0.29, \ x_0 = 1.16, \ y_{-3} = 0.2, \ y_{-2} = 1.3, \ y_{-1} = -0.5, \ y_0 = 0.61, \ z_{-3} = 0.21, \ z_{-2} = 0.26, \ z_{-1} = 0.27, \ z_0 = 1.89, \ t_{-3} = 0.2, \ t_{-2} = 0.09, \ t_{-1} = 0.28 \) and \( t_0 = 0.58 \).

![Figure 2](image-url)

Figure 2. Plot the behavior of the solution of system (2).

3 Systems have a Periodic Solutions:

In this section, we investigate the solutions and periodic nature of the solutions of the following system of four nonlinear difference equations

\[
\begin{align*}
x_{n+1} &= x_{n-3} - t_{n+1} z_{n-1} y_{n-2} x_{n-3}, \\
y_{n+1} &= y_{n-3} - t_{n+1} z_{n-1} y_{n-2} y_{n-3}, \\
z_{n+1} &= z_{n-3} - t_{n+1} z_{n-1} z_{n-2} x_{n-3}, \\
t_{n+1} &= t_{n-3} - t_{n+1} z_{n-1} x_{n-2} t_{n-3}.
\end{align*}
\]

where \( n \in \mathbb{N}_0 \) and the initial conditions are arbitrary real numbers.

Theorem 4 Assume that \( x_{-3} y_{-2} z_{-1} t_0 \neq -1, \ t_{-3} x_{-2} y_{-1} z_0 \neq -1, \ z_{-3} t_{-2} x_{-1} y_0 \neq -1, \ y_{-3} z_{-2} t_{-1} x_0 \neq -1 \) and \( x_{-3} y_{-2} z_{-1} t_0 \neq -2, \ t_{-3} x_{-2} y_{-1} z_0 \neq -2, \ z_{-3} t_{-2} x_{-1} y_0 \neq -2, \)
y_{-3} z_{-2} t_{-1} x_0 \neq -2, then all solutions of the system (4) are unbounded and given by the expressions

\[
\begin{align*}
x_{4n-3} &= \frac{x_{-3}}{(-1-x_{-3} y_{-2} z_{-1} t_0)^n}, \\
x_{4n-1} &= \frac{x_{-3}}{(-1-z_{-3} t_{-2} x_{-1} y_0)^n}, \\
y_{4n-3} &= \frac{y_{-3}}{(-1-y_{-3} z_{-2} t_{-1} x_0)^n}, \\
y_{4n-1} &= \frac{y_{-3}}{(-1-t_{3} x_{-2} y_{-1} z_0)^n}, \\
z_{4n-3} &= \frac{z_{-3}}{(-1-z_{-3} t_{-2} x_{-1} y_0)^n}, \\
z_{4n-1} &= \frac{z_{-3}}{(-1-x_{-3} y_{-2} z_{-1} t_0)^n}, \\
t_{4n-3} &= \frac{t_{-3}}{(-1-t_{3} x_{-2} y_{-1} z_0)^n}, \\
t_{4n-1} &= \frac{t_{-3}}{(-1-y_{-3} z_{-2} t_{-1} x_0)^n}.
\end{align*}
\]
**Proof:** For \( n = 0 \) the result holds. Now suppose that \( n > 0 \) and that our assumption holds for \( n - 1 \). That is

\[
\begin{align*}
x_{4n-7} &= \frac{x_{4n-4}^{x_{4n-4}}}{(1-x_{3}y_{2}z_{1}t_{0})^{x_{4n-4}}}, \quad x_{4n-6} = x_{2}(-1 - x_{3}y_{2}z_{1}t_{0})^{x_{4n-6}}; \\
x_{4n-5} &= \frac{x_{4n-4}^{y_{2}}}{(1-x_{3}y_{2}z_{1}t_{0})^{y_{2}}}, \quad x_{4n-4} = x_{0}(-1 - y_{3}z_{2}t_{1}x_{0})^{y_{2}}; \\
y_{4n-7} &= \frac{y_{2}}{y_{3}}, \quad y_{4n-6} = y_{2}(-1 - y_{3}z_{2}t_{1}x_{0})^{y_{2}}; \\
y_{4n-5} &= \frac{y_{3}}{y_{3}}, \quad y_{4n-4} = y_{0}(-1 - z_{3}t_{2}x_{1}y_{0})^{y_{3}}; \\
z_{4n-7} &= \frac{z_{2}}{z_{3}}, \quad z_{4n-6} = z_{2}(-1 - y_{3}z_{2}t_{1}x_{0})^{z_{2}}; \\
z_{4n-5} &= \frac{z_{3}}{z_{3}}, \quad z_{4n-4} = z_{0}(-1 - t_{3}x_{2}y_{1}z_{0})^{z_{3}}; \\
t_{4n-7} &= \frac{t_{3}}{t_{3}}, \quad t_{4n-6} = t_{2}(-1 - z_{3}t_{2}x_{1}y_{0})^{t_{3}}; \\
t_{4n-5} &= \frac{t_{3}}{t_{3}}, \quad t_{4n-4} = t_{0}(-1 - x_{3}y_{2}z_{1}t_{0})^{t_{3}}.
\end{align*}
\]

It follows from system (4) that

\[
\begin{align*}
x_{4n} &= \frac{x_{4n-4}^{x_{4n-4}}}{(1-x_{3}y_{2}z_{1}t_{0})^{x_{4n-4}}}; \\
&= \frac{y_{2}^{y_{2}}}{y_{3}^{y_{3}}}; \\
y_{4n-7} &= \frac{y_{2}}{y_{3}}; \\
z_{4n-2} &= \frac{z_{2}}{z_{3}}; \\
t_{4n-1} &= \frac{t_{3}}{t_{3}}.
\end{align*}
\]

Also, we can prove the other relations similarly. The proof is complete.

**Theorem 5** If the sequences \( \{x_{n}, y_{n}, z_{n}, t_{n}\} \) are solutions of difference equation system (4) such that \( x_{3}y_{2}z_{1}t_{0} = t_{3}x_{2}y_{1}z_{0} = z_{3}t_{2}x_{1}y_{0} = y_{3}z_{2}t_{1}x_{0} = -2 \),

\[
\frac{t_{3}}{t_{3}}.
\]
then all solutions of the system are periodic with period four and takes the form

\[
\begin{align*}
x_{4n-3} &= x_{-3}, \\
y_{4n-3} &= y_{-3}, \\
z_{4n-3} &= z_{-3}, \\
t_{4n-3} &= t_{-3},
\end{align*}
\]

\[
\begin{align*}
x_{4n-2} &= x_{-2}, \\
y_{4n-2} &= y_{-2}, \\
z_{4n-2} &= z_{-2}, \\
t_{4n-2} &= t_{-2},
\end{align*}
\]

\[
\begin{align*}
x_{4n-1} &= x_{-1}, \\
y_{4n-1} &= y_{-1}, \\
z_{4n-1} &= z_{-1}, \\
t_{4n-1} &= t_{-1},
\end{align*}
\]

\[
\begin{align*}
x_{4n} &= x_{0}, \\
y_{4n} &= y_{0}, \\
z_{4n} &= z_{0}, \\
t_{4n} &= t_{0}.
\end{align*}
\]

Or

\[
\begin{align*}
\{x_n\} &= \{x_{-3}, \ x_{-2}, \ x_{-1}, \ x_0, \ x_{-3}, \ x_{-2}, \ldots\}, \\
\{y_n\} &= \{y_{-3}, \ y_{-2}, \ y_{-1}, \ y_0, \ y_{-3}, \ y_{-2}, \ldots\}, \\
\{z_n\} &= \{z_{-3}, \ z_{-2}, \ z_{-1}, \ z_0, \ z_{-3}, \ z_{-2}, \ldots\}, \\
\{t_n\} &= \{t_{-3}, \ t_{-2}, \ t_{-1}, \ t_0, \ t_{-3}, \ t_{-2}, \ldots\}.
\end{align*}
\]

**Proof:** The proof follows from the previous Theorem and will be omitted.

**Example 3.** We put the initial conditions as follows \(x_{-3} = 0.6, \ x_{-2} = 3, \ x_{-1} = 0.9, \ x_0 = 1.3, \ y_{-3} = 0.22, \ y_{-2} = 1.3, \ y_{-1} = -0.5, \ y_0 = 0.1, \ z_{-3} = 1.1, \ z_{-2} = 0.6, \ z_{-1} = 0.7, \ z_0 = 1.5, \ t_{-3} = 0.02, \ t_{-2} = 0.09, \ t_{-1} = -0.3 \) and \(t_0 = 0.8\), for the difference system (4), see Fig. 3.

![Figure 3. Draw of the behavior of the solution of system (4).](image)

**Example 4.** Figure (4) shows the periodicity behavior of the solution of the difference system (4) with the initial conditions \(x_{-3} = 2, \ x_{-2} = -0.5, \ x_{-1} = 1, \ x_0 = 4, \ y_{-3} = -5, \ y_{-2} = 10, \ y_{-1} = 2, \ y_0 = 0.1, \ z_{-3} = 15, \ z_{-2} = 0.6, \ z_{-1} = -0.7, \ z_0 = 1, \ t_{-3} = 2, \ t_{-2} = -4/3, \ t_{-1} = 1/6 \) and \(t_0 = 1/7\).
Figure 4. Plot the periodicity of the solution of system (4).

The following theorems can be proved similarly.

4 Other Systems:

In this section, we get the solutions of the following systems of the difference equations

\[
\begin{align*}
x_{n+1} &= \frac{x_{n-3}}{1-t_n x_{n-1} y_{n-2} x_{n-3}}, & y_{n+1} &= \frac{y_{n-3}}{1-x_n t_n^{-1} z_{n-1} x_{n-2} y_{n-3}}, \\
z_{n+1} &= \frac{z_{n-3}}{1-y_n x_{n-1} t_{n-2} z_{n-3}}, & t_{n+1} &= \frac{t_{n-3}}{1-z_n y_{n-1} x_{n-2} t_{n-3}}.
\end{align*}
\]

(5)

\[
\begin{align*}
x_{n+1} &= \frac{x_{n-3}}{-1+y_n z_{n-1} x_{n-2} y_{n-3}}, & y_{n+1} &= \frac{y_{n-3}}{-1+x_n t_n^{-1} z_{n-1} z_{n-2} y_{n-3}}, \\
z_{n+1} &= \frac{z_{n-3}}{-1-y_n x_{n-1} t_{n-2} z_{n-3}}, & t_{n+1} &= \frac{t_{n-3}}{-1-z_n y_{n-1} x_{n-2} t_{n-3}}.
\end{align*}
\]

(6)

\[
\begin{align*}
x_{n+1} &= \frac{x_{n-3}}{1+t_n y_{n-1} z_{n-2} x_{n-3}}, & y_{n+1} &= \frac{y_{n-3}}{1-x_n t_n^{-1} z_{n-1} x_{n-2} y_{n-3}}, \\
z_{n+1} &= \frac{z_{n-3}}{1+y_n x_{n-1} z_{n-2} z_{n-3}}, & t_{n+1} &= \frac{t_{n-3}}{1-z_n y_{n-1} x_{n-2} t_{n-3}}.
\end{align*}
\]

(7)

\[
\begin{align*}
x_{n+1} &= \frac{x_{n-3}}{-1+y_n z_{n-1} y_{n-2} x_{n-3}}, & y_{n+1} &= \frac{y_{n-3}}{1+x_n t_n^{-1} z_{n-1} z_{n-2} y_{n-3}}, \\
z_{n+1} &= \frac{z_{n-3}}{1-y_n x_{n-1} y_{n-2} z_{n-3}}, & t_{n+1} &= \frac{t_{n-3}}{1-z_n y_{n-1} x_{n-2} t_{n-3}}.
\end{align*}
\]

(8)

where \( n \in \mathbb{N}_0 \) and the initial conditions are arbitrary real numbers.
Theorem 6 If \( \{x_n, y_n, z_n, t_n\} \) are solutions of difference equation system (5). Then for \( n = 0, 1, 2, \ldots \),

\[
x_{4n-3} = x_{-3} \prod_{i=0}^{n-1} \frac{-(1+4i)x_{-3}y_{-2}z_{-1}t_0}{-(1+4i+1)x_{-3}y_{-2}z_{-1}t_0}, \quad x_{4n-2} = x_{-2} \prod_{i=0}^{n-1} \frac{-(1+4i+1)x_{-3}y_{-2}z_{-1}t_0}{-(1+4i+2)x_{-3}y_{-2}z_{-1}t_0},
\]

\[
x_{4n-1} = x_{-1} \prod_{i=0}^{n-1} \frac{-(1+4i+3)x_{-3}y_{-2}z_{-1}t_0}{-(1+4i+4)x_{-3}y_{-2}z_{-1}t_0}, \quad x_{4n} = x_0 \prod_{i=0}^{n-1} \frac{-(1+4i+3)x_{-3}y_{-2}z_{-1}t_0}{-(1+4i+4)x_{-3}y_{-2}z_{-1}t_0},
\]

\[
y_{4n-3} = y_{-3} \prod_{i=0}^{n-1} \frac{-(1+4i+2)x_{-3}y_{-2}z_{-1}t_0}{-(1+4i+3)x_{-3}y_{-2}z_{-1}t_0}, \quad y_{4n-2} = y_{-2} \prod_{i=0}^{n-1} \frac{-(1+4i+1)x_{-3}y_{-2}z_{-1}t_0}{-(1+4i+2)x_{-3}y_{-2}z_{-1}t_0},
\]

\[
y_{4n-1} = y_{-1} \prod_{i=0}^{n-1} \frac{-(1+4i+3)x_{-3}y_{-2}z_{-1}t_0}{-(1+4i+4)x_{-3}y_{-2}z_{-1}t_0}, \quad y_{4n} = y_0 \prod_{i=0}^{n-1} \frac{-(1+4i+3)x_{-3}y_{-2}z_{-1}t_0}{-(1+4i+4)x_{-3}y_{-2}z_{-1}t_0},
\]

\[
z_{4n-3} = z_{-3} \prod_{i=0}^{n-1} \frac{-(1+4i)x_{-3}y_{-2}z_{-1}t_0}{-(1+4i+1)x_{-3}y_{-2}z_{-1}t_0}, \quad z_{4n-2} = z_{-2} \prod_{i=0}^{n-1} \frac{-(1+4i+1)x_{-3}y_{-2}z_{-1}t_0}{-(1+4i+2)x_{-3}y_{-2}z_{-1}t_0},
\]

\[
z_{4n-1} = z_{-1} \prod_{i=0}^{n-1} \frac{-(1+4i+2)x_{-3}y_{-2}z_{-1}t_0}{-(1+4i+3)x_{-3}y_{-2}z_{-1}t_0}, \quad z_{4n} = z_0 \prod_{i=0}^{n-1} \frac{-(1+4i+3)x_{-3}y_{-2}z_{-1}t_0}{-(1+4i+4)x_{-3}y_{-2}z_{-1}t_0},
\]

\[
t_{4n-3} = t_{-3} \prod_{i=0}^{n-1} \frac{-(1+4i)x_{-3}y_{-2}z_{-1}t_0}{-(1+4i+1)x_{-3}y_{-2}z_{-1}t_0}, \quad t_{4n-2} = t_{-2} \prod_{i=0}^{n-1} \frac{-(1+4i+1)x_{-3}y_{-2}z_{-1}t_0}{-(1+4i+2)x_{-3}y_{-2}z_{-1}t_0},
\]

\[
t_{4n-1} = t_{-1} \prod_{i=0}^{n-1} \frac{-(1+4i+2)x_{-3}y_{-2}z_{-1}t_0}{-(1+4i+3)x_{-3}y_{-2}z_{-1}t_0}, \quad t_{4n} = t_0 \prod_{i=0}^{n-1} \frac{-(1+4i+3)x_{-3}y_{-2}z_{-1}t_0}{-(1+4i+4)x_{-3}y_{-2}z_{-1}t_0},
\]

where \( \prod_{i=0}^{n-1} A_i = 1 \).

Lemma 2. If \( x_i, y_i, z_i, t_i, i = -3, -2, -1, 0 \) arbitrary real numbers and let \( \{x_n, y_n, z_n, t_n\} \) are solutions of system (1) then the following statements are true:

(i) If \( x_{-3} = 0 \), then we have \( x_{4n-3} = 0 \) and \( y_{4n-2} = y_{-2} \), \( z_{4n-1} = z_{-1} \), \( t_{4n} = t_0 \).

(ii) If \( x_{-2} = 0 \), then we have \( x_{4n-2} = 0 \) and \( y_{4n-1} = y_{-1} \), \( z_{4n} = z_0 \), \( t_{4n-3} = t_{-3} \).

(iii) If \( x_{-1} = 0 \), then we have \( x_{4n-1} = 0 \) and \( y_{4n} = y_0 \), \( z_{4n-3} = z_{-3} \), \( t_{4n-2} = t_{-2} \).

(iv) If \( x_0 = 0 \), then we have \( x_{4n} = 0 \) and \( y_{4n-3} = y_{-3} \), \( z_{4n-2} = z_{-2} \), \( t_{4n-1} = t_{-1} \).

(v) If \( y_{-3} = 0 \), then we have \( y_{4n-3} = 0 \) and \( x_{4n} = x_0 \), \( z_{4n-2} = z_0 \), \( t_{4n-3} = t_{-3} \).

(vi) If \( y_{-2} = 0 \), then we have \( y_{4n-2} = 0 \) and \( x_{4n-3} = x_{-3} \), \( z_{4n-1} = z_{-1} \), \( t_{4n} = t_0 \).

(vii) If \( y_{-1} = 0 \), then we have \( y_{4n-1} = 0 \) and \( x_{4n-2} = x_{-2} \), \( z_{4n} = z_0 \), \( t_{4n-3} = t_{-3} \).

(viii) If \( y_{0} = 0 \), then we have \( y_{4n} = 0 \) and \( x_{4n-1} = x_{-1} \), \( z_{4n-3} = z_{-3} \), \( t_{4n-2} = t_{-2} \).

(ix) If \( z_{-3} = 0 \), then we have \( z_{4n-3} = 0 \) and \( x_{4n-1} = x_{-1} \), \( y_{4n} = y_0 \), \( t_{4n-2} = t_{-2} \).

(x) If \( z_{-2} = 0 \), then we have \( z_{4n-2} = 0 \) and \( x_{4n} = x_0 \), \( y_{4n-3} = y_{-3} \), \( t_{4n-1} = t_{-1} \).

(xi) If \( z_{-1} = 0 \), then we have \( z_{4n-1} = 0 \) and \( x_{4n-3} = x_{-3} \), \( y_{4n-2} = y_2 \), \( t_{4n} = t_0 \).

(xii) If \( z_{0} = 0 \), then we have \( z_{4n} = 0 \) and \( x_{4n-2} = x_{-2} \), \( y_{4n-1} = y_{-1} \), \( t_{4n-3} = t_{-3} \).

(xiii) If \( t_{-3} = 0 \), then we have \( t_{4n-3} = 0 \) and \( x_{4n-2} = x_{-2} \), \( y_{4n-1} = y_{-1} \), \( z_{4n} = z_0 \).

(ivx) If \( t_{-2} = 0 \), then we have \( t_{4n-2} = 0 \) and \( x_{4n-1} = x_{-1} \), \( y_{4n} = y_0 \), \( z_{4n-3} = z_{-3} \).
(v) If \( t_{-1} = 0 \), then we have \( t_{4n-1} = 0 \) and \( x_{4n} = x_0; \ y_{4n-3} = y_{-3}; \ z_{4n-2} = z_{-2}. \)
(vi) If \( t_0 = 0 \), then we have \( t_{4n} = 0 \) and \( x_{4n-3} = x_{-3}; \ y_{4n-2} = y_{-2}; \ z_{4n-1} = z_{-1}. \)

**Theorem 7** The form of the solutions of system (6) are given by the following formulae:

\[
\begin{align*}
x_{4n-3} &= \frac{x_{-3}}{(1+x_{-3}y_{-2}z_{-1}t_0)^n}, \quad x_{4n-2} = \frac{(-1)^nx_{-2}(1+t_{-3}x_{-2}y_{-1}z_0)^n}{(1+2t_{-3}x_{-2}y_{-1}z_0)^n}, \\
x_{4n-1} &= \frac{x_{-1}}{(1+x_{-1}z_{-2}t_{-1}y_0)^n}, \quad x_{4n} = x_0 \left(1 - y_{-3}z_{-2}t_{-1}x_0 \right)^n, \\
y_{4n-3} &= y_{-3} \left(1+y_{-3}z_{-2}t_{-1}x_0 \right)^n, \quad y_{4n-2} = y_{-2} \left(1 + x_{-3}y_{-2}z_{-1}t_0 \right)^n, \\
y_{4n-1} &= \frac{(-1)^ny_{-1}(1+2t_{-3}x_{-2}y_{-1}z_0)^n}{(1+y_{-1}z_{-2}t_{-1}y_0)^n}, \quad y_{4n} = y_0 \left(1 - z_{-3}t_{-2}x_{-1}y_0 \right)^n, \\
z_{4n-3} &= \frac{z_{-3}}{(1+z_{-3}t_{-2}x_{-1}y_0)^n}, \quad z_{4n-2} = \frac{(-1)^nz_{-2}(1+y_{-3}z_{-2}t_{-1}x_0)^n}{(1+z_{-2}t_{-1}y_0)^n}, \\
z_{4n-1} &= \frac{z_{-1}}{(1+z_{-1}y_{-2}t_{-1}z_0)^n}, \quad z_{4n} = (-1)^n z_0 \left(1 + t_{-3}x_{-2}y_{-1}z_0 \right)^n, \\
t_{4n-3} &= \frac{t_{-3}}{(1+t_{-3}y_{-2}z_{-1}x_0)^n}, \quad t_{4n-2} = t_{-2} \left(1 + z_{-3}t_{-2}x_{-1}y_0 \right)^n, \\
t_{4n-1} &= \frac{t_{-1}}{(1+t_{-1}z_{-2}t_{-1}y_0)^n}, \quad t_{4n} = (-1)^nt_0 \left(1 + x_{-3}y_{-2}z_{-1}t_0 \right)^n,
\end{align*}
\]

where \( x_{-3}y_{-2}z_{-1}t_0 \neq \pm 1, \ z_{-3}t_{-2}x_{-1}y_0 \neq \pm 1, \ t_{-3}x_{-2}y_{-1}z_0 \neq -1, \ t_{-3}x_{-2}y_{-1}z_0 \neq -\frac{1}{2}, \ y_{-3}z_{-2}t_{-1}x_0 \neq 1, \ y_{-3}z_{-2}t_{-1}x_0 \neq \frac{1}{2}. \)

**Theorem 8** Let \( \{x_n, y_n, z_n, t_n\} \) are solutions of difference equation system (7) with \( x_{-3}y_{-2}z_{-1}t_0 \neq -1, \ y_{-3}z_{-2}t_{-1}x_0 \neq 1, \ t_{-3}x_{-2}y_{-1}z_0 \neq 1, \ z_{-3}t_{-2}x_{-1}y_0 \neq -1 \), then for \( n = 0, 1, 2, ..., \)

\[
\begin{align*}
x_{4n-3} &= \frac{x_{-3}}{(1+x_{-3}y_{-2}z_{-1}t_0)^n}, \quad x_{4n-2} = \frac{(-1)^nx_{-2}(1+t_{-3}x_{-2}y_{-1}z_0)^n}{(1+2t_{-3}x_{-2}y_{-1}z_0)^n}, \\
x_{4n-1} &= \frac{x_{-1}}{(1+x_{-1}z_{-2}t_{-1}y_0)^n}, \quad x_{4n} = (-1)^nx_0 \left(1 - y_{-3}z_{-2}t_{-1}x_0 \right)^n; \\
y_{4n-3} &= \frac{y_{-3}}{(1+y_{-3}z_{-2}t_{-1}x_0)^n}, \quad y_{4n-2} = y_{-2} \left(1 + x_{-3}y_{-2}z_{-1}t_0 \right)^n, \\
y_{4n-1} &= \frac{(-1)^ny_{-1}}{(1+2t_{-3}x_{-2}y_{-1}z_0)^n}, \quad y_{4n} = y_0 \left(1 + z_{-3}t_{-2}x_{-1}y_0 \right)^n, \\
z_{4n-3} &= \frac{z_{-3}}{(1+z_{-3}t_{-2}x_{-1}y_0)^n}, \quad z_{4n-2} = (-1)^nz_{-2} \left(1 + y_{-3}z_{-2}t_{-1}x_0 \right)^n, \\
z_{4n-1} &= \frac{z_{-1}}{(1+y_{-1}z_{-2}t_{-1}y_0)^n}, \quad z_{4n} = (-1)^n z_0 \left(1 + t_{-3}x_{-2}y_{-1}z_0 \right)^n, \\
t_{4n-3} &= \frac{t_{-3}}{(1+t_{-3}y_{-2}z_{-1}x_0)^n}, \quad t_{4n-2} = t_{-2} \left(1 + z_{-3}t_{-2}x_{-1}y_0 \right)^n, \\
t_{4n-1} &= \frac{t_{-1}}{(1+t_{-1}z_{-2}t_{-1}y_0)^n}, \quad t_{4n} = (-1)^nt_0 \left(1 + x_{-3}y_{-2}z_{-1}t_0 \right)^n.
\end{align*}
\]

**Theorem 9** Suppose that the initial conditions of the system (8) are arbitrary real numbers satisfies \( x_{-3}y_{-2}z_{-1}t_0 \neq \pm 1, \ y_{-3}z_{-2}t_{-1}x_0 \neq -1, \ y_{-3}z_{-2}t_{-1}x_0 \neq -\frac{1}{2}, \ t_{-3}x_{-2}y_{-1}z_0 \neq 1, \ t_{-3}x_{-2}y_{-1}z_0 \neq \frac{1}{2}, \ z_{-3}t_{-2}x_{-1}y_0 \neq \pm 1 \), and if \( \{x_n, y_n, z_n, t_n\} \) are solutions of system
(8). Then for \( n = 0, 1, 2, \ldots \),

\[
\begin{align*}
\quad x_{4n-3} & = \frac{(-1)^nx_{3}}{(-1 + x + y - z)}; \quad \quad x_{4n-2} = \frac{x_{2}(-1 + t - x + y)}{(-1 + 2t - x + y)}; \\
\quad x_{4n-1} & = \frac{(-1)^nx_{1}}{(-1 + 2 - t - x + y)}; \quad \quad x_{4n} = x_{0}(1 + y + z - 2t - x)\; , \\
\quad y_{4n-3} & = \frac{y_{3}}{(-1 + x + y - z)}; \quad y_{4n-2} = (-1)^n y_{-2}(-1 + x + y - z - t)\; , \\
\quad y_{4n-1} & = \frac{y_{1}}{(-1 + x + y - z)}; \quad y_{4n} = (-1)^n y_{0}(1 + x + y - z - t)\; , \\
\quad z_{4n-3} & = \frac{z_{3}}{(-1 + x + y - z)}; \quad z_{4n-2} = \frac{z_{2}}{(-1 + x + y - z)}; \quad z_{4n} = (-1)^n z_{0}(1 + x + y - z - t)\; , \\
\quad z_{4n-1} & = \frac{z_{1}}{(-1 + x + y - z)}; \quad t_{4n-2} = t_{2}(1 + x + y - z - t)\; , \\
\quad t_{4n-3} & = \frac{t_{3}}{(-1 + x + y - z)}; \quad t_{4n} = t_{0}(1 + x + y - z - t)\; .
\end{align*}
\]

**Theorem 10** Assume that \( \{x_{n}, y_{n}, z_{n}, t_{n}\} \) are solutions of the system (9), with \( x_{-3y-2z-1t_{0}} \neq 1, z_{-3t_{-2}y_{-1}z_{0}} \neq 1, y_{-3t_{-2}y_{-1}z_{0}} \neq 1, t_{-3t_{-2}x_{-1}z_{0}} = -2, \) then for \( n = 0, 1, 2, \ldots \),

\[
\begin{align*}
\quad x_{4n-3} & = \frac{x_{3}}{(-1 + x + y - z)}; \quad x_{4n-2} = x_{2}(-1 - t - x - y)\; , \\
\quad x_{4n-1} & = \frac{x_{1}}{(-1 + x + y - z)}; \quad x_{4n} = x_{0}(1 + y + z - t)\; , \\
\quad y_{4n-3} & = \frac{y_{3}}{(-1 + x + y - z)}; \quad y_{4n-2} = y_{2}(-1 - x + y + z - t)\; , \\
\quad y_{4n-1} & = \frac{y_{1}}{(-1 + x + y - z)}; \quad y_{4n} = y_{0}(1 + x + y + z - t)\; , \\
\quad z_{4n-3} & = \frac{z_{3}}{(-1 + x + y - z)}; \quad z_{4n-2} = z_{2}(-1 - y - z - t)\; , \\
\quad z_{4n-1} & = \frac{z_{1}}{(-1 + x + y - z)}; \quad z_{4n} = z_{0}(1 + x + y + z - t)\; , \\
\quad t_{4n-3} & = \frac{t_{3}}{(-1 + x + y - z)}; \quad t_{4n-2} = t_{2}(-1 - t - x - y)\; , \\
\quad t_{4n-1} & = \frac{t_{1}}{(-1 + x + y - z)}; \quad t_{4n} = t_{0}(1 + x + y + z - t)\; .
\end{align*}
\]

**Theorem 11** If the sequences \( \{x_{n}, y_{n}, z_{n}, t_{n}\} \) are solutions of difference equation system (9) such that \( x_{-3y-2z-1t_{0}} = \; \vdots z_{-3t_{-2}x_{-1}y_{-1}z_{0}} = -2, \) then \( \{x_{n}, y_{n}\} \) are periodic with period four and \( \{z_{n}, t_{n}\} \) are periodic with period eight and the form

\[
\begin{align*}
\quad x_{4n-3} & = \; x_{3}; \quad x_{4n-2} = \; x_{2}; \quad x_{4n-1} = \; x_{1}; \quad x_{4n} = \; x_{0}; \\
\quad y_{4n-3} & = \; y_{3}; \quad y_{4n-2} = \; y_{2}; \quad y_{4n-1} = \; y_{1}; \quad y_{4n} = \; y_{0}; \\
\quad z_{4n-3} & = \; (-1)^n z_{3}; \quad z_{4n-2} = \; (-1)^n z_{2}; \quad z_{4n-1} = \; (-1)^n z_{1}; \quad z_{4n} = \; (-1)^n z_{0}; \\
\quad t_{4n-3} & = \; (-1)^n t_{3}; \quad t_{4n-2} = \; (-1)^n t_{2}; \quad t_{4n-1} = \; (-1)^n t_{1}; \quad t_{4n} = \; (-1)^n t_{0}.
\end{align*}
\]

**Example 5.** Figure (5) shows the periodicity behavior of the solution of the difference system (9) with the initial conditions \( x_{-3} = 2, \; x_{-2} = -0.5, \; x_{-1} = 1, \; x_{0} = 4, \)
\( y_{-3} = -5, \ y_{-2} = 3, \ y_{-1} = -2, \ y_0 = 0.1, \ z_{-3} = 5, \ z_{-2} = 0.6, \ z_{-1} = -0.1, \ z_0 = 1, \)

\( t_{-3} = -2, \ t_{-2} = 4, \ t_{-1} = -1/6 \) and \( t_0 = -10/3. \)

![Figure 5. Plot the periodicity of the solution of system (9).](image)

Acknowledgements
This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The authors, therefore, acknowledge with thanks DSR technical and financial support.

References


[5] C. Cinar and I. Yaşar, On the positive solutions of the difference equation system \( x_{n+1} = 1/z_n, \ y_{n+1} = y_n/x_{n-1}y_{n-1}, \ z_{n+1} = 1/x_{n-1}, \) J. Inst. Math. Comp. Sci., 18, (2005), 91-93.


[10] A. S. Kurbanli, On the behavior of solutions of the system of rational difference equations: $x_{n+1} = x_{n-1}/x_{n-1}y_n - 1$, $y_{n+1} = y_{n-1}/y_{n-1}x_n - 1$, and $z_{n+1} = x_n/z_{n-1}y_n - 1$, Discrete Dyn. Nat. Soc., 2011, (2011), Article ID 932362, 12 pages.


[18] A. Kurbanli, C. Cinar and M. Erdoğan, On the behavior of solutions of the system of rational difference equations $x_{n+1} = \frac{x_{n-1}}{x_{n-1}y_n - 1}$, $y_{n+1} = \frac{y_{n-1}}{y_{n-1}x_n - 1}$, $z_{n+1} = \frac{x_n}{z_{n-1}y_n}$. Appl. Math., 2, (2011), 1031-1038.

[19] Hui-qi Ma and Hui Feng, On Positive Solutions for the Rational Difference Equation Systems $x_{n+1} = \frac{A}{x_ny_n}$ and $y_{n+1} = \frac{By_n}{x_{n-1}y_{n-1}}$. International Scholarly Research Notices, 2014, (2014), Article ID 857480, 4 pages.


